Mathematics

SOLUTIONS

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Solutions to JEE Advanced Revision Booklet | Mathematics

QUADRATIC EQUATION

1.(A) $5x^2+5xy=60,\ 24xy+36y^2=-60$. Add and factorise to get $(x+4y)(5x+9y)=0$

2.(C) $\left(a^3-2a+c\right)-\left(b^3-2b+c\right)=0 \Rightarrow a^2+ab+b^2=2$

Let $x=a^2\left(2a^2+4ab+3b^2\right),\ y=b^2\left(3a^2+4ab+2b^2\right)$

Then $x+y-8=2(ab+2)\left(a^2+ab+b^2-2\right)=0 \Rightarrow 3+y-8=0$

3.(A) $D \geq 0$ gives $K \in \left[-4, \frac{-4}{3}\right]$ and $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta = 19-(K+5)^2$

So, maximum $\alpha^2+\beta^2 = 19-(-4+5)^2 = 18$

4.(B) The given expression may be written as $\sqrt{(x-3)^2+\left(x^2-2\right)^2}-\sqrt{x^2+\left(x^2-1\right)^2}$, P $(x,\ y)$ on parabola $y = x^2$ and $A(3,2),\ B(0,1)$

$(PA-PB)_{\max}=AB$

5.(C) $x \leq \frac{a}{5},\ x > \frac{b}{6}$ So, $4 \leq \frac{a}{5} < 5$ and $1 \leq \frac{b}{6} < 2$

So, $a \in [20, 25)$ and $b \in [6, 12)$

So, number of pairs (a, b) is ${}^6C_1 \times {}^5C_1$

6.(B) Complete the squares of each expression to get LHS least value = $\sqrt{25}+\sqrt{144}=17$ and RHS maximum value $=\sqrt{289}=17$ So, $x=3$ is the only solution.

7.(D) $(r-a)(r-b)(r-c)(r-d)=4$. Assume $a>b>c>d$ then $r-a<r-b<r-c<r-d$.

So, we must have $r-a=-2,\ r-b=-1,\ r-c=1,\ r-d=2.$ (integers)

8.(A) $\sum\alpha_i=2n$ and $\sum\alpha_i\alpha_j=2n(n-1)$

So, $\sum\alpha_i^2=\left(\sum\alpha_i\right)^2-2\sum\alpha_i\alpha_j=4n$

So, $\sum\left(\alpha_i-2\right)^2=0 \Rightarrow \alpha_i=2$

9.(C) $\alpha+\beta=-a,\ \alpha\beta=2a-1,\ 2a=1+b$

So, $(\alpha+2)(\beta+2)=3 \Rightarrow \alpha+2=3,\ \beta+2=1$ or $\alpha+2=-1,\ \beta+2=-3$

10.(D) Roots $x_1=\frac{a}{2}-\sqrt{4+\frac{a^2}{4}},\ x_2=\frac{a}{2}+\sqrt{4+\frac{a^2}{4}}$

$B \subseteq A \Rightarrow x_1 \geq -2$ and $x_2 < 4$

Simplify to get $a \in [0, 3)$

11.(B) Let $f(x)=4x^2+5x+17,\quad g(x)=x^2+4x+12,\ h(x)=x^2+x+1$

$\frac{f(x)}{g(x)}=\frac{f(x)+h(x)}{g(x)+h(x)}$; Simplify $f(x)=g(x)\quad (h(x)>0) \Rightarrow 3x^2+11x+5=0$

12.(A) Notice that RHS x is always positive and the inner square root term gets simplified to $\sqrt{1+(x+3)(x+5)}=|x+4|$. This way simplifying the whole LHS side amounts to $2(x+1)=x$

13.(C) 14.(D) 15.(A)

Let $f(x)=ax^2+bx+c$

$$f(0)=c,\ f(1)=a+b+c,\ f(-1)=a-b+c$$

$$b=\frac{f(1)-f(-1)}{2},\ a=\frac{f(1)+f(-1)}{2}-f(0)$$

Maximum value of $ax+b=a+b=2$

$\Rightarrow \quad f(1)-f(0)=2;\ \ f(1)-f(0)\le 1-(-1) \quad \Rightarrow\ f(1)=1$ and $f(0)=-1 \ \Rightarrow\ c=-1$

Least value of $f(x)=f(0)=-1$

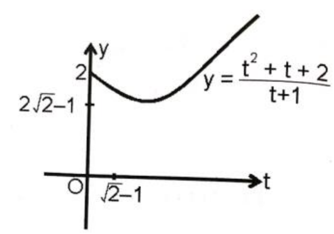
$\Rightarrow \quad -\frac{b}{2a}=0 \ \Rightarrow \ b=0 \ \Rightarrow \ f(1)=f(-1)=1 \quad \therefore \ a=2$

16.(B) 17.(A) 18.(D)

Put $x^2=t$

$$t^2-(k-1)t+2-k=0$$

$$k=\frac{t^2+t+2}{t+1} \qquad t\ge 0$$



From graph it is clear that

$k\in\{2\sqrt{2}-1\}\cup(2,\infty), 2$ real and distinct roots

$k\in\{2\}, 3$ real and distinct roots

$k\in\left(2\sqrt{2}-1, 2\right), 4$ real and distinct roots.

19.(ABCD) $a=\frac{1}{3}[f(2)-f(1)],\ c=\frac{1}{3}[f(2)-4f(1)],$ So, $f(3)=9a-c=\frac{1}{3}[8f(2)-5f(1)]$

20.(ABC) $\sin\theta, \cos\theta=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\sin^2\theta+\cos^2\theta=1\Rightarrow b^2-2ac=a^2\Rightarrow b^2=a(a+2c)$ and $b^2-4ac=a(a-2c)$

So, $a, a-2c, a+2c$ are all perfect squares.

21.(AD) Let α,β,γ be the roots of the cubic then $AM\ge GM$ gives $-a\ge 6$ and real roots of $6x^2-24x-4a=0$ gives the condition $24^2+24\times 4a\ge 0\Rightarrow a=-6$ So, $\alpha=\beta=\gamma=2$

22.(AC) $P(x)-Q(x)=x(a-c)\left(x^2-1\right)$

As $P(0)=Q(0)=1$ So, $x=-1$ and $x=1$ are the common roots.

23.(AD) Let the roots be α,β,γ then $\sum\alpha=-a,\ \sum\alpha\beta=b,\ \alpha\beta\gamma=-c$

Now, $(a+b+c+1)=-45\Rightarrow -a-b-c-1=45$

i.e., $\sum\alpha-\sum\alpha\beta+\alpha\beta\gamma-1=45=(\alpha-1)(\beta-1)(\gamma-1)$ or $(\alpha-1)(\beta-1)(\gamma-1)=3\times 3\times 5$

So, $\alpha=4,\ \beta=4,\ \gamma=6$

24.(ABD) $(x-1)\underbrace{\left(2x^2+3x+5\right)}_{\text{Img. roots}}=0$ and $(x-1)\left(ax^2+bx+c\right)=0$ So, $a+b+c=0$ or $\dfrac{a}{2}=\dfrac{b}{3}=\dfrac{c}{5}$

25.(ABCD) Putting $x=0,1,\dfrac{1}{2}$ one by one in $f(x)$, we have $|c|\le 1$, $|a+b+c|\le 1$, $\left|\dfrac{a}{4}+\dfrac{b}{2}+c\right|\le 1$

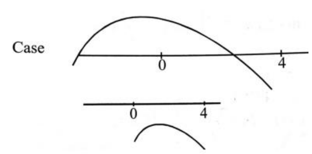
So, $-1\le c\le 1$, $-1\le a+b+c\le 1$ and $-4\le a+2b+4c\le 4$

Rearranging these inequalities, we obtain $-8\le 3a+2b\le 8$, $|a|\le 8$, $|b|\le 8$, $|c|\le 1$

26.(ACD) Let $\sqrt{2x-1}=t$ then the given equation becomes $|t+1|+|t-1|=A\sqrt{2}$
Now consider different cases, $t\le -1$, $-1<t<1$ and $t\ge 1$ etc.

27.(CD) According to given condition, $\dfrac{\left(b^2+4\right)}{2}\le\dfrac{-(4-4c)}{2}$; $c\ge\dfrac{b^2+8}{4}$; $c_{\min}=\dfrac{8}{4}=2$

28.(CD) $f(x)=\dfrac{1}{2bx^2-x^4-3b^2}$ $x\in[-2,1]$



Let $g(t)=2bt-t^2-3b^2$ $t\in[0,4]$

$g(t)<0\,\forall t\in[0,4]$

Case-I if $b\le 0$, then $g(t)$ will decrease in $[0,4]$

Maximum value of $f(t)=f(4)=\dfrac{1}{8b-16-3b^2}$

29.(BD) $a>0, b^2\le 4ac$

$b>0, c^2\le 4ab$

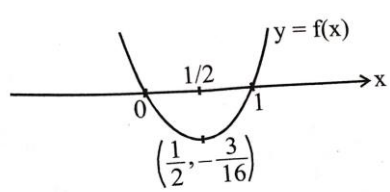
$c>0, a^2\le 4bc$

$\Rightarrow$ $a^2+b^2+c^2<4(ab+bc+ca)$ [$\because$ all equal is impossible]

Also $a^2+b^2+c^2>ab+bc+ca$ ($\because$ a, b, c are distinct)

$\Rightarrow$ $\dfrac{a^2+b^2+c^2}{ab+bc+ca}\in(1,4)$

30.(BCD) $x^4-2x^3+2x^2-x=-\dfrac{a}{8}$



Let $f(x)=x^4-2x^3+2x^2-x=x(x-1)\left(x^2-x+1\right)$

$f'(x)=4x^3-6x^2+4x-1$

$f''(x)=12x^2-12x+4>0$ $\forall x\in R$

$f'\left(\dfrac{1}{2}\right)=0$

Also $f(1-x)=f(x)$

If $-\dfrac{a}{8}\ge-\dfrac{3}{16}$ $\Rightarrow$ $a\le\dfrac{3}{2}$

Then equation has two real roots $\alpha, 1-\alpha$

$\Rightarrow$ sum of all no real roots $= 2-1=1$

If $a > \frac{3}{2}$, then equation has 4 non real roots

$\Rightarrow$ sum of all no real roots = 2.

31. $A-T; B-R,S; C-P,S; D-T$

(A) D$=(2ab\cos C)^2 - 4a^2b^2 = -4a^2b^2 \sin^2 C < 0$

(B) $D = 4\left(b^2 - ax\right) > 0$ $\left[\because \frac{a+c}{2} = b\sqrt{ac} \Rightarrow b^2 > ac\right]$

And $f(x) = ax^2 + abx + c > 0 \quad \forall \quad x \geq 0$

Hence $f(x) = 0$ has both negative roots.

(C) $f(x) = x^2 - (a+1)x - \left(a^2+4\right)$

$f(0) < 0$

(D) $\sqrt{ac} > b \Rightarrow b^2 - ac < 0$; $D = 4\left(b^2 - ac\right) < 0$

32. $A-Q; B-R; C-P; D-T$

$$\sum \alpha\beta = \frac{2^2-(6)}{2} = -1$$

$11 - 3\alpha\beta\gamma = 2\left(6-(1)\right) = 14 \Rightarrow \alpha\beta\gamma = -1$

$\therefore$ $x^2 - 2x^2 - x + 1 = 0 \begin{cases} \alpha \\ \beta \\ \gamma \end{cases}$

$\therefore$ $x^3 = 2x^2 + x - 1$ for $x = \alpha, \beta, \gamma$

$\Rightarrow$ $x^4 = 2x^3 + x^2 - x = 5x^2 + x - 2$ $\Rightarrow$ $\alpha^4 + \beta^4 + \gamma^5 = 5(6) + 2 - 6 = 26$

$x^5 = 5x^3 + x^2 - 2x = 11x^2 + 3x - 5$ for $x = \alpha, \beta, \gamma$

$\Rightarrow$ $\alpha^5 + \beta^5 + \gamma^5 = 11(6) + 3(2) - 15 = 57$

$x^6 = 11x^3 + 3x^2 - 5x = 25x^2 + 6x - 11$ for $x = \alpha, \beta, \gamma$

$\Rightarrow$ $\alpha^6 + \beta^6 + \gamma^6 = 25(6) + 6(2) - 33 = 129$

$-\left(4-\alpha^2\right) + \left(4-\beta^2\right)\left(4-\gamma^2\right) = -2(2-\alpha)(2-\beta)(2-\gamma)(2+\alpha)(2+\beta)(2+\gamma)$

$= \left(8-2(4)-2+1\right)\left((-8-8+2+1)\right) = (-1)(-13) = 13$

33.(2) Let $[x] = t, \{x\} = y, 0 \leq y < 1, x = t + y, x \neq 0$

Simplifying, $8t^2 = 11ty + 10y^2 < 11t + 10$ or $8t^2 - 11t - 10 < 0 \Rightarrow -\frac{5}{8} < t < 2$

So, $t = 1$ and $y = \frac{1}{2}$

34.(0) (Hint: Roots are 1, 2, 3)

35.(5) $P\left(x^5\right)=\left(x^{20}-1\right)+\left(x^{15}-1\right)+\left(x^{10}-1\right)+\left(x^5-1\right)+5$

$P(x)=\dfrac{x^5-1}{x-1}$. Hence, remainder = 5

36.(1) $(x-a)(x-12)+2=0$ should have integral roots.

So, $(x-a)(x-12)=-2=1\times-2$ or -1×2 or -2×1 ; So, $a=9,15$

37.(0) Notice that a, b, c are roots of the equation $x^3-2x^2-x+2=0$

Hence, $x^3-2x^2-x+2=(x-a)(x-b)(x-c)$ *and* Take $\lim\limits_{x\to1}$ on both sides.

38.(4) $f(x)$ will also be a factor of $3\left(x^4+6x^2+25\right)-\left(3x^4+4x^2+28x+5\right)=14\left(x^2-2x+5\right)$

So, $f(x)=x^2-2x+5=(x-1)^2+4$

39.(5) $\Delta>0 \Rightarrow (m-2)^2-\left(m^2-3m+3\right)>0$

$\Rightarrow \quad m<1 \quad \Rightarrow \quad m\in[-1,1)$

$$m\left(\frac{x_1^2}{1-x_1}+\frac{x_2^2}{1-x_2}\right)=m\left[\frac{x_1^2+x_2^2-x_1x_2\left(x_1+x_2\right)}{\left(1-x_1\right)\left(1-x_2\right)}\right]$$

$$=m\left[\frac{\left(x_1+x_2\right)^2-2x_1x_2-x_1x_2\left(x_1+x_2\right)}{\left(1-x_1\right)\left(1-x_2\right)}\right]=m\left[\frac{4(m-2)^2-\left(2-2(m-2)\right)\left(m^2-3m+3\right)}{1+2(m-2)+m^2-3m+3}\right]$$

$$=\left[\frac{2m\left[2\left(m^2-4m+4\right)+(m-3)\left(m^2-3m+3\right)\right]}{m^2-m}\right]=2\left(m^2-3m+1\right)\in(-2,10]$$

40.(7) $p^2-20(66p-1)=k^2$

$\Rightarrow \quad (5p-132)^2-k^2=17404$

$\Rightarrow \quad (5p-132-k)(5p-132+k)=2^2\times10\times229$

$5p-132-k=2\times19$

$5p-132+k=2\times229$

$\Rightarrow \quad p=76$

And $\quad 5p-132-k=2$

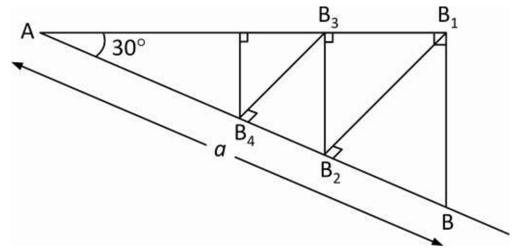
$5p-132+k=2\times19\times229$

Gives on integer solution for p

Hence, $\quad p=76$

TRIGONOMETRY

1.(B) Suppose B_1, B_2, B_3, B_4 . . . are the feet of perpendiculars respectively.



$\therefore$ $BB_1 = a\sin 30°,\ AB_1 = a\cos 30°$.

$B_1B_2 = AB_1 \sin 30° = a\cos 30° \sin 30°$

$AB_2 = AB_1 \cos 30° = a\cos 30° \cos 30°$

$B_2B_3 = a\cos^2 30° \sin 30°,\ AB_3 = a\cos^3 30°$ and so on.

$\therefore$ Length of the infinite polygonal line

$$= BB_1 + B_1B_2 + \ldots = a\ \sin\ 30° + a\ \sin\ 30°\ \cos\ 30° + a\ \sin\ 30° \cos^2 30° + \ldots$$

$$= \frac{a}{2}\left\{1 + \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 + \ldots\right\} = a\left(2+\sqrt{3}\right)$$

2.(B) $\frac{1}{2}(1-\cos A + 1 - \cos B + 1 - \cos C) = \frac{3}{2} - \frac{1}{2}(\cos A + \cos B + \cos C)$ **. . . .(i)**

Let $\cos A + \cos B + \cos C = \lambda$, $\quad\therefore\quad 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} = \lambda$,

$\Rightarrow \quad 2\sin^2\frac{C}{2} - 2\cos\frac{A-B}{2}\sin\frac{C}{2} + \lambda - 1 = 0$.

$\therefore \quad 4\cos^2\frac{A-B}{2} - 8(\lambda - 1) \ge 0 \quad \left(\because\ \sin\frac{C}{2}\ \text{is real}\right), \quad \Rightarrow \quad \lambda \le \frac{2+\cos^2\frac{A-B}{2}}{2} = \frac{2+1}{2} = \frac{3}{2}$

$\therefore$ From (i) : The least value of $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = \frac{3}{2} - \frac{1}{2}.\frac{3}{2} = \frac{3}{4}$

3.(D) Given : A < B < C < D . . . .(i) and

$\sin A = \sin B = \sin C = \sin D = k = (+)ve$ . . . .(ii)

Now, $B = \pi - A, C = 2\pi + A, D = 3\pi - A$

We have chosen the values so as to satisfy the conditions (i) and (ii).

$\therefore \quad 4\sin\frac{A}{2} + 3\sin\frac{1}{2}(\pi - A) + 2\sin\frac{1}{2}(2\pi + A) + \sin\frac{1}{2}(3\pi - A)$

$$= 4\sin\frac{A}{2} + 3\cos\frac{A}{2} - 2\sin\frac{A}{2} - \cos\frac{A}{2} = 2\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)$$

$$= 2\sqrt{\sin^2\frac{A}{2} + \cos^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}} = 2\sqrt{1+\sin A} = 2\sqrt{1+k}$$

4.(C) Let each ratio is equal to $\frac{1}{k}$

$\therefore \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = k\left\{\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right)\right\}$

$$= k\left\{2\sin\left(\theta + \frac{2\pi}{3}\right)\cos\frac{2\pi}{3} + \sin\left(\theta + \frac{2\pi}{3}\right)\right\} = 0 \qquad \left(\text{As } \cos\frac{2\pi}{3} = \frac{-1}{2}\right)$$

$\therefore \quad \sum xy = xy + yz + zx = 0$.

5.(A) $u^2 = \left(a^2+b^2\right)+2\left\{a^2b^2\left(1-2\sin^2\theta\cos^2\theta\right)+\sin^2\theta\cos^2\theta\left(a^4+b^4\right)\right\}^{1/2}$

$$= \left(a^2+b^2\right)+2\left\{a^2b^2+\sin^2\theta\cos^2\theta\left(a^2-b^2\right)^2\right\}^{1/2} = \left(a^2+b^2\right)+\left\{4a^2b^2+\sin^2 2\theta\left(a^2-b^2\right)^2\right\}^{1/2}$$

When $\sin 2\theta = 0$ or 1, then u^2 is min. or maximum respectively.

$\therefore$ Maximum $u^2 = \left(a^2+b^2\right)+\left(a^2+b^2\right)=2\left(a^2+b^2\right)$ and Minimum $u^2 = a^2+b^2+2ab$.

$\therefore$ Difference is $a^2+b^2-2ab=(a-b)^2$

6.(D) Given : $\sin x+\sin y=\sin(x+y)$

$\Rightarrow$ $2\sin\frac{x+y}{2}\left(\cos\frac{x-y}{2}-\cos\frac{x+y}{2}\right)=0 \Rightarrow 4\sin\frac{x+y}{2}\sin\frac{x}{2}\sin\frac{y}{2}=0$

$\Rightarrow$ $\sin\frac{x+y}{2}=0$ or $\sin\frac{x}{2}=0$ or $\sin\frac{y}{2}=0$ **. . . .(i)**

Given : $|x|+|y|=1$ **. . . .(ii)** $|x|\le 1$ and $|y|\le 1$

From (i) : $x+y=0, x=0, y=0$

Putting $x=0$ in (ii) : $y=\pm 1$ and also putting y = 0 in (ii) : $x=\pm 1$

Again, putting $x+y=0 \Rightarrow y=-x$ in (ii) : $2|x|=1 \Rightarrow x=\pm\frac{1}{2}$

We get 6 pairs of (x, y) such as $(0,\pm 1),(\pm 1,0),\left(\frac{1}{2},-\frac{1}{2}\right),\left(-\frac{1}{2},\frac{1}{2}\right)$

7.(B) $T_n=\cot^{-1}\left(2^{n+1}+\frac{1}{2^n}\right)=\cot^{-1}\frac{2^{n+1}.2^n+1}{2^n}=\tan^{-1}\frac{2^n(2-1)}{1+2^{n+1}.2^n}=\tan^{-1}\frac{2^{n+1}-2^n}{1+2^{n+1}.2^n}=\tan^{-1}2^{n+1}-\tan^{-1}2^n$.

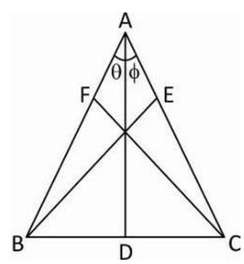
Putting $n=1,2,3,\ldots,n$ and then adding, $S_n=\tan^{-1}2^{n+1}-\tan^{-1}2$

$\therefore$ $S_\infty=\tan^{-1}\infty-\tan^{-1}2=\frac{\pi}{2}-\tan^{-1}2=\cot^{-1}2=\tan^{-1}\frac{1}{2}$

8.(A) $\frac{AD}{BD}=\frac{\sin 60^\circ}{\sin\theta}$ and $\frac{AD}{DC}=\frac{\sin 45^\circ}{\sin\phi}$

$\therefore$ $\frac{DC}{BD}=\frac{AD}{BD}\times\frac{DC}{AD}=\frac{\sin 60^\circ}{\sin 45^\circ}.\frac{\sin\phi}{\sin\theta}$

$\Rightarrow$ $\frac{3}{1}=\frac{\sqrt{3}/2}{1/\sqrt{2}}.\frac{\sin\phi}{\sin\theta}$ $\therefore$ $\frac{\sin\theta}{\sin\phi}=\frac{1}{\sqrt{6}}$



9.(C) Given : $AB=2, BC=5$ and $\angle ABC=60^\circ$

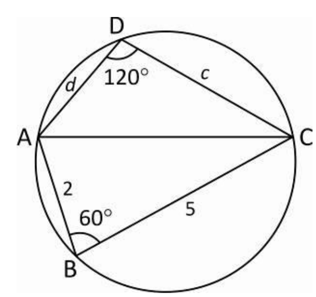
$\angle CDA=180^\circ-60^\circ=120^\circ$ ($\because$ $ABCD$ is cyclic quadrilateral).

Let CD = c and DA = d.

$\therefore$ Area of quadrilateral ABCD = Area of ΔABC

+ Area of $\Delta ACD=\frac{1}{2}AB.BC\sin 60^\circ+\frac{1}{2}CD.DA\sin 120^\circ$

$=\frac{1}{2}.2.5.\frac{\sqrt{3}}{2}+\frac{1}{2}.cd.\frac{\sqrt{3}}{2}=4\sqrt{3}$ (Given)



Thus $\frac{\sqrt{3}}{4}cd = 4\sqrt{3} - \frac{5\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \Rightarrow cd = 6$ **. . . .(i)**

Now, $AB^2 + BC^2 - 2AB\,.\,BC\cos 60° = AC^2 = CD^2 + DA^2 - 2CD\,.\,DA\cos 120°$.

$\Rightarrow$ $4 + 25 - 2.2\,.\,5\,.\,\frac{1}{2} = c^2 + d^2 + cd \Rightarrow c^2 + d^2 + cd = 19$ **. . . .(ii)**

From (i) and (ii) : $c^2 + d^2 = 13$ and $c^2d^2 = 36$

$\therefore$ c^2 and d^2 are the roots of $t^2 - 13t + 36 = 0$

$\therefore$ $t = 9, 4 \Rightarrow c^2 = 9, d^2 = 4$ or $c^2 = 4, d^2 = 9$

Hence, $c = 3, d = 2$ or $c = 2, d = 3$. Thus the other two sides are 2 and 3.

10.(C) $3\sin x - 4\sin^3 x - k = 0 \Rightarrow \sin 3x - k = 0$ **. . . .(i)**

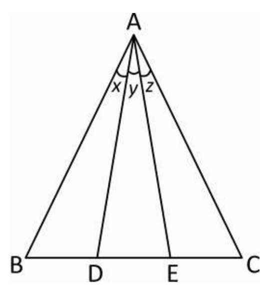
$\because$ A and B satisfy (i),

$\therefore$ $\sin 3A - k = 0$ and $\sin 3B - k = 0$, i.e.,

$\sin 3A = \sin 3B \Rightarrow 3A = 180° - 3B\left(\because\ A > B\right) \Rightarrow A + B = 60°$

$\therefore$ $C = 180° - 60° = \frac{2\pi}{3}$

11.(A) From ΔADC, $\frac{\sin(y+z)}{DC} = \frac{\sin C}{AD}$



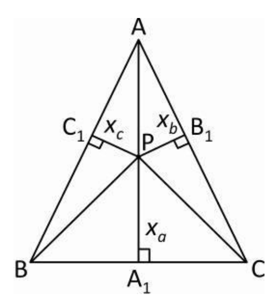
From ΔABD, $\frac{\sin x}{BD} = \frac{\sin B}{AD}$

From ΔAEC, $\frac{\sin z}{EC} = \frac{\sin C}{AE}$

And from ΔABE, $\frac{\sin(x+y)}{BE} = \frac{\sin B}{AE}$

$\therefore$ $\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z} = \frac{BE}{AE} \times \frac{DC}{AD} \times \frac{AD}{BD} \times \frac{AE}{EC} = \frac{2BD \times 2EC}{BD \times EC} = 4$

12.(A) Let A_1, B_1, C_1 be the feet of altitudes drawn from P to sides BC, CA, AB respectively.



Given : $PA_1 = x_a$, $PB_1 = x_b$ and $PC_1 = x_c$

$\therefore$ $\Delta ABC = \Delta BPC + \Delta CPA + \Delta APB$

$\Rightarrow$ $\frac{\sqrt{3}}{4}\,.\,4 = \frac{1}{2}\,.\,2\left(x_a + x_b + x_c\right)$

$\therefore$ $x_a + x_b + x_c = \sqrt{3}$

13.(B) We know that $\cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$ and $\sin\theta = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

$\therefore$ $\tan\frac{\theta}{2}$, $\tan\frac{\phi}{2}$ (are unequal) are the roots of $(x - a)\frac{1 - t^2}{1 + t^2} + y\,.\,\frac{2t}{1 + t^2} = a \Rightarrow xt^2 - 2yt + 2a - x = 0$

$\therefore$ $\tan\frac{\phi}{2}\tan\frac{\theta}{2} = \frac{2a - x}{x}$ **. . . .(i)** and $\quad tan\dfrac{\theta}{2}+tan\dfrac{\phi}{2}=\dfrac{2y}{x}$ $\quad$ **. . . .(ii)**

Now $\quad \left(tan\dfrac{\theta}{2}-tan\dfrac{\phi}{2}\right)^2=\left(tan\dfrac{\theta}{2}+tan\dfrac{\phi}{2}\right)^2-4tan\dfrac{\theta}{2}tan\dfrac{\phi}{2}$. But $tan\dfrac{\theta}{2}-tan\dfrac{\phi}{2}=2e$

$\therefore \quad (2e)^2=\left(\dfrac{2y}{x}\right)^2-4\left(\dfrac{2a-x}{x}\right) \Rightarrow 4e^2=\dfrac{4y^2}{x^2}-4\left(\dfrac{2a-x}{x}\right)$

$\Rightarrow \quad e^2=\dfrac{y^2}{x^2}-\dfrac{2a-x}{x}$, i.e., $y^2=2ax-\left(1-e^2\right)x^2$

14.(B) $\left(x^2-\dfrac{3+x}{2}\right)+(x+1)sin\dfrac{\pi x}{6}=0 \Rightarrow \left(2x^2-x-3\right)+2(x+1)sin\dfrac{\pi x}{6}=0$

$\Rightarrow \quad (x+1)\left\{(2x-3)+2\,sin\dfrac{\pi x}{6}\right\}=0 \Rightarrow x+1=0$ $\quad$ **. . . .(i)**

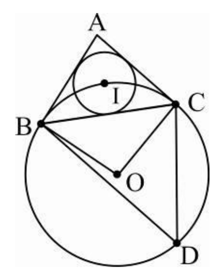
or $\quad (2x-3)+2\,sin\dfrac{\pi x}{6}=0$ $\quad$ **. . . .(ii)**

From (i) : $x=-1$ and from (ii) : $2\,sin\dfrac{\pi x}{6}=3-2x \Rightarrow sin\dfrac{\pi x}{6}=\dfrac{3-2x}{2}$

On drawing the graphs of $y=sin\dfrac{\pi x}{6}$ and $y=\dfrac{3-2x}{2}$, we see that they intersect at the point $x=1, y=\dfrac{1}{2}$. But x = 1 does not lie in the given domain $-2\le x\le 0$

Hence $x=-1$ is the only solution.

15.(B) Let O be the centre of the required circle and R_1 is the radius and $BC = a$



Then $\angle BOC=2\angle BDC=(180°-A) \Rightarrow \angle BDC=90^o-\dfrac{A}{2}$

Now using Sine Law in ΔBOC

$$\frac{a}{\sin\angle BDC}=2R_1$$

$$R_1=\frac{a}{2}\sec\frac{A}{2}$$

16.(B) Let $\dfrac{AF}{FC}=\dfrac{\lambda}{1}$

And let $D(0, 0)$, $B(-a,0), C(a,0), A(h,k)$, then

$$E=\left(\frac{h}{2},\frac{k}{2}\right) \text{ and } F=\left(\frac{\lambda a+h}{\lambda+1},\frac{k}{\lambda+1}\right)$$

Now, $\quad B, E, F$ are collinear

$$\Rightarrow \quad \begin{vmatrix} \dfrac{\lambda a+h}{\lambda+1} & \dfrac{k}{\lambda+1} & 1 \\ \dfrac{h}{2} & \dfrac{k}{2} & 1 \\ -a & 0 & 1 \end{vmatrix}=0 \Rightarrow \lambda=\frac{1}{2} \Rightarrow \frac{AF}{AC}=\frac{1}{3}$$

17.(A) $\sum sin3A=0$

$\Rightarrow \quad sin3A+sin3B+sin3C=0 \qquad \Rightarrow \quad 2\sin\left[\frac{3A+3B}{2}\right]\cos\left[\frac{3A-3B}{2}\right]+2\sin\frac{3C}{2}\cos\frac{3C}{2}=0$

$\Rightarrow \quad 2\sin\frac{3}{2}(\pi-C)\cos\frac{3}{2}(A-B)+2\sin\frac{3C}{2}\cos\frac{3C}{2}=0$

$\Rightarrow \quad 2\sin\left[\frac{3\pi}{2}-\frac{3C}{2}\right]\cos\left(\frac{3A}{2}-\frac{3B}{2}\right)+2\sin\frac{3C}{2}\cos\frac{3C}{2}=0$

$\Rightarrow \quad -2\cos\frac{3C}{2}\cos\left(\frac{3A}{2}-\frac{3B}{2}\right)+2\sin\frac{3C}{2}\cos\frac{3C}{2}=0$

$\Rightarrow \quad 2\cos\frac{3C}{2}\left[\sin\frac{3C}{2}-\cos\left(\frac{3A}{2}-\frac{3B}{2}\right)\right]=0 \qquad \Rightarrow \quad 2\cos\frac{3C}{2}\left[\sin3\left[\frac{\pi}{2}-\frac{(A+B)}{2}\right]-\cos\left(\frac{3A}{2}-\frac{3B}{2}\right)\right]=0$

$\Rightarrow \quad 2\cos\frac{3C}{2}\left[-\cos\left(\frac{3A}{2}+\frac{3B}{2}\right)-\cos\left(\frac{3A}{2}-\frac{3B}{2}\right)\right]=0$

$\Rightarrow \quad -2\cos\frac{3C}{2}\left(2\cos\frac{3A}{2}\cos\frac{3B}{2}\right)=0 \qquad \Rightarrow \quad 4cos\frac{3A}{2}cos\frac{3B}{2}cos\frac{3C}{2}=0$

$\Rightarrow \quad cos\frac{3A}{2}$ or $cos\frac{3B}{2}$ or $cos\frac{3C}{2}=0 \qquad \Rightarrow \quad \frac{3A}{2}=\frac{\pi}{2}$ or $\frac{3B}{2}=\frac{\pi}{2}$ or $\frac{3C}{2}=\frac{\pi}{2}$

$\Rightarrow \quad A=\frac{\pi}{3}$ or $B=\frac{\pi}{3}$ or $C=\frac{\pi}{3} \qquad \Rightarrow$ At least one angle of ΔABC is $60°$

18.(A) We have $\left[sin^{-1}cos^{-1}sin^{-1}tan^{-1}x\right]=1$

$\Rightarrow \quad 1\le sin^{-1}cos^{-1}sin^{-1}tan^{-1}x\le\frac{\pi}{2} \qquad \Rightarrow \quad sin1\le cos^{-1}sin^{-1}tan^{-1}x\le 1$

$\Rightarrow \quad cos\ sin1\ge sin^{-1}tan^{-1}x\ge cos1 \qquad \Rightarrow \quad sin\ cos\ sin1\ge tan^{-1}x\ge sin\ cos1$

$\Rightarrow \quad tan\ sin\ cos\ sin1\ge x\ge tan\ sin\ cos1$ Hence $x\in\left[tan\ sin\ cos1, tan\ sin\ cos\ sin1\right]$

19.(B) $sin\,x+cosec\ x\ge 2$ when $x\in\left[0,\frac{\pi}{2}\right]$ and $tan\,y+cot\,y\ge 2$ when $x\in\left[0,\frac{\pi}{2}\right]$

$\Rightarrow$ Solution of the equation possible only when $sin\,x+cosec\ x=2$ and $tan\,y+cot\,y=2$

$\Rightarrow \quad sin\,x=1$ and $tan\,y=1 \qquad \Rightarrow \quad x=\frac{\pi}{2},\ y=\frac{\pi}{4} \qquad \Rightarrow \quad tan\left(\frac{y}{2}\right)=tan\left(\frac{\pi}{8}\right)=\sqrt{2}-1$

Note that $\left(\sqrt{2}-1\right)$ satisfies $\alpha^2+2\alpha-1=0$

20.(B) $\theta\in\left(0,\frac{\pi}{4}\right)\Rightarrow tan\,\theta\in(0,1)$ and $cot\,\theta\in(1,\infty) \qquad \Rightarrow \quad t_1=(tan\,\theta)^{tan\,\theta}\in(0,1)$ and $t_2=(tan\,\theta)^{cot\,\theta}\in(0,1)$

Also $t_2<t_1$ since numbers between 0 and 1 decrease with increasing power.

$t_3=(cot\,\theta)^{tan\,\theta}\in(1,\infty)$ and $t_4=(cot\,\theta)^{cot\,\theta}\in(1,\infty)$

Also $t_4>t_3$ since numbers greater than 1 increase with increasing power. So, $t_4>t_3>t_1>t_2$

21.(D) $sin^2x+sin^2\left(x+\frac{\pi}{3}\right)+cos\,x\,cos\left(x+\frac{\pi}{3}\right)=sin^2x+\left(\frac{sin\,x}{2}+\frac{\sqrt{3}}{2}cos\,x\right)^2+cos\,x\left(\frac{cos\,x}{2}-\frac{\sqrt{3}}{2}sin\,x\right)$

$=sin^2x+\frac{sin^2x}{4}+\frac{3}{4}cos^2x+\frac{\sqrt{3}}{2}sin\,x\,cos\,x+\frac{cos^2x}{2}-\frac{\sqrt{3}}{2}sin\,x\,cos\,x=\frac{5}{4}\left(sin^2x+cos^2x\right)=\frac{5}{4}$

$\Rightarrow \quad f(x)$ is a constant. So, its period is undefined.

22.(C) $\sin x+\sin y+\sin z-\sin t=(\sin y+\sin z)+(\sin x-\sin t)$

$$=2\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right)+2\sin\left(\frac{x-t}{2}\right)\cos\left(\frac{x+t}{2}\right)$$

$$=2\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right)+2\sin\left(\frac{-(y+z)}{2}\right)\cos\left(\frac{x+t}{2}\right)=2\sin\left(\frac{y+z}{2}\right)\left[\cos\left(\frac{y-z}{2}\right)-\cos\left(\frac{x+t}{2}\right)\right]$$

$$=2\sin\left(\frac{y+z}{2}\right)\left[2\sin\left(\frac{y-z+x+t}{4}\right)\sin\left(\frac{x+t-y+z}{4}\right)\right]=4\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x+z}{2}\right)$$

23.(B) Let $\cos^{-1}x=y \Rightarrow x=\cos y, \frac{1}{2}\le x\le 1 \Rightarrow 0\le y\le\frac{\pi}{3}$

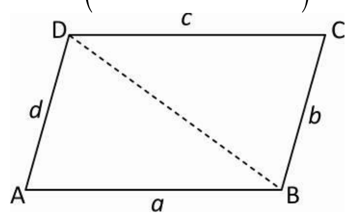
$$\Rightarrow \quad \frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}=\frac{x}{2}+\frac{\sqrt{3}}{2}\sqrt{1-x^2}=\frac{\cos y}{2}+\frac{\sqrt{3}}{2}\sin y=\cos\left(\frac{\pi}{3}-y\right) \Rightarrow \cos^{-1}\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right)=\frac{\pi}{3}-y$$

$$\Rightarrow \quad \cos^{-1}x+\cos^{-1}\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right)=y+\frac{\pi}{3}-y=\frac{\pi}{3}$$

24.(B) Since a circle can be inscribed in the quadrilateral, we have $a+c=b+d$

Since quadrilateral is cyclic, $C=\pi-A$

$$\cos A=\frac{a^2+d^2-BD^2}{2ad} \quad \Rightarrow \quad 2ad\cos A=a^2+d^2-\left(b^2+c^2-2bc\cos C\right)$$



$$\Rightarrow \quad 2(ad+bc)\cos A=a^2+d^2-b^2-c^2$$

$$\Rightarrow \quad \cos A=\frac{a^2+d^2-b^2-c^2}{2(ad+bc)}$$

$a+c=b+d \Rightarrow (a-d)=(b-c) \Rightarrow (a-d)^2=(b-c)^2$

$$\Rightarrow \quad a^2+d^2-b^2-c^2=2ad-2bc \quad \Rightarrow \quad \cos A=\frac{2(ad-bc)}{2(ad+bc)}=\frac{ad-bc}{ad+bc}$$

25.(D) $a_1+a_2\cos 2x+a_3\sin^2 x=1$

$$\Rightarrow \quad a_1+a_2\left(1-2\sin^2 x\right)+a_3\sin^2 x=1 \quad \Rightarrow \quad \sin^2 x(a_3-2a_2)+(a_1+a_2-1)=0$$

Since the above equation is satisfied by every real value of *x*, we have

$a_3-2a_2=0$ and $a_1+a_2-1=0$

3 unknowns and 2 equations $\Rightarrow$ infinite number of solutions

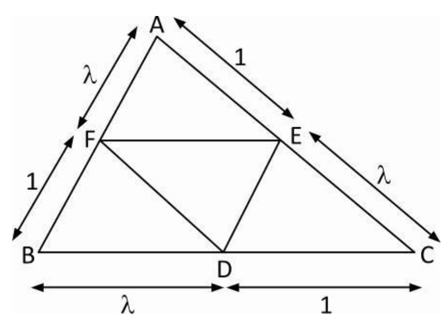
26.(BC) Case 1 : When $2c=a+b$ $\quad\therefore\quad$ $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{b^2+c^2-(2c-b)^2}{2bc}=\frac{4bc-3c^2}{2bc}=\frac{4b-3c}{2b}$

Case 2 : When $2b=a+c$ $\quad\therefore\quad$ $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{b^2+c^2-(2b-c)^2}{2bc}=\frac{4bc-3b^2}{2bc}=\frac{4c-3b}{2c}$

27.(BD) $ar(\Delta AEF)=\frac{1}{2}\times AF\times AE\sin A=\frac{1}{2}\left(\frac{\lambda c}{\lambda+1}\right)\left(\frac{b}{\lambda+1}\right)\sin A$

$$=\frac{\lambda}{(\lambda+1)^2}\left(\frac{1}{2}bc\sin A\right)=\frac{\lambda\Delta}{(\lambda+1)^2}$$

Similarly, $ar(\Delta BDF)=ar(\Delta CDE)=\frac{\lambda\Delta}{(\lambda+1)^2}$



Now, $\dfrac{ar(\Delta DEF)}{ar(\Delta ABC)}=\dfrac{2}{5} \Rightarrow \dfrac{\Delta-\dfrac{3\lambda\Delta}{(\lambda+1)^2}}{\Delta}=\dfrac{2}{5}$

or $1-\dfrac{3\lambda}{(\lambda+1)^2}=\dfrac{2}{5}$ or $\dfrac{\lambda}{(\lambda+1)^2}=\dfrac{1}{5}$ or $5\lambda=\lambda^2+2\lambda+1$ or $\lambda^2-3\lambda+1=0 \Rightarrow \lambda=\dfrac{3\pm\sqrt{5}}{2}$

28.(A) Greatest side is $a \Rightarrow \angle A=90^\circ \Rightarrow b^2+c^2=a^2 \Rightarrow sin\,C=\dfrac{c}{a} \Rightarrow \dfrac{2tan\left(\dfrac{C}{2}\right)}{1+tan^2\left(\dfrac{C}{2}\right)}=\dfrac{c}{a}$

Put $tan\left(\dfrac{C}{2}\right)=t$ to get : $2at=c+ct^2$ or $ct^2-2at+c=0 \Rightarrow t=\dfrac{2a\pm\sqrt{4a^2-4c^2}}{2c}=\dfrac{2a\pm\sqrt{4b^2}}{2c}=\dfrac{a\pm b}{c}$

$\dfrac{a+b}{c}>\dfrac{c}{c}=1$ (Due to triangle inequality)

$\Rightarrow$ If $tan\left(\dfrac{C}{2}\right)=\dfrac{a+b}{c}$, then $tan\left(\dfrac{C}{2}\right)>1$ or $\dfrac{C}{2}>\dfrac{\pi}{4}$ or $C>\dfrac{\pi}{2}$ which is not possible.

$\Rightarrow tan\left(\dfrac{C}{2}\right)=\dfrac{a-b}{c}$

29.(ABD) Since $\left|tan^{-1}x\right|=\begin{cases} tan^{-1}x & \text{if } 0\le tan^{-1}x<\dfrac{\pi}{2} \\ -tan^{-1}x & \text{if } -\dfrac{\pi}{2}<tan^{-1}x<0 \end{cases}=\begin{cases} tan^{-1}x & \text{if } x\ge 0 \\ -tan^{-1}x & \text{if } x<0 \end{cases}$

$\Rightarrow \left|tan^{-1}x\right|=tan^{-1}|x|$ for all $x\in R \Rightarrow tan\left|tan^{-1}x\right|=tan\,tan^{-1}|x|=|x|$ Hence, (A) is correct.

Likewise $sin\left|sin^{-1}x\right|=sin\,sin^{-1}|x|=|x|$ for all $|x|\le 1$. Hence (D) is correct.

$cot\left|cot^{-1}x\right|=cot\,cot^{-1}x=x$ Hence (B) is correct.

Since $|tan\,x|$ is not necessarily always equal to $tan|x|$ Hence $tan^{-1}|tan\,x|\ne|x|$

30.(BD) $\left(sin^{-1}x\right)^3+\left(cos^{-1}x\right)^3=$

$\left(sin^{-1}x+cos^{-1}x\right)^3-3sin^{-1}x\,cos^{-1}x\left(sin^{-1}x+cos^{-1}x\right)$

$=\left(\dfrac{\pi}{2}\right)^3-3sin^{-1}x\,cos^{-1}x\left(\dfrac{\pi}{2}\right)=\dfrac{\pi^3}{8}-\dfrac{3\pi}{2}sin^{-1}x\left(\dfrac{\pi}{2}-sin^{-1}x\right)$

$=\dfrac{\pi^3}{8}-\dfrac{3\pi^2}{4}sin^{-1}x+\dfrac{3\pi}{2}\left(sin^{-1}x\right)^2=\dfrac{\pi^3}{8}+\dfrac{3\pi}{2}\left[\left(sin^{-1}x\right)^2-\dfrac{\pi}{2}sin^{-1}x\right]$

$=\dfrac{\pi^3}{8}+\dfrac{3\pi}{2}\left[\left(sin^{-1}x-\dfrac{\pi}{4}\right)^2\right]-\dfrac{3\pi^3}{32}=\dfrac{\pi^3}{32}+\dfrac{3\pi}{2}\left(sin^{-1}x-\dfrac{\pi}{4}\right)^2$

$\Rightarrow$ The least value is $\dfrac{\pi^3}{32}$ and since $\left(sin^{-1}x-\dfrac{\pi}{4}\right)^2\le\left(\dfrac{3\pi}{4}\right)^2$

$\Rightarrow$ The greatest value is $\dfrac{\pi^3}{32}+\dfrac{9\pi^2}{16}\times\dfrac{3\pi}{2}=\dfrac{7\pi^3}{8}$

31.(AD) For $f(x)$ to be defined, $\left[1+\cos^2 x\right] \in (-\infty, -1] \cup [1, \infty)$

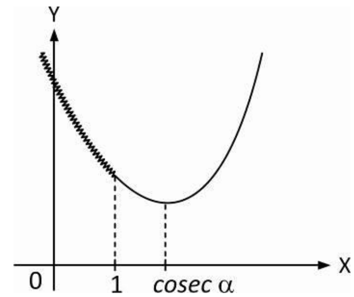
$\cos^2 x \in [0, 1] \Rightarrow 1+\cos^2 x \in [1, 2] \Rightarrow \left[1+\cos^2 x\right] \in \{1, 2\}$

As $\left[1+\cos^2 x\right] \in (-\infty, 1] \cup [1, \infty)\ \forall\ x \in R$, the domain of f is R.

As we have seen above $\left[1+\cos^2 x\right] \in \{1, 2\}\ \forall\ x \in R \quad \Rightarrow \quad$ The range of f is $\left\{\sec^{-1}1, \sec^{-1}2\right\}$

32.(CD) x-coordinate corresponding to the vertex is given by

$$x = \frac{1}{\sin\alpha} = \operatorname{cosec}\alpha > 1.$$



The graph of $f(x) = (\sin\alpha)x^2 - 2x + (b-2),\ \forall\ x \le 1$ is shaded above.

Minimum of $f(x)$ must be greater than 0.
Minimum occurs at $x = 1$.

$\Rightarrow \quad \sin\alpha - 2 + b - 2 \ge 0 \Rightarrow b \ge 4 - \sin\alpha, \alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow \quad b > 3$

33.(BD) $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{3\sqrt{2}}{5}$ and $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{4\sqrt{2}}{5}$

Dividing the above two equations we get : $\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{3}{4}$

$$\Rightarrow \quad \sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{2\times\frac{3}{4}}{1+\frac{9}{16}} = \frac{24}{25} \text{ and } \cos(\alpha+\beta) = \frac{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{1-\frac{9}{16}}{1+\frac{9}{16}} = \frac{7}{25}$$

34.(ABC) Triangle must be obtuse because in an acute angled triangle all the four centres lie inside the triangle and in a right angled triangle two lie inside and other two lie on the triangle.

In an obtuse angled triangle circumcentre and orthocentre lie outside the triangle.

If the triangle is isosceles obtuse angled, then incentre may be collinear with the other three centres.

$$r = \frac{\Delta}{s}, r_1 = \frac{\Delta}{s-a}, r_2 - \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c} \qquad \Rightarrow \qquad r < r_1, r_2 \text{ and } r_3.$$

35.(AB) Applying the operations $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we have $\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \cos^2\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$

$\Rightarrow \quad \left(1+4\sin 4\theta + \sin^2\theta\right) - \left(-\cos^2\theta\right) = 0$ or $1+4\sin 4\theta + 1 = 0$ or $\sin 4\theta = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow \quad 4\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$ or $\theta = \frac{n\pi}{4} + \frac{1}{4}(-1)^n\left(-\frac{\pi}{6}\right)$

$0 \le \theta \le \frac{\pi}{2} \Rightarrow \theta = \frac{7\pi}{24}, \frac{11\pi}{24}$

36.(BD) $x = sin\left(2tan^{-1}2\right)$

Assume that $tan^{-1}2 = \theta \Rightarrow x = sin(2\theta) = \dfrac{2tan\theta}{1+tan^2\theta} = \dfrac{2(2)}{1+2^2} = \dfrac{4}{5} \Rightarrow y = sin\left(\dfrac{1}{2}tan^{-1}\dfrac{4}{3}\right)$

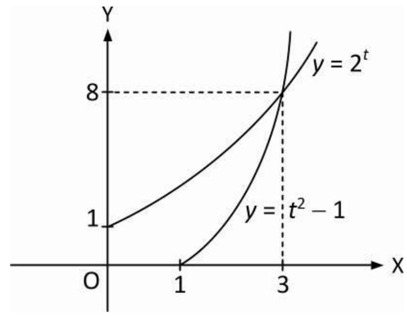
Assume that $tan^{-1}\dfrac{4}{3} = \alpha \Rightarrow y = sin\left(\dfrac{\alpha}{2}\right)$

Now $tan\alpha = \dfrac{4}{3} \Rightarrow sin\alpha = \dfrac{4}{5}$ and $cos\alpha = \dfrac{3}{5}$

$cos\alpha = 1 - 2sin^2\dfrac{\alpha}{2} \Rightarrow \dfrac{3}{5} = 1 - 2sin^2\dfrac{\alpha}{2} \Rightarrow sin\dfrac{\alpha}{2} = \pm\dfrac{1}{\sqrt{5}}$

Since $\alpha = tan^{-1}\dfrac{4}{3}, 0 < \alpha < \dfrac{\pi}{2} \Rightarrow sin\left(\dfrac{\alpha}{2}\right) > 0 \Rightarrow y = sin\dfrac{\alpha}{2} = \dfrac{1}{\sqrt{5}}$

37.(BD) $\left(1-tan^2\theta\right)\left(1+tan^2\theta\right)sec^2\theta + 2^{tan^2\theta} = 0$



$\Rightarrow \left(1-tan^2\theta\right)\left(1+tan^2\theta\right) + 2^{tan^2\theta} = 0$

$\Rightarrow \left(1-tan^4\theta\right) + 2^{tan^2\theta} = 0$

Put $tan^2\theta = t$ to get :

$2^t = t^2 - 1,\ t > 0$

Solving the above equation graphically $t = 3$ is a solution

$\Rightarrow tan^2\theta = 3$ or $tan\theta = \pm\sqrt{3} \Rightarrow \theta = \pm\dfrac{\pi}{3}$ are solutions.

Also after t = 3 at some value of t, 2^t becomes above $t^2 - 1$

$\Rightarrow$ One root also exist after t = 3

Hence two solution in $\left(0, \dfrac{\pi}{2}\right)$ and in $\left(-\dfrac{\pi}{2}, 0\right)$

38.(AB) $f(x) = tan3x \,.\, cot\,x = \dfrac{3tan\,x - tan^3x}{1-3tan^2x} \cdot \dfrac{1}{tan\,x} = \dfrac{3-tan^2x}{1-3tan^2x}$

Let $y = \dfrac{3-tan^2x}{1-3tan^2x} \Rightarrow tan^2x = \dfrac{y-3}{3y-1}$ As $tan^2x > 0, \dfrac{y-3}{3y-1} > 0 \Rightarrow y \in \left(-\infty, \dfrac{1}{3}\right) \cup (3, \infty)$

39.(AD) $sin(B-C) = \dfrac{3}{5} \Rightarrow cos(B-C) = \dfrac{4}{5} = \dfrac{1-t^2}{1+t^2}$, where $t = tan\left(\dfrac{B-C}{2}\right)$

$\Rightarrow t^2 = \dfrac{1}{9} \Rightarrow t = \pm\dfrac{1}{3} = tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}cot\left(\dfrac{A}{2}\right)$

Put $b = 2c$ to get $\pm\dfrac{1}{3} = \dfrac{1}{3}\cot\left(\dfrac{A}{2}\right) \Rightarrow \cot\left(\dfrac{A}{2}\right) = \pm 1$

$\cot\left(\dfrac{A}{2}\right)$ can't be negative $\Rightarrow \cot\left(\dfrac{A}{2}\right) = 1 = \cot\left(\dfrac{\pi}{4}\right) \Rightarrow \dfrac{A}{2} = \dfrac{\pi}{4}$ or $A = \dfrac{\pi}{2}$

$a = \sqrt{b^2+c^2} = \sqrt{4c^2+c^2} = \sqrt{5}c \Rightarrow a : c = \sqrt{5} : 1.$

40.(ABD) **(A)** $\tan 1 > \tan^{-1} \Rightarrow \tan 1 > \frac{\pi}{4}$

(B) $\sin 1 > \cos 1$

$\sin 57.3° > \cos 57.3°$

(C) $\tan 1 > \sin 1$ (not possible)

Because $\tan 57.3° > 1 > \sin 57.3°$

(D) $\cos 1 < \frac{\pi}{4} \Rightarrow \cos(\cos 1) > \cos\left(\frac{\pi}{4}\right)$

41.(BD) **(A)** $\tan 1 > 1$ and $\sin 1 < 1$, then $\log_{\sin 1} \tan 1 < 0$

(B) $1 + \tan 3 < 1$ and $\cos 1 < 1$, then $\log_{\cos 1}(1+\tan 3) > 0$

(C) $\cos\theta + \sec\theta > 2$ and $\log_{10} 5 < 1$, then $\log_{\log_{10} 5}(\cos\theta + \sec\theta) < 0$

(D) $2\sin 18° < 1$ and $\tan 15° < 1$, then $\log_{\tan 15°} 2\sin 18° > 0$

42.(BD) $2\Sigma(\cos x \cos y) + 2\Sigma(\sin x \sin y) + 3 = 0$

$(\Sigma \cos x)^2 + (\Sigma \sin x)^2 = 0 \Rightarrow \Sigma \cos x = 0$ and $\Sigma \sin x = 0$

$\cos 3x + \cos 3y + \cos 3z = 4(\cos^3 x + \cos^3 y + \cos^3 z) - 3(\cos x + \cos y + \cos z) = 12\cos x \cos y \cos z$

43.(ABD) **(A)** $2(a+d) = 2(b+c)$

(B) $\tan 50° = \frac{1}{\tan 40°} = \frac{1-\tan^2 20°}{2\tan 20°}$

$\Rightarrow \tan 20° + 2\tan 50° = \tan 70° \Rightarrow 2a + 2b = 2c$

(D) $\tan 20° - 2\tan 10° = \tan 20° \tan^2 10° > 0 \Rightarrow \tan 20° > 2\tan 10° \Rightarrow b > a$ and $d > c$

44.(ABCD) $\frac{(1-\cot x)}{\sin^2 x} = (1-\cot x)\cdot \operatorname{cosec}^2 x = (1-\cot x)(1+\cot^2 x)$

45.(ABCD) $4\sin 3x + 5 \geq 4\cos 2x + 5\sin x \Rightarrow (\sin x - 1)(4\sin x + 1)^2 \leq 0 \ \forall\ x \in R$

46.(ACD) $\cos x \cos 6x = -1$

Case-I : $\cos x = 1$ and $\cos 6x = -1$ Not possible

Case-2 : $\cos 6x = 1$ and $\cos x = -1 \Rightarrow x = (2n-1)\pi,\ (n \in I)$

47.(ABCD) Verify in given intervals

48.(ABC) $2k = \sin^2 2x - 2\sin 2x - 2$

Let $\sin 2x = t$ $t \in [-1, 1]$

$2k = t^2 - 2t - 2 \Rightarrow k \in \left[-\frac{3}{2}, \frac{1}{2}\right]$

49.(BCD) $f(\theta) = \left(\cos\theta - \cos\frac{\pi}{8}\right)\left(\cos\theta - \cos\frac{3\pi}{8}\right)\left(\cos\theta - \cos\frac{5\pi}{8}\right)\left(\cos\theta - \cos\frac{7\pi}{8}\right) = \cos^4\theta - \cos^2\theta + \frac{1}{8}$

50.(CD) $x^2 - r(r_1r_2 + r_2r_3 + r_1r_3)x + (r_1r_2r_3 - 1) = 0$

$x^2 - (r_1r_2r_3)x + (r_1r_2r_3 - 1) = 0 \Rightarrow$ Roots are 1 and $r_1r_2r_3 - 1$

51.(BC) $D+E+F=\frac{\pi}{2}$

52.(ABC) If $\frac{r}{r_1}=\frac{r_2}{r_3} \Rightarrow \frac{s-a}{s}=\frac{s-c}{s-b} \Rightarrow a^2+b^2=c^2 \Rightarrow \angle C=90°$

53.(BC) $\angle BOC = 2\angle A$

$\angle BIC = \pi/2 + A/2$

$\angle BHC = \pi - A$

54.(AC) $\sqrt{3}x^2-4x+\sqrt{3}<0$

$\Rightarrow (\sqrt{3}x-1)(x-\sqrt{3})<0 \Rightarrow \frac{1}{\sqrt{3}}<x<\sqrt{3}$

$30°<A,\ B<60° \Rightarrow 60°<C<120°$

55.(ABC) Use graphs of $\sin^{-1}(\sin x),\ \cos^{-1}(\cos x)$

56.(AC) $\cos^{-1}x=\tan^{-1}x \Rightarrow x\in[0,1]$

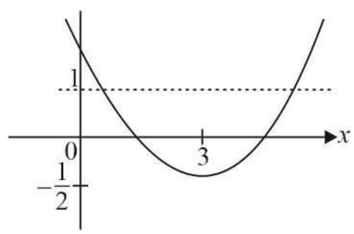
$\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\tan^{-1}x \Rightarrow x^2=\sqrt{1-x^2} \Rightarrow x^4+x^2-1=0$ **;** $x^2=\frac{\sqrt{5}-1}{2}$

57.(AB) Check with the options

58.(ABD) $\sin^{-1}\left(x^2-6x+\frac{17}{2}\right)=\sin^{-1}k$

where $-1\le k\le 1$

$y=x^2-6x+\frac{17}{2}$



59.(ABD) $(\sin^{-1}x-\cos^{-1}x)((\sin^{-1}x)^2+(\cos^{-1}x)^2+2\sin^{-1}x\cos^{-1}x)=\frac{\pi^3}{16}$

$\Rightarrow \sin^{-1}x-\cos^{-1}x=\frac{\pi}{4} \Rightarrow \cos^{-1}x=\frac{\pi}{8} \Rightarrow x=\cos\frac{\pi}{8}$

60.(2) $a\sin\theta-b\cos\theta=-\sin 4\theta=-2\sin 2\theta\cos 2\theta=-4\sin\theta\cos\theta\left(\cos^2\theta-\sin^2\theta\right)=-4\sin\theta\cos^3\theta+4\sin^3\theta\cos\theta$ **. . . .(i)**

$a\cos\theta+b\sin\theta=\frac{5-3\cos 4\theta}{2}=\frac{2+3-3\cos 4\theta}{2}=1+3\sin^2 2\theta$

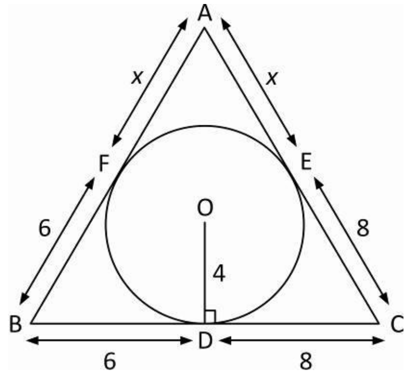
$=\left(\sin^2\theta+\cos^2\theta\right)^2+3\,.\,(2\sin\theta\cos\theta)^2=\sin^4\theta+\cos^4\theta+14\sin^2\theta\cos^2\theta$ **. . . .(ii)**

Now solving, (i) and (ii) we get : $a=\cos^5\theta+10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$ and

$b=5\sin\theta\cos^4\theta+10\cos^2\theta\sin^3\theta+\sin^5\theta$

Thus $a+b=(\cos\theta+\sin\theta)^5$ and $a-b=(\cos\theta-\sin\theta)^5$

$\therefore (a+b)^{2/5}+(a-b)^{2/5}=(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2=2$



61.(3) $s=\dfrac{(x+6)+(x+8)+14}{2}=x+14$

$\Rightarrow \quad \Delta=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{(x+14)(8)(6)(x)}$

Now $\quad r=\dfrac{\Delta}{s} \Rightarrow 4=\sqrt{\dfrac{48x}{x+14}}$

$\Rightarrow \quad 16(x+14)=48x$ or $x=7$

$\Rightarrow \quad \Delta=\sqrt{48\times7\times21}=84$

$\Rightarrow \quad \sqrt{\sqrt{\Delta-3}}=3$

62.(6) $tan\{x\}=cot\{x\}$

$\Rightarrow \quad \dfrac{tan\{x\}}{cot\{x\}}=1 \quad \Rightarrow \quad tan^2\{x\}=1 \quad \Rightarrow \quad tan\{x\}=\pm1$

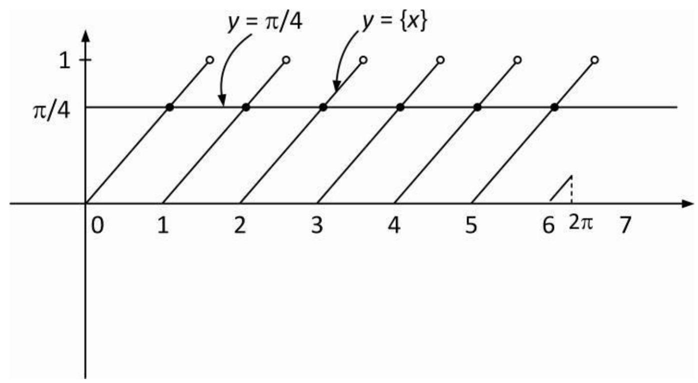
But $\{x\}\in(0,1) \Rightarrow tan\{x\}\to\in[0,tan1) \quad \Rightarrow \quad tan\{x\}\neq-1$

$\therefore \quad tan\{x\}=1$

$\Rightarrow \quad \{x\}=tan^{-1}1=\dfrac{\pi}{4} \quad$ **. . . .(i)**

But $x\in[0,2\pi]$

$\therefore$ Clearly, (i) holds where graphs of $y=\{x\}$ and $y=\dfrac{\pi}{4}$ intersect each other in $[0,2\pi]$ which are 6 points as shown below in diagram.



63.(6) $tan^{-1}x+cos^{-1}\dfrac{y}{\sqrt{1+y^2}}=sin^{-1}\left(\dfrac{3}{\sqrt{10}}\right) \quad \Rightarrow \quad tan^{-1}x+tan^{-1}\left(\dfrac{1}{y}\right)=tan^{-1}(3)$

$\Rightarrow \quad \dfrac{xy+1}{y-x}=3$ or $xy+1=3y-3x$ or $y=\dfrac{3x+1}{3-x}$

The positive integral solutions are $(1,2),(2,7) \quad \Rightarrow \quad \alpha=1+2=3$ and $\beta=2+7=9 \Rightarrow \beta-\alpha=9-3=6$

64.(4) $sin\,x+sin^2x+sin^3x=1$

$\Rightarrow \quad sin\,x(1+sin^2x)=1-sin^2x=cos^2x \quad \Rightarrow \quad sin^2x(1+sin^2x)^2=cos^4x$

$\Rightarrow \quad (1-cos^2x)(2-cos^2x)^2=cos^4x \quad \Rightarrow \quad (1-cos^2x)(4+cos^4x-4cos^2x)=cos^4x$

$\Rightarrow \quad 4-8cos^2x+5cos^4x-cos^6x=cos^4x \quad \Rightarrow \quad cos^6x-4cos^4x+8cos^2x=4$

65.(1) For $sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ to exist, $\left|\dfrac{1+x^2}{2x}\right|\le1$

$\Rightarrow \quad |1+x^2|\le|2x| \Rightarrow 1+x^2\le2|x| \Rightarrow |x|^2-2|x|+1\le0 \Rightarrow (|x|-1)^2\le0 \quad \Rightarrow \quad |x|=1$ or $x=\pm1$

For x = 1, the equation is satisfied. For $x=-1$, the equation is not satisfied. So, only one solution.

66.(2) For a triangle inscribed in a circle, we have

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R \quad \Rightarrow \quad sin^2 A+sin^2 B+sin^2 C=\frac{a^2+b^2+c^2}{4R^2}$$

According to the question $(2R)^2=\frac{a^2+b^2+c^2}{2}$ or $a^2+b^2+c^2=8R^2 \quad \Rightarrow \quad sin^2 A+sin^2 B+sin^2 C=\frac{8R^2}{4R^2}=2$

67.(9) $\tan\left(142\frac{1}{2}°\right)=-\tan\left(37\frac{1}{2}°\right)=-\tan\left(60°-22\frac{1}{2}°\right)=\tan\left(22\frac{1}{2}°-60°\right)=\frac{\left(\sqrt{2}-1\right)-\sqrt{3}}{1+\left(\sqrt{2}-1\right)\left(\sqrt{3}\right)}=\frac{\sqrt{2}-\sqrt{3}-1}{1+\sqrt{6}-\sqrt{3}}$

$$=\frac{\left(\sqrt{2}-\sqrt{3}-1\right)\left(1-\sqrt{6}+\sqrt{3}\right)}{\left(1+\sqrt{6}-\sqrt{3}\right)\left(1-\sqrt{6}+\sqrt{3}\right)}=\frac{4\sqrt{2}-4\sqrt{3}+2\sqrt{6}-4}{6\sqrt{2}-8}=\frac{\left(2\sqrt{2}-2\sqrt{3}+\sqrt{6}-2\right)\left(3\sqrt{2}+4\right)}{\left(3\sqrt{2}-4\right)\left(3\sqrt{2}+4\right)}=2+\sqrt{2}-\sqrt{3}-\sqrt{6}$$

$\Rightarrow \quad \mu=3, \lambda=6 \;\Rightarrow\; \mu+\lambda=9$

68.(8) $\tan\left(\frac{2\pi}{3}-x\right)=\frac{2\sin\left(\frac{\pi}{3}-\frac{x}{2}\right)\cos\left(\frac{\pi}{3}+\frac{x}{2}\right)}{-2\sin\left(\frac{\pi}{3}-\frac{x}{2}\right)\sin\left(\frac{\pi}{3}+\frac{x}{2}\right)}=-\cot\left(\frac{\pi}{3}+\frac{x}{2}\right)=tan\left(\frac{\pi}{2}+\frac{\pi}{3}+\frac{x}{2}\right)=tan\left(\frac{5\pi}{6}+\frac{x}{2}\right)$

$\Rightarrow \quad \frac{2\pi}{3}-x=n\pi+\frac{5\pi}{6}+\frac{x}{2} \Rightarrow \frac{3x}{2}=-n\pi-\frac{\pi}{6} \quad \Rightarrow \quad x=\frac{-2n\pi}{3}-\frac{\pi}{9}$

$\Rightarrow \quad x=\frac{5\pi}{9},\frac{11\pi}{9} \quad \Rightarrow \quad \frac{12}{\pi}\left|x_2-x_1\right|=\frac{12}{\pi}\times\frac{6\pi}{9}=8$

69.(6) $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\left(\frac{C}{2}\right)=\frac{1}{9}\cot\left(\frac{C}{2}\right)$

$$\cos(A-B)=\frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)}=\frac{1-\frac{1}{81}\cot^2\left(\frac{C}{2}\right)}{1+\frac{1}{81}\cot^2\left(\frac{C}{2}\right)}=\frac{31}{32} \quad \Rightarrow \quad \cot^2\left(\frac{C}{2}\right)=\frac{9}{7}$$

$$\cos(C)=\frac{1-\tan^2\left(\frac{C}{2}\right)}{1+\tan^2\left(\frac{C}{2}\right)}=\frac{1-\frac{7}{9}}{1+\frac{7}{9}}=\frac{2}{16}=\frac{1}{8} \quad \Rightarrow \quad \cos(C)=\frac{a^2+b^2-c^2}{2ab}=\frac{25+16-c^2}{40}=\frac{1}{8} \Rightarrow c=6$$

70.(3) $10(1-\cos 2\alpha)^2+15(1+\cos 2\alpha)^2=24 \quad \Rightarrow (5\cos 2\alpha+1)^2=0 \quad \Rightarrow \quad \cos 2\alpha=-\frac{1}{5} \quad \Rightarrow \quad \tan^2\alpha=\frac{3}{2}$

71.(2) $a^2\sec^2 200°=c^2\tan^2 200°+d^2+2cd\tan 200°$

$b^2\sec^2 200°=c^2+d^2\tan^2 200°-2cd\tan 200°$

$\Rightarrow \quad a^2+b^2=c^2+d^2$

$(a\sec 200°-c\tan 200°)^2=d^2$

$(b\sec 200°+d\tan 200°)^2=c^2$

$\Rightarrow \quad (c^2+d^2)(\sec^2 200°+\tan^2 200°)+(2bd-2ac)\sec 200°\tan 200°=c^2+d^2$

$\Rightarrow \quad (c^2+d^2)(2\tan^2 200°)=(2ac-2bd)\sec 200°\tan 200° \quad \Rightarrow \quad \frac{2(c^2+d^2)}{ac-bd}=\frac{2\sec 200°}{\tan 200°}=\frac{2}{\sin 200°}=\frac{-2}{\sin 20°}$

72.(3) $\displaystyle\sum_{r=1}^{n}\frac{\sin(2^r-2^{r-1})}{\cos 2^r\cos 2^{r-1}}=\sum_{r=1}^{n}(\tan 2^r-\tan 2^{r-1})=\tan 2^n-\tan 1$

73.(1) $3^{\sin 2x+2\cos^2 x}+\dfrac{3^3}{3^{\sin 2x+2\cos^2 x}}=28$

Let $3^{\sin 2x+2\cos^2 x}=t,\ t^2-28t+27=0 \quad\Rightarrow\quad t=1,\ 27$

If $t=1 \quad\Rightarrow\quad \sin 2x+2\cos^2 x=0$

$2\cos x(\sin x+\cos x)=0 \quad\Rightarrow\quad x=\dfrac{\pi}{2},\dfrac{3\pi}{4},\dfrac{7\pi}{4}$

If $t=27 \quad\Rightarrow\quad \sin 2x+2\cos^2 x=3$ (Not possible)

$(\sin 2\alpha-\cos 2\alpha)^2+8\sin 4\alpha=1+7\sin 4\alpha=1$ (at $\alpha=\dfrac{\pi}{2},\dfrac{3\pi}{4},\dfrac{7\pi}{4}$)

74.(3) $\cos\dfrac{2\pi}{7}+\cos\dfrac{4\pi}{7}+\cos\dfrac{6\pi}{7}-1=\dfrac{\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}\cdot\cos\dfrac{4\pi}{7}-1=\dfrac{-3}{2}$

75.(9) $\sin y-2014\cos y=1 \quad\Rightarrow\quad y=\dfrac{\pi}{2}$

76.(6) $\dfrac{2\sin 6x}{\sin x-1}<0 \quad\Rightarrow\quad \sin 6x>0 \quad\Rightarrow\quad x\in\left(0,\dfrac{\pi}{6}\right)\cup\left(\dfrac{\pi}{3},\dfrac{\pi}{2}\right)$

77.(7) $\sin^4 x-4\sin^2 x+(2+k)=0$

Let $\sin^2 x=t \qquad t\in[0,1]$

$t^2-4t+(2+k)=0$

$f(0)f(1)\le 0$

$(k+2)(k-1)\le 0 \quad\Rightarrow\quad -2\le k\le 1$

78.(8) $2\sin^2 x+\sin^2 2x=2$

$2\sin^4 x-3\sin^2 x+1=0 \quad\Rightarrow\quad (2\sin^2 x-1)(\sin^2 x-1)=0$

$\sin 2\mathrm{x}+\cos 2\mathrm{x}=\tan\mathrm{x}$

$2\tan x+1-\tan^2 x=\tan x(1+\tan^2 x)$

$\Rightarrow \quad \tan^3 x+\tan^2 x-\tan x-1=0 \quad\Rightarrow\quad (1+\tan x)(\tan^2 x-1)=0$

$2\cos^2 x+\sin x\le 2$

$2\sin^2 x-\sin x\ge 0$

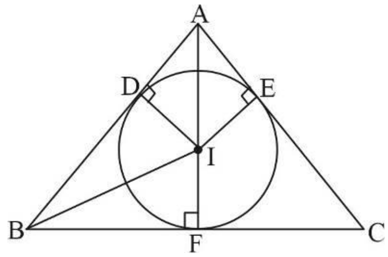
$\sin x(2\sin x-1)\ge 0$

79.(8) $(8\cos 4\theta-3)(\cot\theta-\tan\theta)^2=12$

$8(2\cos^2 2\theta-1)-3\left(\dfrac{4\cos^2 2\theta}{\sin^2 2\theta}\right)=12$

$16\cos^4 2\theta-8\cos^2 2\theta-3=0 \quad\Rightarrow\quad (4\cos^2 2\theta-3)(4\cos^2 2\theta+1)=0 \quad\Rightarrow\quad \cos 2\theta=\pm\dfrac{\sqrt{3}}{2}$

80.(3) $\frac{1}{2}r(AD+AE)=5$ **;** $\frac{1}{2}r(BF+BD)=10$



$$\Rightarrow \quad \frac{BF+BD}{AD+AE}=2 \quad \Rightarrow \quad \frac{r\cot\frac{B}{2}+r\cot\frac{B}{2}}{r\cot\frac{A}{2}+r\cot\frac{A}{2}}=2$$

Applying C and D, $\dfrac{\cos\frac{C}{2}}{\sin\frac{A-B}{2}}=3$

81.(1) $\dfrac{\Delta_1\Delta_2\Delta_3}{\Delta^3}=\dfrac{(r_1r_2r_3)^2}{r^6}=\left(\dfrac{s}{s-a}\times\dfrac{s}{s-b}\times\dfrac{s}{s-c}\right)^2$

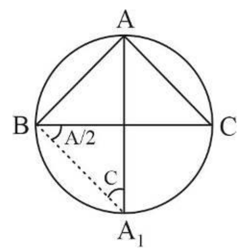
$$\frac{(s-a)+(s-b)+(s-c)}{3}\geq[(s-a)(s-b)(s-c)]^{1/3} \quad \Rightarrow \quad \frac{s^3}{(s-a)(s-b)(s-c)}\geq 27$$

Minimum value $=1$

82.(2) $\dfrac{AH}{AD}+\dfrac{BH}{BE}+\dfrac{CH}{CF}=\dfrac{R}{\Delta}(a\cos A+b\cos B+c\cos C)=\dfrac{R^2}{\Delta}(\sin 2A+\sin 2B+\sin 2C)$

$$=\frac{4R^2}{\Delta}\sin A\sin B\sin C=\frac{bc\sin A}{\Delta}=2$$

83.(2) $\dfrac{c}{\sin C}=\dfrac{AA_1}{\sin\left(B+\frac{A}{2}\right)}$



$AA_1\cos\frac{A}{2}=\sin B+\sin C \qquad (\because R=1)$

$$\Rightarrow \quad \frac{AA_1\cos\frac{A}{2}+BB_1\cos\frac{B}{2}+CC_1\cos\frac{C}{2}}{\sin A+\sin B+\sin C}=2$$

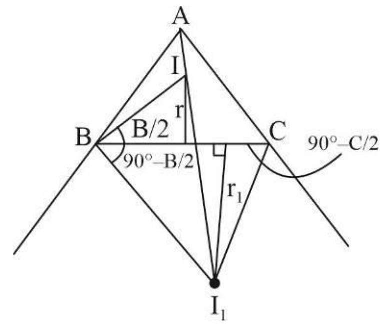
84.(7) $(R^2-4Rr+4r^2)+(4r^2-12r+9)=0$

$$(R-2r)^2+(2r-3)^2=0 \quad \Rightarrow \quad r=\frac{3}{2};\ R=2r$$

ΔABC is an equilateral triangle.

85.(9) $5-2\pi>x^2-4x$ **,** $x^2-4x+(2\pi-5)<0$**,** $2-\sqrt{9-2\pi}<x<2+\sqrt{9-2\pi} \quad \Rightarrow \quad \lambda=9$

86.(16) $\sin\frac{B}{2} = \frac{r}{IB}$



$IB = 4R\sin\frac{A}{2}\sin\frac{C}{2}$

$\sin\left(90° - \frac{B}{2}\right) = \frac{r_1}{BI_1} \quad \Rightarrow \quad BI_1 = 4R\sin\frac{A}{2}\cos\frac{C}{2}$

$(II_1)^2 = (BI)^2 + (BI_1)^2 = 16R^2\sin^2\frac{A}{2}$ **....... (1)**

$I_2I_3\cos\left(90° - \frac{A}{2}\right) = a$ (by using pedal triangle)

$I_2I_3 = 4R\cos\frac{A}{2}$

$(I_2I_3)^2 = 16R^2\cos^2\frac{A}{2}$ **....... (2)**

From (1) & (2) we get $\lambda = 16$

87.(65) $2\tan^{-1}\left(\frac{1}{5}\right) - \sin^{-1}\left(\frac{3}{5}\right)$

$\tan^{-1}\left(\frac{5}{12}\right) - \sin^{-1}\left(\frac{3}{5}\right)$

$\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}\left(\frac{3}{4}\right) = -\left(\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{5}{12}\right)\right) = -\tan^{-1}\left(\frac{16}{63}\right) = -\cos^{-1}\left(\frac{63}{65}\right) \Rightarrow \lambda = 65$

88.(1) $\sum_{n=0}^{\infty} 2\tan^{-1}\left(\frac{2}{n^2+n+4}\right) = \sum_{n=0}^{\infty} 2\tan^{-1}\left(\frac{\frac{1}{2}}{\frac{n^2}{4}+\frac{n}{4}+1}\right) = \sum_{n=0}^{\infty} 2\tan^{-1}\left(\frac{\left(\frac{n}{2}+\frac{1}{2}\right)-\frac{n}{2}}{\frac{n}{2}\left(\frac{n}{2}+\frac{1}{2}\right)+1}\right) = \sum_{n=0}^{\infty} 2\left(\tan^{-1}\left(\frac{n}{2}+\frac{1}{2}\right) - \tan^{-1}\left(\frac{n}{2}\right)\right)$

89.(8) $\cos^{-1}(|3\log_6^2(\cos x) - 7|) = \cos^{-1}(|\log_6^2(\cos x) - 1|)$

$|3\log_6^2(\cos x) - 7| = |\log_6^2(\cos x) - 1|$

Let $|3t - 7| = |t - 1| \quad \Rightarrow \quad t = 3$ and $t = 2$

$\Rightarrow \quad \cos x = 6^{-\sqrt{3}}$ and $6^{-\sqrt{2}}$

SEQUENCE AND SERIES

1.(D) $\frac{1}{x}=a+(p-1)d \quad \frac{1}{y}=a+(q-1)d \quad \frac{1}{z}=a+(r-1)d$

$$\frac{1}{x}-\frac{1}{y}=(p-q)d$$

$$\Rightarrow \quad xy(p-q)=\frac{y-x}{d};yz(q-r)=\frac{z-y}{d};zx(r-p)=\frac{x-z}{d}$$

$$\Rightarrow \quad (p-q)xy+yz(q-r)+zx(r-p)=\frac{1}{d}(y-x+z-y+x-z)=0$$

2.(D) $(a+nd)^2=(a+md)(a+rd)$

$$\Rightarrow \quad \left(n^2-mr\right)d^2=ad(r+m-2n) \quad \Rightarrow \quad \frac{d}{a}=\frac{\left(\frac{2mr}{n}-2n\right)}{n^2-mr} \qquad \left[\because\ n=\frac{2mr}{m+r}\right]$$

$$\Rightarrow \quad \frac{d}{a}=-\frac{2}{n}$$

3.(B) $\sum_{r=1}^{99} r!\left((r+1)^2-r\right)=\sum_{r=1}^{99}\left((r+1)(r+1)!-rr!\right)=100(100!)-1$

4.(D) $\sum_{k=1}^{n}\frac{k^2-\frac{1}{2}}{\left(k^2-k+\frac{1}{2}\right)\left(k^2+k+\frac{1}{2}\right)}=\sum_{k=1}^{n}\left(\frac{k-\frac{1}{2}}{k^2-k+\frac{1}{2}}-\frac{k+\frac{1}{2}}{k^2+k+\frac{1}{2}}\right)=1-\frac{n+\frac{1}{2}}{n^2+n+\frac{1}{2}}=1-\frac{2n+1}{2n^2+2n+1}$

5.(B) $k^4+\frac{1}{4}=\left(k^2-k+\frac{1}{2}\right)\left(k^2+k+\frac{1}{2}\right)$

$$\prod_{k=1}^{n}\frac{\left((2k-1)^2-(2k-1)+\frac{1}{2}\right)\left((2k-1)^2+(2k-1)+\frac{1}{2}\right)}{\left((2k)^2-2k+\frac{1}{2}\right)\left((2k)^2+2k+\frac{1}{2}\right)}=\prod_{k=1}^{n}\frac{(2k-1)^2-(2k-1)+\frac{1}{2}}{(2k)^2+2k+\frac{1}{2}}=\frac{\frac{1}{2}}{4n^2+2n+\frac{1}{2}}=\frac{1}{8n^2+4n+1}$$

6.(B) $\sum_{k=1}^{n}\frac{1}{\sqrt{k}\sqrt{k+1}\left(\sqrt{k+1}+\sqrt{k}\right)}=\sum_{k=1}^{n}\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}}=\sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)=1-\frac{1}{\sqrt{n+1}}$

7.(C) $S_{2n}-S_n=\frac{2n}{2}\left(2a+(2n-1)d\right)-\frac{n}{2}\left(2a+(n-1)d\right)=\frac{n}{2}\left(2a+(3n-1)d\right)=\frac{1}{3}S_{3n}$

8.(B) $S_n=\sum_{r=1}^{n}\frac{\left(r^2+6r+12\right)}{(r)(r+1)(r+2)}\frac{1}{2^{r+1}}=\sum_{r=1}^{n}\frac{2(r+2)(r+3)-r(r+4)}{(r)(r+1)(r+2)^{r+1}}$

$$=\sum_{r=1}^{n}\left(\frac{r+3}{r(r+1)2^r}-\frac{r+4}{(r+1)(r+2)2^{r+1}}\right)=1-\frac{n+4}{(n+1)(n+2)2^{n+1}}$$

9.(D) $S=\sum_{r=1}^{\infty} r^2\left(-\frac{1}{5}\right)^{r-1}$

$$\sum_{r=0}^{\infty} x^r=\frac{1}{1-x}$$

$$\Rightarrow \quad \sum_{r=1}^{\infty} r x^r=\frac{x}{(1-x)^2}=\frac{1}{x-1}+\frac{1}{(x-1)^2} \Rightarrow \sum_{r=1}^{\infty} r^2 x^{r-1}=-\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3} \Rightarrow \sum_{r=1}^{\infty} r^2\left(-\frac{1}{5}\right)^{r-1}=\frac{25}{54}$$

10.(A) $\sum_{r=1}^{n} \frac{n-(r-1)}{r(r+1)(r+2)}=\frac{(n+1)}{2} \sum_{r=1}^{n} \frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}-\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)$

$$=\frac{(n+1)}{2}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right)-\left(\frac{1}{2}-\frac{1}{n+2}\right)=\frac{1}{2(n+2)}+\frac{n+1}{4}-\frac{1}{2}$$

11.(D) **12.(B)** **13.(C)**

$$\sum_{r=1}^{n}\left((r+1)^2-r^2\right) \phi(r)=\sum_{r=1}^{n}(r+1)^2(\phi(r)-\phi(r+1))+\sum_{r=1}^{n}\left((r+1)^2 \phi(r+1)-r^2 \phi(r)\right)$$

$$=-\sum_{r=1}^{n}(r+1)+(n+1)^2 \phi(n+1)-1^2 \phi(1)=-(1+2+3+\ldots+(n+1))+(n+1)^2 \phi(n+1)$$

$$=-(1+2+3+\ldots+(n+1))+(n+1)^2 \phi(n+1)=-\frac{(n+1)(n+2)}{2}+(n+1)^2 \phi(n+1)$$

$$\sum_{r=0}^{9} P(r)=1^2+2^2+\ldots 10^2=\frac{10 \times 11 \times 21}{6}=385$$

$$\sum_{r=0}^{\infty} \frac{1}{Q(r)}=\sum_{r=0}^{\infty} \frac{2}{(r+1)(r+2)}=2 \sum_{r=0}^{\infty}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)=2$$

$$P(n)-Q(n)=(n+1)^2-\frac{(n+1)(n+2)}{2}=\frac{n(n+1)}{2}$$

14.(B)
15.(C)
16.(D)
17.(AD) $a_k a_{n-(k-1)}=\left(a_1+(k-1) d\right)\left(a_n-(k-1) d\right)$

$$=a_1 a_n+d(k-1)\left(a_n-a_1\right)-(k-1)^2 d^2$$

$$=a_1 a_n+d^2(k-1)(n-k)$$

$$\leq \begin{cases} a_1 a_n+d^2\left(\frac{n+1}{2}-1\right)\left(n-\frac{n+1}{2}\right)=a_1 a_n+\frac{(n-1)^2}{4} d^2 & \text{if } n \text{ is odd} \\ a_1 a_n+\frac{d^2}{4} n(n-2) & \text{if } n \text{ is even}\end{cases}$$

18.(BCD) $y=\dfrac{\tan 3x}{\tan x}=\dfrac{3-\tan^2 x}{1-3\tan^2 x}=\dfrac{8}{3\left(1-3\tan^2 x\right)}+\dfrac{1}{3}$

$$\Rightarrow \; y\in\left(-\infty,\frac{1}{3}\right)\cup(3,\infty)$$

$$a=\frac{1}{3}, b=3, s_\infty=\frac{6}{1-\frac{1}{3}}=9$$

19.(CD) $\dfrac{a_1+2}{2}=\sqrt{2a_1} \;\Rightarrow\; (a_1-2)^2=0 \;\Rightarrow\; a_1=2$

$$(a_n+2)^2=8s_n$$

$$(a_{n-1}+2)^2=8s_{n-1}$$

$$\Rightarrow \quad (a_n+2)^2-(a_{n-1}+2)^2=8a_n \;\Rightarrow\; (a_n-a_{n-1})(a_n+a_{n-1}+4)=8a_n$$

$$\Rightarrow \quad a_n^2-a_{n-1}^2=4(a_n+a_{n-1}) \;\Rightarrow\; a_n-a_{n-1}=4$$

20.(AB) $\dfrac{3x+\dfrac{4xz}{x+z}}{2x-\dfrac{2xz}{x+z}}+\dfrac{3z+\dfrac{4xz}{x+z}}{2z-\dfrac{2xz}{z+x}}=\dfrac{3x^2+7xz}{2x^2}+\dfrac{7xz+3z^2}{2z^2}=3+\dfrac{7}{2}\left(\dfrac{z}{x}+\dfrac{x}{z}\right)\in(10,\infty)$

21.(ABD) $\dfrac{5^{a_{n+1}}}{5^{a_n}}=\dfrac{3n+5}{3n+2}$

$$\frac{5^{a_2}}{5^{a_1}}\times\frac{5^{a_3}}{5^{a_2}}\times.....\frac{5^{a_n}}{5^{a_{n-1}}}=\frac{8}{5}\times\frac{11}{8}\times\frac{14}{11}\times.....\frac{3n+2}{3n-1}$$

$$\frac{5^{a_n}}{5^{a_1}}=\frac{3n+2}{5} \quad\Rightarrow\quad 5^{a_n}=3n+2 \quad\Rightarrow\quad a_n=\log_5(3n+2)$$

$$[a_n]=3 \quad\Rightarrow\quad \log_5(3n+2)\in[3,4)$$

$$\Rightarrow \; n\in\{41,42,.....,......207\}$$

$$[a_n]=4 \;\Rightarrow\; \frac{623}{3}\le n<\frac{3123}{3} \;\Rightarrow\; n\in\{208,209,...,1041\}$$

22. (ABC) $\dfrac{a^2+1}{b+c}+\dfrac{b^2+1}{c+a}+\dfrac{c^2+1}{a+b}\ge 2\left(\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\right)$

$$=2\left[(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)-3\right]\ge 2\left(\frac{9}{2}-3\right)=3$$

$$\frac{3}{\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)}\le\frac{(b+c)+(c+a)+(a+b)}{3} \qquad\Rightarrow\qquad (a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\ge\frac{9}{2}$$

23.(AD) $p = a + n\left(\frac{b-a}{n+1}\right) = \frac{nb+a}{n+1}$

$$\frac{1}{q} = \frac{1}{a} + n\left(\frac{\frac{1}{b}-\frac{1}{a}}{n+1}\right) = \frac{na+b}{(n+1)ab}$$

$$q = \frac{(n+1)ab}{na+b}$$

$\therefore \quad q\left(na + \frac{p(n+1)-a}{n}\right) = (n+1)a\left(\frac{(n+1)p-a}{n}\right)$

$\Rightarrow \quad q\left(\left(n^2-1\right)a + p(n+1)\right) = (n+1)a\left((n+1)p - a\right)$

$\Rightarrow \quad a^2 + a\left(q(n-1)-(n+1)p\right) + pq = 0$

$D \ge 0 \quad \Rightarrow \quad \left(q(n-1)-(n+1)p\right)^2 - 4pq \ge 0$

$\Rightarrow \quad (q-p)^2 n^2 + (q+p)^2 - 2n\left(q^2-p^2\right) - 4pq \ge 0$

$\Rightarrow \quad (q-p)^2 n^2 + (q+p)^2 - 2n\left(q^2+p^2\right) \ge 0$

$\Rightarrow \quad (q-p)\left((q-p)n^2 + (q-p) - 2n(q+p)\right) \ge 0$

$\Rightarrow \quad (q-p)\left(q(n-1)^2 - p(n+1)^2\right) \ge 0$

$$q \in (-\infty, p] \cup \left[\left(\frac{n+1}{n-1}\right)^2 p, \infty\right)$$

24.(AB) $\dfrac{\underbrace{(x+x+.....x)}_{\text{x times}} + \underbrace{y+y+.....y}_{\text{y times}} + \underbrace{z+z+.....z}_{\text{z times}}}{(x+y+z)} \ge \left(x^x y^y z^z\right)^{\frac{1}{x+y+z}}$

$$\ge \frac{x+y+z}{\underbrace{\left(\frac{1}{x}+\frac{1}{x}+.....\frac{1}{x}\right)}_{\text{x times}} + \underbrace{\left(\frac{1}{y}+\frac{1}{y}+....\frac{1}{y}\right)}_{\text{y times}} + \underbrace{\left(\frac{1}{z}+\frac{1}{z}+.....\frac{1}{z}\right)}_{\text{z times}}}$$

25.(CD) $\frac{1}{x-1} + \left(\frac{1}{x+1} - \frac{1}{x-1}\right) + \frac{2}{x^2+1} + \frac{4}{x^4+1} + ... + \frac{2^n}{x^{2n}+1}$

$$= \frac{1}{x-1} + 2\left(\frac{1}{x^2+1} - \frac{1}{x^2-1}\right) + \frac{4}{x^4+1} + ... + \frac{2^n}{x^{2n}+1}$$

$$= \frac{1}{x-1} + 4\left(\frac{1}{x^4+1} - \frac{1}{x^4-1}\right) + \frac{8}{x^8+1} + ... + \frac{2^n}{x^{2^n}+1} = \frac{1}{x-1} - \frac{2^{n+1}}{x^{2^{n+1}}-1} \quad \Rightarrow \quad f(x) = \frac{1}{x-1} \quad \forall x > 1$$

26.(AC) $S(n)=1+\underbrace{\left(\frac{1}{2}+\frac{1}{3}\right)}_{2\text{ terms}}+\underbrace{\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)}_{4terms}+\underbrace{\left(\frac{1}{8}+\frac{1}{9}++....+\frac{1}{15}\right)}_{8\text{terms}}+.....\underbrace{\left(\frac{1}{2^n-1}+\frac{1}{2^{n-1}+1}....\frac{1}{2^n-1}\right)}_{2^{n-1}\text{terms}}$

$<+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+...+\frac{1}{8}\right)+...+\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+...\frac{1}{2^{n-1}}\right)=1+1+1+1+1+...+1=n$

$S(n)>\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+...+\frac{1}{26}\right)+.....+\left(\frac{1}{2^{n-1}+1}+\frac{1}{2^{n-1}+2}+\frac{1}{2^n}\right)$

$>\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+.....+\frac{1}{16}\right)+....\left(\frac{1}{2^n}+\frac{1}{2^n}+...+\frac{1}{2^n}\right)$

$=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+...+\frac{1}{2}=\frac{n}{2} \Rightarrow S(2n)>n$

27.(BC) $\frac{1}{a^2\left(1-r^2\right)}\left[1+\frac{1}{r^2}+\frac{1}{r^4}+...\frac{1}{r^{2(n-2)}}\right]=\frac{1-\frac{1}{r^{2(n-1)}}}{\left(1-\frac{1}{r^2}\right)a^2\left(1-r^2\right)}=\frac{\left(1-r^{2n}\right)}{a^2\left(1-r^2\right)^2 r^{2n-4}}$

$\frac{1}{a^m\left(1+r^m\right)}\left[1+\frac{1}{r^m}+\frac{1}{r^{2m}}+...+\frac{1}{r^{(n-2)m}}\right]=\frac{\left(1-\frac{1}{r^{(n-1)m}}\right)}{\left(1-\frac{1}{r^m}\right)a^m\left(1+r^m\right)}=\frac{\left(1-r^{mn-m}\right)}{\left(1-r^m\right)\left(r^{mn-n}-r^{mn-2m}\right)}$

28.(ABCD) $\frac{1}{\alpha}+\frac{1}{\gamma}=\frac{2}{\beta} \Rightarrow \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{3}{\beta}=\frac{q}{q}=1 \Rightarrow \beta=3$

$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\geq\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha} \Rightarrow 3\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)\geq\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^2=1 \Rightarrow \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\geq\frac{1}{3}$

Again $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\geq\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha} \Rightarrow 3\left(\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}\right)\leq\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^2=1$

Put $\beta=3$ in the equation $27+9p+3q-q=0 \Rightarrow 9p+2q+27=0$

$\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}\leq\frac{1}{3} \Rightarrow \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}=\frac{-p}{q}\leq\frac{1}{3} \Rightarrow \frac{p}{q}\geq-\frac{1}{3}$

29.(BC) $S=\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{k}{2^{n+k}}=\sum_{k=1}^{\infty}\left(\frac{k}{2^k}\sum_{n=k+1}^{\infty}\frac{1}{2^n}\right)=\sum_{k=1}^{\infty}\frac{k}{2^k}\times\frac{2}{2^{k+1}}\times 2$

$\sum_{k=1}^{\infty}\frac{k}{4^k}=\frac{4}{9}$

30.(ACD) $n^2<n^2+1<n^2+2<....<n^2+n$

$\Rightarrow \frac{1}{n^2+1}+\frac{2}{n^2+2}+....+\frac{n}{n^2+n}<\frac{1}{n^2}+\frac{2}{n^2}+...\frac{n}{n^2}=\frac{1}{2}+\frac{1}{2n}$

and $\frac{1}{n^2+1}+\frac{2}{n^2+2}+....+\frac{n}{n^2+n}>\frac{1}{n^2+n}+...\frac{n}{n^2+n}=\frac{1}{2}$

31.(ABC) (a) $\left(1\cdot 3\cdot 5...(2n-1)\right)^{1/n} < \dfrac{1+3+5+...(2n-1)}{n} = n \Rightarrow 1\cdot 3\cdot 5....(2n-1) < n^n$

(b) $\dfrac{1+2+2^2+....+2^{n-1}}{n} > \left(2^{1+2+...(n-1)}\right)^{1/n} = 2^{\frac{n-1}{2}} \Rightarrow 2^n\cdot > 1+n\,2^{\frac{n-1}{2}}$

(c) $\dfrac{1}{n+1}+\dfrac{1}{n+2}+...+\dfrac{1}{n+n} > \dfrac{1}{2n}+\dfrac{1}{2n}+...+\dfrac{1}{2n} = \dfrac{1}{2}$

32.(BCD) $\dfrac{1}{a_{n+1}-1} = \dfrac{1}{a_n(a_n-1)} = \dfrac{1}{a_n-1}-\dfrac{1}{a_n}$

$$\frac{1}{a_n-1}-\frac{1}{a_{n+1}-1} = \frac{1}{a_n}$$

$$\Rightarrow \sum_{r=1}^{2018}\frac{1}{a_r} = \frac{1}{a_1-1}-\frac{1}{a_{2019}-1} = 1-\frac{1}{a_{2019}-1} < 1$$

$$a_{2019}-1 = a_{2018}(a_{2018}-1) = a_{2018}\,a_{2017}(a_{2017}-1)$$

$$\Rightarrow a_{2019}-1 = a_{2018}\,a_{2017}\,a_{2016}...a_2\,a_1(a_1-1) = a_1a_2a_3...a_{2018}$$

$$2018 < \frac{2018}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_{2018}}} < (a_1a_2....a_{2018})^{\frac{1}{2018}}$$

$$\Rightarrow a_{2019}-1 > (2018)^{2018} \Rightarrow 1-\frac{1}{a_{2019}-1} > 1-\frac{1}{(2018)^{2018}}$$

33.(BC) $\dfrac{k}{(k-1)^{4/3}+k^{4/3}+(k+1)^{4/3}} < \dfrac{k}{(k-1)^{4/3}+\left((k-1)(k+1)\right)^{2/3}+(k+1)^{4/3}}$

$$= \frac{\left((k+1)^{2/3}-(k-1)^{2/3}\right)k}{(k+1)^2-(k-1)^2} = \frac{1}{4}\left((k+1)^{2/3}-(k-1)^{2/3}\right)$$

$$\Rightarrow \quad S_n < \frac{1}{4}\sum_{k=1}^{n}\left((k+1)^{2/3}-(k-1)^{2/3}\right) = \frac{1}{4}\left((n+1)^{2/3}+n^{2/3}-1\right)$$

$$\Rightarrow \quad S_{999} < \frac{1}{4}\left((1000)^{2/3}+(999)^{2/3}-1\right) < \frac{1}{4}(100+100-1) < 50$$

$$S_{26} < \frac{1}{4}\left((27)^{2/3}+(26)^{2/3}-1\right) < \frac{1}{4}(9+9-1) = \frac{17}{4}$$

34.(BC) $S = \dfrac{1}{\sqrt{1}+\sqrt{3}}+\dfrac{1}{\sqrt{5}+\sqrt{7}}+....+\dfrac{1}{\sqrt{9997}+\sqrt{9999}}$

$$> \frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{7}+\sqrt{9}}+\frac{1}{\sqrt{9999}+\sqrt{10001}}$$

$$\Rightarrow \quad 2s > \frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{7}}+....+\frac{1}{\sqrt{9999}+\sqrt{10001}}$$

$$= \frac{1}{2}\left(\sqrt{3}-\sqrt{1}+\sqrt{5}-\sqrt{3}+\sqrt{7}-\sqrt{5}+....+\sqrt{10001}-\sqrt{9999}\right) = \frac{1}{2}\left(\sqrt{10001}-\sqrt{1}\right) > \frac{1}{2}(100-1) \Rightarrow S > \frac{99}{4} > 24$$

35.(BD) $a_n, 2 = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} 2^i 2^j = \frac{1}{2}\left[\left(1+2+2^n+\ldots+2^n\right)^2 - \left(1^2+2^2+2^4+\ldots.+2^{2n}\right)\right]$

$$= \frac{\left(2^{n+1}-1\right)\left(2^{n+1}-2\right)}{3}$$

36.(ABCD) $$\frac{\underbrace{(a+a+\ldots a)}_{\text{a times}}+\underbrace{(b+b+\ldots b)}_{\text{b times}}+\underbrace{(c+c+\ldots.c)}_{\text{c times}}}{a+b+c} \geq \left(a^a b^b c^c\right)^{\frac{1}{a+b+c}}$$

$$\Rightarrow \quad \left(a^a b^b c^c\right)^{1/n} \leq \frac{a^2+b^2+c^2}{n}$$

$$\frac{\underbrace{(a+a+\ldots a)}_{\text{b times}}+\underbrace{(b+b+\ldots b)}_{\text{c times}}+\underbrace{(c+c+\ldots c)}_{\text{a times}}}{a+b+c} \geq \left(a^b b^c c^a\right)^{\frac{1}{a+b+c}}$$

$$\frac{a^2+b^2+c^2}{n} \geq \frac{ab+bc+ca}{n} \geq \left(a^b b^c c^a\right)^{\frac{1}{n}}$$

$$\left(a^a b^b c^c\right)^{\frac{1}{n}} + \left(a^b b^c c^a\right)^{\frac{1}{n}} + \left(a^c b^a c^b\right)^{\frac{1}{n}} \leq \frac{a^2+b^2+c^2}{n} + \frac{ab+bc+ca}{n} + \frac{ac+ba+cb}{b} = n$$

37.(ABCD) $S = 3(1+2+3+4++\ldots+2016) = 2^4 \times 3^3 \times 7 \times 2017$

$\left[\because\ 6^2+5^2+4^2-3^2-2^2-1^2 = 3\left[(6+3)+(5+2)+(4+1)\right]\right.$

$\left.= 3(1+2+3+4+5+6)\right]$

38.(BD) $f(x) = \sum_{r=1}^{n} \frac{rx^{r-1}}{(x+1)(x+2)\ldots(x+r)}$

$$f(x) = \left(1-\frac{x}{1+x}\right) + \sum_{r=2}^{n} \frac{(x+r)x^{r-1}-x^r}{(x+1)(x+2)\ldots.(x+r)}$$

$$= \left(1-\frac{x}{1+x}\right) + \sum_{r-2}^{n}\left(\frac{x^{r-1}}{(x+1)(x+2)\ldots(x+r-1)} - \frac{x^r}{(x+1)(x+2)\ldots(x+r)}\right) = 1 - \frac{x^n}{(x+1)(x+2)\ldots(x+n)}$$

$$1-f(x) = \prod_{r=1}^{n} \frac{x}{r+x}$$

$$\Rightarrow \quad \frac{-f'(x)}{1-f(x)} = \frac{1}{x}\sum_{r=1}^{n} \frac{r}{x+r}$$

$$\Rightarrow \quad f'(x) = -\frac{x^{n-1}}{(x+1)(x+2)\ldots.(x+n)} \sum_{r=1}^{n} \frac{r}{x+r}$$

39. **[A-Q] [B-T] [C-P] [D-R]**

(A) $$A=\frac{n(n+1)(2n+1)}{6}$$

$$B=\sum_{m=1}^{n}\frac{m(m+1)}{2}-\frac{1}{2}\frac{n(n+1)}{2}$$

$$\frac{1}{6}\sum_{m=1}^{n}(m+2)m(m+1)-m(m+1)(m-1)-\frac{n(n+1)}{4}$$

$$=\frac{n(n+1)(n+2)}{6}-\frac{n(n+1)}{4}=\frac{n(n+1)(2n+1)}{12}$$

(B) $$\frac{\sum a\left(b^2+c^2\right)}{abc}\ge\frac{a(2bc)+b(2ca)+c(2ab)}{abc}=6$$

(C) $$2\sum\frac{x^2-1+1}{1-x}=2\sum-(x+1)+\frac{1}{1-x}=-8+2\sum\frac{1}{1-x}\ge-8+2\left(\frac{9}{2}\right)=1$$

$$\left[\because\ \frac{3}{\sum\frac{1}{1-x}}\le\frac{\sum(1-x)}{3}=\frac{2}{3}\ \Rightarrow\ \sum\frac{1}{1-x}\ge\frac{9}{2}\right]$$

(D) $\because\ x,y,z\in N \quad\Rightarrow\quad \frac{1}{xy}\le 1 \quad\Rightarrow\quad \frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\le 3 \qquad\therefore\qquad x=y=z=1$

40. **[A-T] [B-Q] [C-S] [D-P]**

(A) $$\frac{\left(\frac{49a}{3}\right)3+\left(\frac{3b}{2}\right)2+c}{6}\ge\left(\left(\frac{49a}{3}\right)^3\left(\frac{3b}{2}\right)^2c\right)^{1/6}=\left(\frac{7^6a^3b^2c}{12}\right)^{1/6}\Rightarrow\ 49a+3b+c\ge 6\times 7=42$$

(B) $x=-t,t>0$

$$2x^3-\frac{3}{x^2}=-\left(2t^3+\frac{3}{t^2}\right)\le -5$$

$$\because\qquad \frac{t^3+t^3+\frac{1}{t^2}+\frac{1}{t^2}+\frac{1}{t^2}}{5}\ge\left(\left(t^3\right)^2\frac{1}{\left(t^2\right)^3}\right)^{1/5}=1\ \Rightarrow\qquad 2t^3+\frac{3}{t^2}\ge 5$$

(C) $$\left(\frac{\frac{x^3}{5}+\frac{x^3}{5}+\frac{x^3}{5}+\frac{x^3}{5}+\frac{x^3}{5}+\frac{8-x^3}{3}+\frac{8-x^3}{3}}{8}\right)^8\ge\left(\frac{8-x^3}{3}\right)\left(\frac{x^3}{5}\right)\ \Rightarrow\ \left(8-x^3\right)x^5\le\left(3^35^5\right)^{1/3}=3\times 5^{5/3}$$

(D) $$\frac{(x+x+x+x+x+x+x)+(y+y+y+y+y)}{12}\ge\left(x^7y^5\right)^{1/12}\Rightarrow(7x+5y)\ge 12a^{1/2}$$

$\therefore\qquad 12a^{1/12}\ge 12 \qquad\Rightarrow\quad a\ge 1$

41.(2) $S_n = \sum_{r=1}^{n} a_r = \sum_{k=1}^{k}\sum_{j=1}^{k} 2j = \sum_{k=1}^{n} k(k+1) = \frac{1}{3}\sum_{k=1}^{n}\left(\left(k(k+1)(k+2) - (k-1)k(k+1)\right)\right)$

$$S_n = \frac{1}{3}n(n+1)(n+2)$$

$$a_n = S_n - S_{n-1} = \frac{1}{3}(n+1)(n)\left((n+2)-(n-1)\right) = n(n+1)$$

$$\sum_{r=1}^{n}\frac{1}{a_r} = \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = e^{\lim\limits_{n\to\infty} -\frac{n}{n+1}} = e^{-1}$$

$$\frac{1}{\lambda} = e \quad \Rightarrow \quad \left[\frac{1}{\lambda}\right] = 2$$

42.(6) $a + b = \frac{1}{6}\sum_{i=1}^{6}(a_i + b_i) = 1$

$$\sum_{i=1}^{6}(a_i - a)^2 + \sum_{i=1}^{6} a_i b_i = 6a^2 - 2a\sum a_i + \sum a_i^2 + \sum a_i b_i$$

$$= 6a^2 - 2a(6a) + \sum_{i=1}^{6} a_i(a_i + b_i) = -6a^2 + 6a = 6a(1-a) = 6ab$$

43.(661750) $S = \sum_{k=1}^{99} k\left((k+1)^2 - k^2\right) + 100 = \frac{100\times 99\times 401}{6} + 100 = 661750$

44.(289) $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i 3^i 3^k} = \left(\frac{1}{1-\frac{1}{3}}\right)^3 - 3\left(\sum_{i=0}^{\infty}\frac{1}{3^{2i}}\frac{1}{\left(1-\frac{1}{3}\right)} - \sum_{i=0}^{\infty}\frac{1}{3^{3i}}\right) - \sum_{i=0}^{\infty}\frac{1}{3^i}$

$(i \neq j \neq k)$

$$= \frac{27}{8} - \frac{3}{2}\times 3\left(\frac{1}{1-\frac{1}{9}}\right) + 2\frac{1}{1-\frac{1}{27}} = \frac{27}{8} - \frac{81}{16} + \frac{27}{13} = \frac{81}{208}$$

45.(69) $\frac{n(n+1)}{2} - (m+k) = 17(n-2)3 \le \frac{n^2 - 33n + 68}{2} = m + k \le 2n - 1$

$\Rightarrow \quad n \in [31, 35] \qquad n = \{31, 32, 33, 34, 35\} \qquad \therefore \qquad$ Maximum value of $m + k = 35 + 34 = 69$

46.(352) $x^4 - 16x^3 + px^2 - 256x + q = 0$

x_1, x_2, x_3, x_4 are in G.P. $\Rightarrow \qquad x_1 x_4 = x_2 x_3$

$$(x_1 + x_4) + (x_2 + x_3) = 16$$

$x_1x_4(x_2+x_3)+x_2x_3(x_1+x_4)=256$

$\Rightarrow \quad x_1x_4(16)=256 \quad \Rightarrow \quad x_1x_4=x_2x_3=16$

$$P=x_1x_4+x_2x_3+(x_1+x_4)(x_2+x_3)$$

$$x_1x_2x_3x_4=(16)^2 \quad \Rightarrow \quad (x_1x_2x_3x_4)^{1/4}=2^2=4$$

$$\frac{x_1+x_2+x_3+x_4}{4}=4$$

$\Rightarrow \quad x_1=x_2=x_3=x_4=4 \quad \Rightarrow \quad P=6\times 16=96, q=256$

47.(0) $\displaystyle\sum_{r=1}^{2018}\frac{x_r^2}{1-x_r}=\sum_{r=1}^{2018}\left(-(x_r+1)+\frac{1}{1-x_r}\right)$

$$=\sum_{r=1}^{2018}\left[-(x_r+1)+\frac{1-x_r+x_r}{1-x_r}\right]=\sum_{r=1}^{2018}\left[-(x_r+1)+1+\frac{x_r}{1-x_r}\right]=-1+1=0$$

48.(2017) $\displaystyle\sum_{i=1}^{2018}\frac{x_i^2}{1-x_i}=\sum_{i=1}^{2018}-(x_i+1)+\frac{1}{1-x_i}=-2019+\sum_{i=1}^{2018}\frac{1}{1-x_i}$

$$\frac{(1-x_1)+(1+x_2)+....(1-x_{2018})}{2018}\geq\frac{2018}{\frac{1}{1-x_1}+\frac{1}{1-x_2}+...+\frac{1}{1-x_{2018}}} \quad \Rightarrow \quad \sum_{i=1}^{2018}\frac{1}{1-x_i}\geq\frac{(2018)^2}{2017}$$

$$\therefore \quad \sum_{i=1}^{2018}\frac{x_i^2}{1-x_i}\geq\frac{(2018)^2}{2017}-2019=\frac{(2018)^2-\left((2018)^2-1\right)}{2017}=\frac{1}{2017}$$

$\Rightarrow$ the least value of k satisfying $k\displaystyle\sum_{i=1}^{2018}\frac{x_i^2}{1-x_i}\geq 1 \;=2017$

49.(2018) Adding, $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{3}{2018}$

$$2xz-2yz+1=\frac{z}{2018}; \quad 2yz-2zx+1=\frac{x}{2018}; \quad 2yz-2xy+1=\frac{y}{2018}$$

Adding, $x+y+z=2018\times 3$

$\because \quad \dfrac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\leq\dfrac{x+y+z}{3} \quad \Rightarrow \quad x=y=z=2018$

50.(4) $-\dfrac{a^3}{(a-b)(c-a)}-\dfrac{b^3}{(b-c)(a-b)}-\dfrac{c^3}{(c-a)(b-c)}$

$$=-\frac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{(a-b)(b-c)(c-a)}=-\frac{(b-c)(c-a)(a-b)(-a-b-c)}{(a-b)(b-c)(c-a)}=a+b+c\geq 4$$

COMPLEX NUMBER

1.(D) Since $|z + a| \le a$ implies z lies on or inside a circle with centre (-a, 0) and radius a, we have $|z_1|+|z_2|+|z_3| \le 14$.

2.(A) Hence, $i\left\{\log\left(\frac{x-i}{x+i}\right)\right\}-\pi+2\tan^{-1}x=k(\text{say})$

$\therefore \quad \log\left(\frac{x+i}{x-i}\right)=i\left(k+\pi-2\tan^{-1}x\right)$ Or $\frac{x+i}{x-i}=e^{i\theta}$, where $\theta=k+\pi-2\tan^{-1}x$

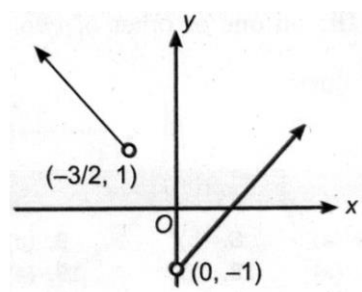
$\Rightarrow \quad x+i=(x\cos\theta+\sin\theta)+i(x\sin\theta-\cos\theta) \quad \Rightarrow \quad x=x\cos\theta+\sin\theta$

And $1=x\sin\theta-\cos\theta$

$\Rightarrow \quad x=\cot\frac{\theta}{2} \quad \Rightarrow \quad \theta=2\cot^{-1}x$ Or $k+\pi-2\tan^{-1}x=2\cot^{-1}x$

$\Rightarrow \quad k+\pi=2\left(\cot^{-1}x+\tan^{-1}x\right)=2\left(\frac{\pi}{2}\right) \quad \Rightarrow \quad k+\pi=\pi$ or $k=0$.

3.(D) $\arg(z + i) = \frac{\pi}{4}$



$\Rightarrow \quad \arg\{x+iy+i\}=\frac{\pi}{4} \qquad \{x>0, y+1>0\}$

$\Rightarrow \quad \frac{y+1}{x}=1, x>0, y>-1$

Now, $x = y + 1, x > 0, y > -1$ …(1)

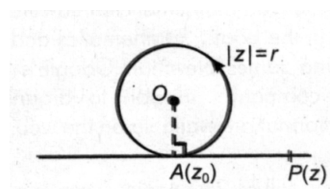
Also, $\arg\{(2z+3-2i)\}=\frac{3\pi}{4}, \{2x+3<0, 2y-2>0\}$

$\Rightarrow \quad \frac{2y-2}{2x+3}=-1, x<-\frac{3}{2}, y>1 \quad \Rightarrow \quad 2y - 2 = -2x - 3$

$\Rightarrow \quad 2x + 2y = -1$

$\Rightarrow \quad x+y=-\frac{1}{2}, x<-\frac{3}{2}, y>1$ ….(2) Hence, (1) and (2) have no solutions.

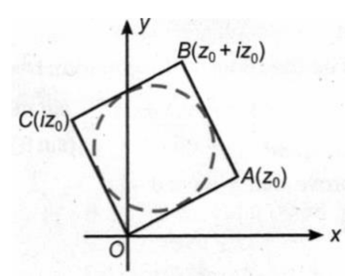
4.(A) $\angle OAP=\frac{\pi}{2}$



$\Rightarrow \quad \frac{z-z_0}{z_0}$ is purely imaginary. $\quad \Rightarrow \quad \frac{z-z_0}{z_0}+\frac{\bar{z}-\bar{z}_0}{\bar{z}_0}=0$

$\Rightarrow \quad \frac{z}{z_0}+\frac{\bar{z}}{\bar{z}_0}=2 \quad \Rightarrow \quad 2\,\text{Re}\left(\frac{z}{z_0}\right)=2$

$\Rightarrow \quad \text{Re}\left(\frac{z}{z_0}\right)=1.$



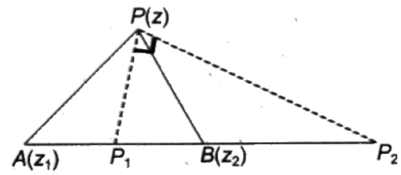
5.(B) Clearly, mid-point of OB is on centre of the circle whose radius is equal to $\frac{|z_0|}{2}$.

$\Rightarrow$ Required equation of circle is $\left|z-\frac{z_0(1+i)}{2}\right|=\frac{|z_0|}{2}$.

6.(B) Let internal and external bisectors of $\angle APB$ meet the line joining A and B at P_1 and P_2, respectively.

Now, $\frac{AP_1}{P_1B} = \frac{PA}{PB} = \frac{3}{1}$ (internal division)

And $\frac{AP_2}{P_2B} = \frac{PA}{PB} = \frac{3}{1}$ (external division)



Thus, P_1 and P_2 are fixed points

Also, $\angle P_1PP_2 = \frac{\pi}{2}$

Thus, P lies on a circle having P_1P_2 as its diameter. Clearly, $B(z_2)$ lies inside this circle.

7.(A) $z_1 + z_2 = -1$ and $z_1z_2 = \frac{b}{3}$.

As z_1, z_2 and origin form an equilateral triangle,

We have $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

$\Rightarrow \quad z_1^2 + z_2^2 + 0^2 = z_1z_2 + 0 + 0 \quad \Rightarrow \quad (z_1 + z_2)^2 = 3z_1z_2 \quad \Rightarrow \quad 1 = b$

8.(B) As the roots are of opposite signs, product of roots < 0.

$\Rightarrow \quad a^2 + b^2 - 1 < 0 \quad \Rightarrow \quad a^2 + b^2 < 1 \quad \Rightarrow \quad |a + ib| < 1$

So, the point a + ib lies inside a circle of centre (0, 0) and radius 1.

9.(C) $\left|\frac{z+1}{z}\right| = 1^{1/n} = 1$

$\Rightarrow \quad |z + 1| = |z| \quad \Rightarrow$ z lies on the right bisector of the line joining the points (-1, 0) and (0, 0).

$\Rightarrow$ Roots are collinear.

10.(B) Given, $1 \geq |z - (4 - 3i)| \geq \begin{cases} |z| - |4 - 3i| \\ |4 - 3i| - |z| \end{cases}$

$\Rightarrow \quad |z| \leq 6$ and $|z| \geq 4 \quad \Rightarrow \quad 4 \leq |z| \leq 6 \quad \Rightarrow \alpha = 4, \beta = 6$

Let $\quad y = \frac{x^4 + x^2 + 4}{x} = x^3 + x + \frac{4}{x} = x^3 + x + \frac{1}{x} + \frac{1}{x} + \frac{1}{x} + \frac{1}{x}$

Since $x \in (0, \infty)$, therefore $x^3, x, \frac{1}{x}, \frac{1}{x}, \frac{1}{x}, \frac{1}{x}$ are positive. Sum will be the least when $x^2 = x = \frac{1}{x} \Rightarrow x = 1$.

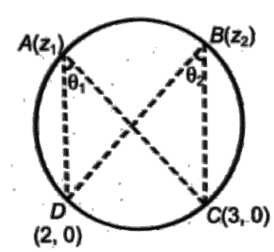
$\therefore \quad K = 6 \quad$ Hence, $K = \beta$.

11.(D) Let (cos x – i sin 2x) = sin x + i cos 2x $\quad \Rightarrow \quad$ cos x + i sin 2x = sin x + i cos 2x

$\therefore \quad$ cos x = sin x and sin 2x = cot 2x $\quad \Rightarrow \quad$ tan x = 1 and tan 2x = 1

Which is impossible.

12.(A) $arg\left(\frac{3 - z_1}{2 - z_1}\right) + arg\left(\frac{2 - z_2}{3 - z_2}\right) = arg\left(\frac{3 - z_1}{2 - z_1} \cdot \frac{2 - z_2}{3 - z_2}\right)$



Now if, $\left(\frac{3 - z_1}{2 - z_1} \cdot \frac{2 - z_2}{3 - z_2}\right)$ is a positive real number, then its argument will be zero.

So, angle θ_1 and θ_2 are equal in magnitude but opposite in signs.

So, chord DC subtends equal angles at A and B. So, points are concyclic for K > 0.

13.(A) $z^2 + ax + a^2 = 0 \Rightarrow z = a\omega, a\omega^2$ (where ' ω ' is non-real root of unity)

$\Rightarrow$ Locus of z is a pair of straight lines and arg(z) = arg(a) + arg(ω) or arg(z) = arg(a) + arg(ω^2)

$\Rightarrow$ $\arg(z) = \pm\frac{2\pi}{3}$ Also, $|z| = |a||\omega|$ or $|z| = |a||\omega^2|$ $\Rightarrow$ $|z| = |a|$

14.(D) Required locus is the set of all points, sum of whose distance from (-1, 0) and (1, 0) is equal to 2. Again distance between (-1, 0) and (1, 0) is also 2.

$\therefore$ The required locus is the portion of the real axis between the points (-1, 0) and (1, 0).

15.(C) $|z - 2| = \min\{|z-1|, |z-5|\}$

i.e., $|z-2| = |z-1|$, where $|z-1| < |z-5|$ $\Rightarrow$ $\text{Re}(z) = \frac{3}{2}$ which satisfy $|z-1| < |z-5|$

Also, $|z-2| = |z-5|$, where $|z-5| < |z-1|$ $\Rightarrow$ $\text{Re}(z) = \frac{7}{2}$ which satisfy $|z-5| < |z-1|$

16.(C) $\therefore$ $z^{n-1} + z^{n-2} + z^{n-3} + ... + z + 1 = 0$

$\Rightarrow$ $(z-1)(z^{n-1} + z^{n-2} + ... + z + 1) = 0$ $\Rightarrow$ $z^n - 1 = 0$ $\Rightarrow$ $z^n = e^{f\,2\pi r}, (r \in N)$ $\Rightarrow$ $z = e^{f\,2\pi r/n}$

$\Rightarrow$ The roots are $e^{i2\pi/n}, e^{i4\pi/n}, ..., e^{i(2n-n)\pi/n}$

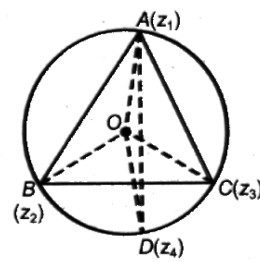
Which is a GP with a common ratio $e^{i\frac{2\pi}{n}}$. Also, $a^{\frac{z_r+1}{z_r}} = a^{e^{i2\pi/n}}$ which is a constant.

17.(A) Now, $\angle BOD = 2\angle BAD = A$

And $\angle COD = 2\angle CAD = A$

From the rotation about the point O, we get,

$$\frac{z_4}{z_2} = e^{iA}, \frac{z_3}{z_4} = e^{iA} \Rightarrow z_4^2 = z_2 z_3.$$



18.(C) $|z - 2 - i| = |z|\sin\left(\frac{\pi}{4} - \arg z\right)$

$\Rightarrow$ $|(x-2) + i(y-1)| = |z|\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)$ (where $\theta = \arg z$)

$\Rightarrow$ $\sqrt{(x-2)^2 + (y-1)^2} = \frac{1}{\sqrt{2}}|z|\cos\theta - \frac{1}{\sqrt{2}}|z|\sin\theta$ $\Rightarrow$ $\sqrt{(x-2)^2 + (y-1)^2} = \frac{1}{\sqrt{2}}(x-y)$

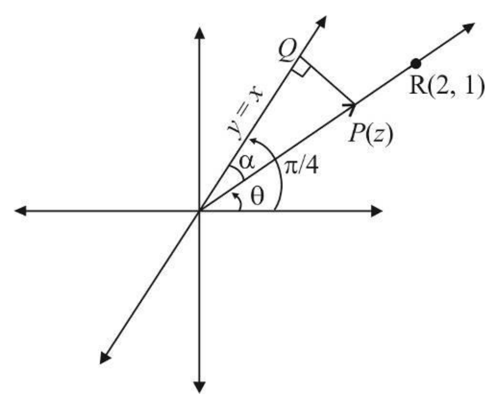
Which is a parabola with vertex (2, 1) and directrix x – y = 0.

Alternate approach

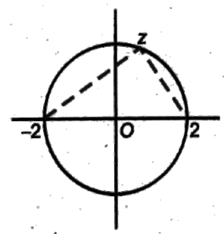
$\alpha = \frac{\pi}{4} - \theta$

$PQ = |z|\sin\left(\frac{\pi}{4} - \theta\right)$ is distance of $P(z)$ from $y = x$ line on argand plane which is equal to *PR*.

As distance of a point from a fixed line is equal to its distance from a fixed point so locus of given point is a parabola whose directrix is $y = x$ and focus is (2, 1)



19.(C) Clearly, $\arg\left(\frac{z-2}{z+2}\right)=\pm\frac{\pi}{2}$



$\Rightarrow \quad \arg\left(\frac{z_1-z_2}{z_1-z_3}\right)=\pm\frac{\pi}{2}$

$\Rightarrow$ z_1, z_2, z_3 will be the vertices of a right-angled triangle.

20.(B) We are finding out sum of distances of complex number z from origin and $(\cos\alpha,\sin\alpha)$. This sum will be minimum if z lies on the line joining the two points and minimum value of sum will be distance between those two points.

i.e., $\sqrt{(\cos\alpha-0)^2+(\sin\alpha-0)^2}=1$.

21-23. **21.(A)** **22.(D)** **23.(C)**

$f(x)=x^4-6x^3+26x^2-46x+65$.

Since complex roots occur in conjugate pairs.

$a_1=a_2,\ a_3=a_4,\ b_1=-b_2\ \ \&\ \ b_3=-b_4$

Product $=\left(a_1^2+b_1^2\right)\left(a_3^2+b_3^2\right)=65$ and sum of roots $=2(a_1+a_3)=6\Rightarrow a_1+a_3=3$

Without loss of generality let $a_1^2+b_1^2<a_3^2+b_3^2$

Case I $a_1^2+b_1^2=1,\ a_3^2+b_3^2=65\Rightarrow \quad a_1=0,\ b_1=1 \ \Rightarrow \quad a_3=3\ (\because a_1+a_3=3)$

Not possible $\because\ b_3=\sqrt{65-9}$ not an integer

Case II $a_1^2+b_1^2=5,\ a_3^2+b_3^2=13$

Sub-case I $a_1=2,\ b_1=1$

$\Rightarrow \quad a_3=1,$ not possible $\because\ b_3=\sqrt{12}$ not an integer

Sub-case II $a_1=1,\ b_1=2$

$\Rightarrow \quad a_3=2,\ b_3=3$ integer (only possible solution)

$\Rightarrow$ roots of equation are $1\pm 2i,\ 2\pm 3i$.

$\Rightarrow \quad \mu=1+1+2+2=6,\ \lambda=2+2+3+3=10$ and $S=\{1,\ 2,\ -2,\ 3,\ -3\}$

$\Rightarrow$ Number of functions from S to C are 8^5

24.(ABC) $\lambda=\alpha^6$ is a root of

$1+x+x^2+x^3+\ldots+x^{10}=0 \qquad \ldots..(1)$

$\therefore \quad 1+\lambda+\lambda^2+\ldots+\lambda^{10}=0$

$\Rightarrow \quad 1+\lambda+\lambda^2+\lambda^3+\lambda^4+\lambda^5+\bar{\lambda}+\overline{\lambda^2}+\overline{\lambda^3}+\overline{\lambda^4}+\overline{\lambda^5}=0 \quad \Rightarrow \quad \mathrm{Re}\left(\lambda+\lambda^2+\ldots.+\lambda^5\right)=-\frac{1}{2}$

$\beta,\beta^2,\beta^3,\ldots.,\beta^{10}$ are roots of (1)

$\therefore \quad (x-\beta)\left(x-\beta^2\right)\left(x-\beta^3\right)\ldots\left(x-\beta^{10}\right)=1+x+x^2+\ldots.+x^{10} \qquad \ldots..(2)$

Now put x = μ

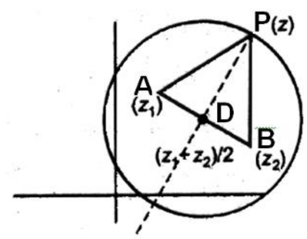
$(\mu-\beta)\left(\mu-\beta^2\right)\ldots\left(\mu-\beta^{10}\right)=1+\mu+\ldots+\mu^{10}=0 \qquad [\therefore \mu \text{ is a root of Eq. } (1)]$

Put x = i in Eq. (2), $(i-\beta)\left(i-\beta^2\right)\ldots.\left(i-\beta^{10}\right)=i$.

25.(BD) $\sum\sum_{1\le i<j\le 100} \alpha_i^5\alpha_j^5 = \left(\alpha_1^5+\alpha_2^5+\alpha_3^5+\alpha_4^5+\alpha_5^5+....\right)^2-\left(\alpha_1^{10}+\alpha_2^{10}+\alpha_3^{10}+\alpha_4^{10}+\alpha_5^{10}+...\right)=0-0=0$

Because $\left(\alpha_1^r+\alpha_2^r+....\alpha_{100}^r\right)=\begin{cases}100, & \text{if } r=100k\\ 0, & \text{if } r\ne 100k\end{cases}$

26.(ABCD) Maximum area of $(\Delta PAB)=\frac{1}{2}(PD)(AB)=\frac{1}{2}k\,|z_1-z_2|$



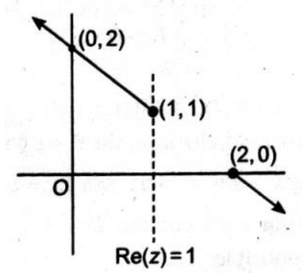
When ΔPAB is equilateral triangle of maximum area, then

$PD=\frac{\sqrt{3}}{2}AB \Rightarrow 4k^2\,|z_1-z_2|^2$

For maximum area of ΔPAB, P has two possible positions on the circle.

For the given area of triangle less than the maximum area there are four possible positions of P.

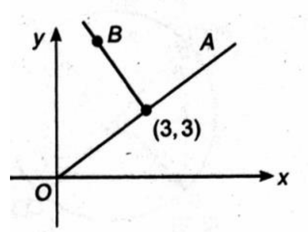
27.(D) The given equation is written is $\arg\left(z-(1+i)\right)=\begin{cases}\frac{3\pi}{4} & \text{when }, x\le 2\\ \frac{-\pi}{4} & \text{when}, x>2\end{cases}$



$\Rightarrow$ The locus is a set of two rays.

28.(BD) We can observe that $3+3i\in A$ but $\notin B$

$\therefore$ $n(A\cap B)=0$.



29.(BC) Let y = $|z_1-z_2|^2+|z_2-z_3|^2+|z_3-z_1|^2=(z_1-z_2)(\bar z_1-\bar z_2)+(z_2-z_3)(\bar z_2-\bar z_3)+(z_3-z_1)(\bar z_3-\bar z_1)$

$= 6-(z_1\bar z_2+z_2\bar z_1+z_2\bar z_3+z_3\bar z_2+z_3\bar z_1+z_1\bar z_3)....(1)$

Now, we know $|z_1+z_2+z_3|^2\ge 0 \quad\Rightarrow\quad 3+(z_1\bar z_2+z_2\bar z_1+z_1\bar z_3+z_3\bar z_1+z_2\bar z_3+z_3\bar z_2)\ge 0....(2)$

From Eqs. (1) and (2), we get : $y\le 9$

30.(AD) $f(i)=\alpha,\ f(-i)=\beta$

Now $f(z)=\text{Quotient}(z^2+1)+(az+b)$ where $g(z)=az+b$ is remainder and $a, b\in C$

Now $f(i)=ai+b$ and $f(-i)=-ai+b \quad\Rightarrow\quad a=\frac{\alpha-\beta}{2i}$ and $b=\frac{\alpha+\beta}{2}$

Then, $g(z)=i\left(\frac{z}{2}+1\right)+\frac{1}{2}$ when $\alpha=i$ and $\beta=1+i$

and $\alpha=i$ and $\beta=1-i\Rightarrow g(z)=\left(z+\frac{1}{2}\right)+\frac{iz}{2}$

31.(ABCD) Now, $a_1^2 + b_1^2 = 1$,

$a_2^2 + b_2^2 = 4$ and $a_1a_2 = b_1b_2$

$\therefore \quad a_2^2 + b_2^2 = 4a_1^2 + 4b_1^2 \quad \Rightarrow \quad (a_2 + 2ia_1)^2 = (2b_1 + ib_2)^2 \quad \Rightarrow \quad a_2 = \pm 2b_1$

$2a_1 = \pm b_2$

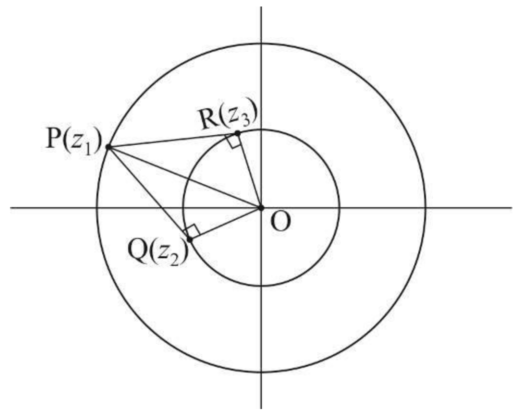
$|\omega_1|^2 = a_1 + a_1^2 + \frac{a_2^2}{4} = a_1^2 + b_1^2 = 1 \Rightarrow |\omega_1| = 1$

And $|\omega_2|^2 = 4b_1^2 + b_2^2 = 4 \Rightarrow |\omega_2| = 2$

$\text{Re}(\omega_1\omega_2) = 2a_1b_1 - \frac{a_2b_2}{2} = 0$

$\text{Im}(\omega_1\omega_2) = a_1b_2 + a_2b_1 = 0$

32.(ABCD) Option (A), (D) are true as PQR is an equilateral triangle. So orthocentre, circumcentre and centroid, all will coincide.



(B) $\left|\frac{z_1 + z_2 + z_3}{3}\right| = 1 \quad \Rightarrow \quad |z_1 + z_2 + z_3|^2 = 9$

$(z_1 + z_2 + z_3)(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) = 9$

$\left(\frac{4}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)\left(\frac{4}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}\right) = 9$.

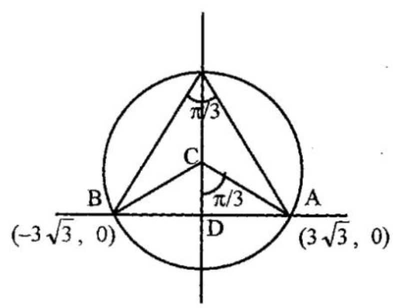
(C) $\angle QOR = 120^\circ$.

33.(AB) $|z + \bar{z}| + |z - \bar{z}| = 2 \quad \Rightarrow \quad |2\,\text{Re}(z)| + |2i\,\text{Im}(z)| = 2 \Rightarrow |x| + |y| = 1$

Which is the locus of a square.

Also, $|z + i| + |z - i| = 2$ represents a line. Hence, (a), (b) are the correct answer.

34.(BD) $\tan 60^\circ = \frac{3\sqrt{3}}{y} \Rightarrow y = 3$



$|z - 3i| = \sqrt{9 + 27} = 6, I_m\ Z > 0$

35.(BC) $i^i = \left(e^{\frac{i\pi}{2}}\right)^i = e^{-\frac{\pi}{2}}$

$\log i^i = -\frac{\pi}{2}$

36.(AB) $x^6 = (4 - 3i)^5 \qquad = -5^5 (\cos 5\theta + i \sin 5\theta)$

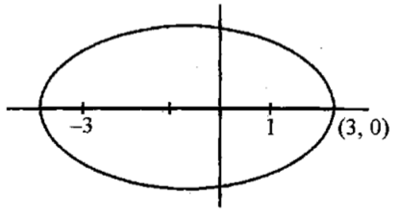
Hence $x_1x_2....x_6 = -(4 - 3i)^5$

37.(CD) $2ae = 4$

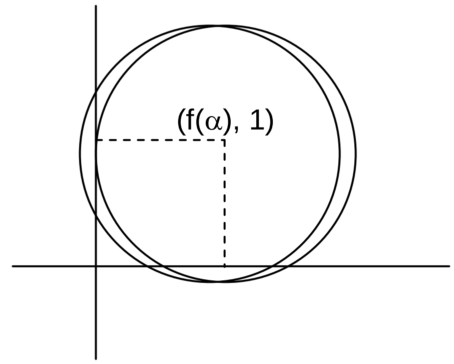
$8\,e = 4$

$e = 1/2$

$|Z-4| \in [1, 9]$



38.(ABC) From the figure it is clear that for $\arg z \geq \frac{\pi}{2}$ circle should either lie in second quadrant or touch the positive imaginary axis.



$\Rightarrow \quad f(\alpha) \leq 1$

$\Rightarrow \quad \alpha^2 - 7\alpha + 11 \leq 1$

$\Rightarrow \quad \alpha^2 - 7\alpha + 10 \leq 0$

$\Rightarrow \quad (\alpha-2)(\alpha-5) \leq 0$

$2 \leq \alpha \leq 5$

39.(AD) $\log_{\sqrt{3}}\left|1+\alpha+\alpha^2+\alpha^3-\frac{2}{\alpha}\right| = \log_{\sqrt{3}}\left|-\alpha^4-\frac{2}{\alpha}\right| = \log_{\sqrt{3}} 3 = 2$

40.(BD) $\sum_{r=0}^{3n-1} \alpha^{2^r} = n(\alpha+\alpha^2+\alpha^4)$

$1+(\alpha+\alpha^2+\alpha^4)+(\alpha^3+\alpha^5+\alpha^6) = 0$

$(\alpha+\alpha^2+\alpha^4)(\alpha^3+\alpha^5+\alpha^6) = 3+(\alpha+\alpha^2+.......\alpha^6) = 2$

Hence, $(\alpha+\alpha^2+\alpha^4)$ is root of $t^2+t+2=0$

$t = \frac{-1\pm\sqrt{7}i}{2}$

$\Rightarrow \quad \left|\sum_{r=0}^{3n-1} \alpha^{2^r}\right|^2 = n^2\,|\alpha+\alpha^2+\alpha^4|^2 = 2n^2 = 32$

$n = 4$

41.(BD) $z^{15} = 1 \Rightarrow z = \left(\cos\frac{2\aleph k}{15} + i\sin\frac{2\aleph k}{15}\right), k = 0, \pm1, \pm2, ..., \pm7$

$\Rightarrow \quad |\arg\ z| = \left|\frac{2\pi k}{15}\right| < \frac{\pi}{2} \quad \Rightarrow \quad |k| < \frac{15}{4} \quad \Rightarrow \quad k = 0, \pm1, \pm2, \pm3.$

42.(AB) $|z|+1\ \ |z|-2\ < 0 \Rightarrow -1 < |z| < 2$

$\left|z^2 + z\sin\theta\right| \leq |z|^2 + |z|\sin\theta < 6$

43.(AC) $z_1.z_2.....z_n = e^{i\pi\left\{\frac{1}{1.2.3.}+\frac{1}{2.3.4}+...+\frac{1}{k\ k+1\ \ k+2}\right\}}\ 1$

Let $S_k = \frac{1}{1.2.3}+\frac{1}{2.3.4}+...+\frac{1}{k(k+1)(k+2)}$

Consider $t_r = \frac{1}{2}\left\{\frac{r+2-r}{r(r+1)(r+2)}\right\} = \frac{1}{2}\left\{\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right\}$

$S_k = \sum t_r = \frac{1}{2}\left\{\frac{1}{1.2}-\frac{1}{(k+1)(k+2)}\right\} = \frac{1}{4}-\frac{1}{2(k+1)(k+2)}$

$\therefore \quad \underset{k\to\infty}{\text{Lt}}\ S_k = \frac{1}{4}$ and hence from (1) $\underset{k\to\infty}{\text{Lt}}\ z_1 ... z_k = e^{i\pi/4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

44.(BC) (i) $xy > 0,\ 3x^2 - xy - 2y^2 = -7$ or $(3x+2y)(x-y) = -7$ x and y being integers, we can take

$3x + 2y = 7$ and $x - y = -1$

$x = 1, y = 2$

If x and y are charged to $-x, -y$, equation remains same.

$x = -1,\ y = -2$ is also a solution pair.

$3x + 2y = -1$ and $x + y = 7$ do not give integral solutions.

(ii) $xy = 0$ will not give any integral solution.

(iii) $xy < 0\ 3x^2 + xy - 2y^2 = -7$

$(3x - 2y)(x - y) = -7$

$3x - 2y = -7$ and $x + y = 1$ leads to $x = -1\ y = 2$

$3x - 2y = 7$ and $x + y = -1$ lead to x = 1 and $y = -2$.

Required complex numbers are $1+2i, 1-2i, -1+2i, -1-2i$ which form two conjugate pairs.

45.(ABC) Let $z_1 = x_1,\ z_2,\ z_3 = x_2 \pm iy_2$

$\Rightarrow \quad z_1 + z_2 + z_3 = -\frac{b}{a} \quad \Rightarrow \quad x_1 + 2x_2 = -\frac{b}{a} < 0 \Rightarrow ab < 0$

Also $z_1z_2z_3 = x_1\left[x_2^2 + y_2^2\right] = -\frac{d}{a} < 0 \Rightarrow ad > 0$

Also $z_1z_2 + z_2z_3 + z_1z_3 = \frac{c}{a}$

$\Rightarrow \quad x_1(x_2 + iy_2) + x_1(x_2 - iy_2) + x_2^2 + y_2^2 = 2x_1x_2 + x_2^2 + y_2^2 > 0 \quad \Rightarrow \quad \frac{c}{a} > 0 \quad \Rightarrow \quad \frac{b}{a}\frac{c}{a} > 0 \Rightarrow bc > 0$

46.(CD) $y = |z_1 - z_2|^2 + |z_1 - z_3|^2 + |z_3 - z_1|^2$

$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) + (z_2 - z_3)(\bar{z}_2 - \bar{z}_3) + (z_3 - z_1)(\bar{z}_3 - \bar{z}_1) = 6 - (z_1\bar{z}_2 + \bar{z}_2\bar{z}_1 + z_2\bar{z}_3 + z_3\bar{z}_2 + z_3\bar{z}_1 + z_1\bar{z}_3)$

Also $|z_1 + z_2 + z_3|^2 \ge 0 \quad \Rightarrow \quad 3 + (z_1 + \bar{z}_2 + z_2\bar{z}_1 + z_1\bar{z}_3 + z_3\bar{z}_1 + z_2\bar{z}_3 + z_3\bar{z}_2) \ge 0 \quad \Rightarrow \quad y \le 9$

47.(ABCD) Re $(z_1 z_2) = 0$

$\Rightarrow \quad a_1a_2 = b_1b_2$

$|z_1| = 1,\ |z_2| = 2$

$\Rightarrow \quad a_1^2 + b_1^2 = 1,\ a_2^2 + b_2^2 = 4$

$w_1w_2 = (2a_1b_1 - (a_2b_2)/2) + i(a_2b_1 + a_1b_2)$

$a_2^2 + b_2^2 = 4a_1^2 + 4b_1^2$

$\Rightarrow \ (a_2 + 2ia_1)^2 = (2b_1 + ib_2)^2 \ \Rightarrow \ a_2 = \pm\, 2b_1$ and $b_2 = \pm 2a_1$

$|w_1|^2 = a_1^2 + \dfrac{a_2^2}{4} = a_1^2 + b_1^2 = 1$

$|w_1|^2 = 4b_1^2 + b_2^2 = 4\left(a_1^2 + b_1^2\right) = 4$

$\text{Re}(w_1 w_2) = \left(2a_1 b_1 - \dfrac{1}{2}(2b_1)(2a_1)\right) = 0$

$\text{Im}(w_1 w_2) = \left(2b_1^2 + 2a_1^2\right) = 2\left(a_1^2 + b_1^2\right) = 2$

48.(AB) For complex number z and w, we are given that

$|z|^2 w - |w|^2 z = z - w$ …(i)

$\Rightarrow \quad z\left(1 + |w|^2\right) = w\left(1 + |z|^2\right) \quad \Rightarrow \quad \dfrac{z}{w} = \dfrac{1+|z|^2}{1+|w|^2} = a$ real number

$\Rightarrow \quad \overline{\left(\dfrac{z}{w}\right)} = \dfrac{z}{w} \quad \Rightarrow \quad \dfrac{\bar{z}}{\bar{w}} = \dfrac{z}{w}$

$\Rightarrow \quad \bar{z}w = z\bar{w}$ …(ii)

Again from equation (i), we get $z\,\bar{z}w = w\,\bar{w}z = z - w$

$z(\bar{z}w - 1) - w(\bar{w}z - 1) = 0$

$z(\bar{z}w - 1) - w(z\bar{w} - 1) = 0$ (using equation (ii))

$\Rightarrow \quad (z\bar{w} - 1)\ (z - w) = 0 \ \Rightarrow \ z\bar{w} = 1$ or $z = w$

49.(ABD) Let $z = \alpha$ be the real root of the given equation. Then $\alpha^3 (3 + 2i)\alpha + (-1 + i\alpha) = 0$

$\Rightarrow \ \left(\alpha^3 + 3\alpha - 1\right) + i(\alpha + 2\alpha) = 0 \ \Rightarrow \ \alpha^3 + 3\alpha - 1 = 0$ and $a = -\dfrac{\alpha}{2} \ \Rightarrow \ -\dfrac{a}{8} - \dfrac{3a}{2} - 1 = 0 \ \Rightarrow \ a^3 + 12a + 8 = 0$

Let $f(a) = a^3 + 12a + 8$

$f(-1) < 0,\ f(0) > 0,\ f(-2) < 0,\ f(3) > 0,\ f(1) > 0 \quad \Rightarrow \quad a \in (-2, 1)$ or $(-1, 0)\ (-2, 3)$

50.(AB) $p\left(x^3\right) + xq\left(x^3\right) + x^2 r\left(x^3\right) = \left(1 + x + x^2\right) s(x)$

Putting $x = 1$, we get

$\Rightarrow \quad p(1) + q(1) + r(1) = 3s(1)$ …(i)

Putting $x = \omega$, where ω is cube root of unity, we get

$\Rightarrow \quad p(1) + \omega q(1) + \omega^2 r(1) = 0$

Putting $x = \omega^2$, we get

$\Rightarrow \quad p(1) + \omega^2 q(1) + \omega r(1) = 0$

Adding (i), (ii), (iii) we get

$3p(1) = 3s(1) \ \Rightarrow p(1) = s(1)$

Multiplying (ii) by $(-\omega)$ and (iii) by $\left(-\omega^2\right)$, we get

$$-\omega p(1)-\omega^2 q(1)-r(1)=0$$

$$-\omega^2 p(1)-\omega q(1)-r(1)=0$$

Adding (ii), (iii), (iv) and (v) we get $\left(2-\omega-\omega^2\right)p(1)+\left(\omega+\omega^2-2\right)r(1)=0 \quad \Rightarrow \quad p(1)=r(1)$

51.(AB) We have $z^4+\lambda z^3+(-36+15i)z^2+mz=0$

Clearly, one root is $z = 0$

Let the other roots be z, iz and $z+iz$ which are the roots of the equation

$$z^3+\lambda z^2+(-36+15i)z+m=0$$

From the equation, $z+iz+z+iz=-\lambda \quad \Rightarrow \quad 2(1+i)z=\lambda$

Sum of the product of roots taken two at a time.

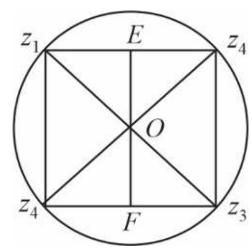
$iz^2+z^2+iz^2+iz^2-z^2=-36+15i$

$\Rightarrow \quad 3iz^2=-36+15i \quad \Rightarrow \quad z^2=5+12i \quad \Rightarrow \quad z=3+2i$ or $-3-2i$

Product of all roots, $z^3 i(1+i)=-m$

So, (λ, m) can be $(-2-10i, 37+55i)$ or $(2+10i, -37-55i)$

52.(AD) $z_1+z_2+z_3+z_4=0 \quad \Rightarrow \quad \dfrac{z_1+z_2}{2}=-\dfrac{z_3+z_4}{2}$



$\dfrac{z_3+z_4}{2}$ represents mid-point F of DC.

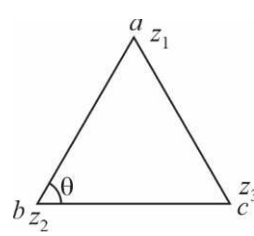
From (1), OE and OF are in opposite direction and $OE = OF$.

Thus, ΔOAB is isosceles, since $OA = OB = 1$ [A, B lie on $|z|=1$].

Hence, OE is perpendicular to AB. Also, OF is perpendicular to CD.

Therefore, $ABCD$ is a parallelogram inscribed in a circle and hence, it is a rectangle.

53.(CD) Triangles are similar



$\Rightarrow \quad -z_1z_2z_3=-abc$

$\Rightarrow \quad \arg\left(\dfrac{z_1-z_2}{z_3-z_2}\right)=\arg\left(\dfrac{a-b}{c-b}\right)$ and $\left|\dfrac{z_1-z_2}{z_3-z_2}\right|=\left|\dfrac{a-b}{c-b}\right|$

$\therefore \quad \dfrac{z_1-z_2}{z_3-z_2}=\dfrac{a-b}{c-b}$

$\therefore \quad (z_1-z_2)(c-b)=(a-b)(z_3-z_2)$

$\Rightarrow \quad z_1(b-c)+z_2(c-a)+z_3(a-b)=0$ and $a(z_2-z_3)+b(z_3-z_1)+c(z_1-z_2)=0$

54.(BC) $k+\left|k+z^2\right|=|z|^2$

$\Rightarrow \quad \left|k+z^2\right|^2=\left(|z|^2-k\right)^2 \quad \Rightarrow \quad \left(k+z^2\right)\left(k+\bar{z}^2\right)=|z|^4+k^2-2k|z|^2$

$\Rightarrow \quad k^2+(z\bar{z})^2+k\left(z^2+\bar{z}^2\right)=(z\bar{z})^2+k^2-2k(z\bar{z}) \quad \Rightarrow \quad k\left(z^2+\bar{z}^2+2z\bar{z}\right)=0 \quad \Rightarrow \quad k(z+\bar{z})^2=0$

As $k\neq 0$, $z+\bar{z}=0$ i.e., Re(z) = 0 So, z is a purely imaginary number. $\Rightarrow$ $\arg z=\pm\dfrac{\pi}{2}$

55.(ABC) Since z_1 and z_2 lie on $|z|=1$ and $|z|=2$, respectively, then $|z_1|=1$ and $|z_2|=2$

Now, $|2z_1+z_2|\le 2|z_1|+|z_2|\le 4 \Rightarrow \max|2z_1+z_2|=4$

Also, $|z_1-z_2|\ge ||z_1|-|z_2||=|1-2|=1 \Rightarrow \min|z_1-z_2|=1$

And, $\left|z_2+\frac{1}{z_1}\right|\le |z_2|+\frac{1}{|z_1|}=2+1=3 \Rightarrow \left|z_2+\frac{1}{z_1}\right|\le 3$

56.(AD) If z lies on the line joining a and ib, then

$$\arg\left(\frac{z-ib}{z-a}\right)=0 \text{ or } \pi$$

$$\Rightarrow \frac{z-ib}{z-a}=\frac{\bar{z}+ib}{\bar{z}-a} \Rightarrow z(-a-ib)+\bar{z}(-ib+a)+2iab=0$$

$$\Rightarrow z\left(-\frac{1}{2ib}-\frac{1}{2a}\right)+\bar{z}\left(-\frac{1}{2a}+\frac{1}{2ib}\right)+1=0 \Rightarrow z\left(\frac{1}{2a}+\frac{1}{2ib}\right)+\bar{z}\left(\frac{1}{2a}-\frac{1}{2ib}\right)=1 \Rightarrow z\left(\frac{1}{2a}-\frac{i}{2ib}\right)+\bar{z}\left(\frac{1}{2a}-\frac{i}{2b}\right)=1$$

57.(BD) $a+\bar{c}\ne 0$

Now, $(a+\bar{c})w_1^2+(b+\bar{b})w_1+(\bar{a}+c)=0 \Rightarrow (\bar{a}+c)\bar{w}_1^2+(b+\bar{b})w_1+(\bar{a}+c)=0$

Since, $\bar{w}_1\ne 0$, we get $(a+\bar{c})\frac{1}{\bar{w}_1^2}+(b+\bar{b})\frac{1}{\bar{w}_1}+(\bar{a}+c)=0$

Hence, $\frac{1}{\bar{w}_1}$ is also a root, but $\frac{1}{\bar{w}_1}$, so $w_1=\frac{1}{\bar{w}_1} \Rightarrow |w_1|=1$ Similarly $|w_2|=1$

58.(AC) $|Z^2-9|+|Z^2|=41$ N $\Rightarrow |Z+3|+|Z-3|=10 \Rightarrow 10=|Z+3|+|Z-3|\ge|Z+3+Z-3| \Rightarrow |Z|\le 5$

59.(AB) $|z_1+z_2|=\left|-\frac{b}{a}\right|=1$

$|z_1z_2|=\left|\frac{c}{a}\right|=1$

$\therefore \quad |z_2|=1$

$\quad |z_1+z_2|^2=1$

$\therefore \quad 2+z_1\bar{z}_2+z_2\bar{z}_1=1$

Now $z_2=z_1e^{i\theta}$

$\therefore \quad |z_1+z_1e^{i\theta}|=|z_1||1+e^{i\theta}|=1$

$\therefore \quad 2\cos\frac{\theta}{2}=1$

$\therefore \quad \theta=\frac{2\pi}{3}$

Now, $\frac{(z_1+z_2)^2}{z_1z_2}=1 \Rightarrow \frac{b^{2a}}{a^2}=\frac{c}{a} \Rightarrow b^2=ac$

$PQ=|z_1-z_2|=\sqrt{3}$

60.(ABCD) $Z_1 = e^{i\theta_1}$, $Z_2 = e^{i\theta_2}$, $\text{Re}(Z_1 Z_2) = 0 \Rightarrow \theta_1 + \theta_2 = -\frac{\pi}{2}$, (as z_1, z_2 lie in fourth quadrant)

$Z_3 = e^{-i\theta_1}$, $Z_4 = -ie^{i\theta_1}$, $Z_5 = \cos\theta_1(1-i)$, $Z_6 = \sin\theta_1(-1+i)$

61.(AC) Let $z - 3 = re^{i\phi}$ $\quad\therefore\quad$ $\frac{z-3}{e^{i\theta}} + \frac{e^{i\theta}}{z-3} = re^{i(\phi-\theta)} + \frac{1}{r}e^{i(\phi-\theta)}$

Imaginary part of the above = $r\sin(\phi-\theta) - \frac{1}{r}\sin(\phi-\theta) - \frac{1}{r}\sin(\phi-\theta)$

Given that r sin $(\phi-\theta) - \frac{1}{r}\sin(\phi-\theta) = 0$

$\Rightarrow \quad r - \frac{1}{r} = 0$ or $\sin(\phi-\theta) = 0$

$r - \frac{1}{r} = 0 \Rightarrow r = 1$

$\Rightarrow \quad |z-3| = 1$

$\Rightarrow \quad$ z lies on a circle with unit radius.

$\sin(\phi-\theta) = 0 \Rightarrow \phi = \theta$

$\therefore \quad z - 3 = re^{i\theta}$

$z - 3 = r\cos\theta,\ y = r\sin\theta \quad \Rightarrow \quad x - 3 = r\cos\theta,\ y = r\sin\theta$

$\frac{x-3}{\cos\theta} = \frac{y}{\sin\theta} = r$

The represents a straight line through (3, 0)

62.(AB) Given equation $z^6 = (z+1)^6$

$\Rightarrow \quad |z^6| = |(z+1)^6| \quad \Rightarrow \quad |z| = |z+1| \quad \Rightarrow$ Roots are collinear $\quad \Rightarrow \quad \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = 0 \text{ or } \pi$

63.(CD) P is centre of square $\Rightarrow$ $AP = PC$ and $-APC = \frac{\pi}{2} - ABC$ is also $\frac{\pi}{2}$.

$\Rightarrow \quad$ $ABCP$ is a cyclic quadrilateral.

$\Rightarrow \quad$ A, B, C, P lie on a circle.

$\Rightarrow \quad$ In circle chord, $AP = PC$

$\Rightarrow \quad$ arc AP = arc PC

$\Rightarrow \quad$ $-ABP = -PBC$

$\Rightarrow \quad$ BP is angular bisector.

Let $\arg\left(\frac{z_1 - z_2}{z_4 - z_2}\right) = -\alpha$

$\Rightarrow \quad \arg\left(\frac{z_3 - z_2}{z_4 - z_2}\right) = \alpha$

Add (i) and (ii), $\arg\left(\frac{z_1 - z_2}{z_4 - z_2}\right) + \arg\left(\frac{z_3 - z_2}{z_4 - z_2}\right) = 0$

64. **[A-r] [B-s] [C-q] [D-p]**

If k = 2, line segment

If 0 < k < 2, No locus

If k = 5

Let $z_1 = 1$ $z_2 = -1$

$|z_1 - z_2| = 2 = 2ae$

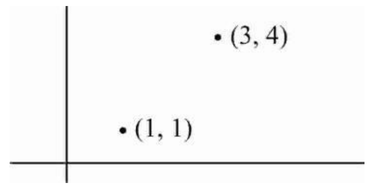
$5 = 2a \Rightarrow e = \frac{2}{5}$

65. **[A-q] [B-s] [C-r] [D-p]**

If $\theta = 0$, outside portion of line $\therefore$ line ray

If $\theta = \pi$, middle portion of line $\therefore$ line segment

If $\theta = \frac{2\pi}{3}$ (Obtuse) $\therefore$ minor arc



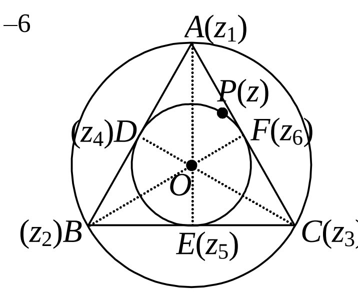
66. **[A-q] [B-p] [C-r] [D-s]**

(A) As we know,

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}\, z_1\bar{z}_2$$

$\Rightarrow$ $AB^2 = OA^2 + OB^2 - 2\text{Re}\; z_1\bar{z}_2$ $\Rightarrow$ $\text{RE}(z_1\bar{z}_2) = -2$

Similarly, $\text{Re}(z_1\bar{z}_3) = \text{Re}(z_3\bar{z}_1) = -2$ $\Rightarrow$ $\text{Re}(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1) = -6$



(B) Since $\angle AOC = \frac{2\pi}{3}$

$$\therefore \quad \frac{z_1 - 0}{z_3 - 0} = \frac{2}{2} e^{\frac{2\pi i}{3}}$$

$$\Rightarrow \quad \frac{4z_1}{z_3} = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \times 4 = 2(-1 + i\sqrt{3}) \quad \Rightarrow \quad a = 2$$

(C) We have $|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2$

$$= 2(|z_1|^2 + |z_2|^2 + |z_3|^2) + 2\,\text{Re}(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1) = 12$$

(D) $DP^2 + EP^2 + FP^2 = |z - z_4|^2 + |z - z_5|^2 + |z - z_6|^2 = 3|z|^2 + |z_4|^2 + |z_5|^2 + |z_6|^2 = 6$

67.(1) Here $z^2 - z = |z|^2 + \frac{64}{|z|^5}$ ....(i)

$\Rightarrow$ $z^2 - z = \bar{z}^2 - \bar{z}$ ($\because z^2 - z$ is purely real number)

$\Rightarrow$ $(z - \bar{z})(z + \bar{z} - 1) = 0$

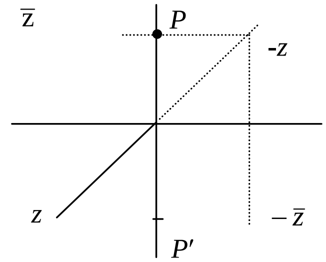
$\Rightarrow$ $z = \bar{z}$ as $z + \bar{z} = 1$ is not possible $\left(\because x \neq \frac{1}{2}\right)$

$\Rightarrow$ $z = x$

$\therefore$ Equation (i), given as $x^2 - x - |x|^2 - \frac{64}{|x|^5} = 0 \Rightarrow x = -2$

$\therefore$ Only one solution

68.(5) Clearly $P\left(\dfrac{\bar{z}-z}{2}\right)$ will lie on imaginary axis



$P'\left(\dfrac{z-\bar{z}}{2}\right) \Rightarrow \arg\left(\dfrac{z-\bar{z}}{2}\right)$ is $-\dfrac{\pi}{2}$

69.(7) For $|z-1|$ maximum $z=-4 \Rightarrow$ centroid $(\alpha) = \dfrac{z-w-\bar{w}}{3} = \dfrac{-4+1}{3} = -1 \Rightarrow \text{Re}(\alpha) = -1$

70.(6) $|z|+|z-1|+|2z-3| = |z|+|z-1|+|3-2z| \geq |z+z-1+3-2z| = 2$

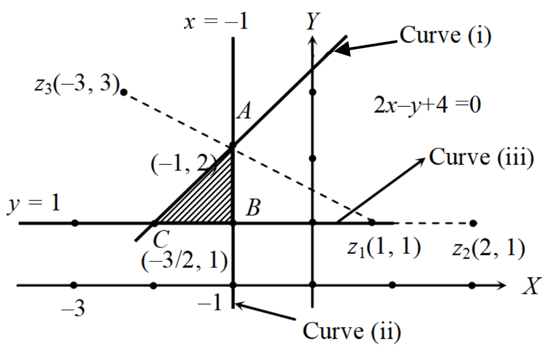
$\therefore \quad \lambda = 2$

$2[x]+3 = 3([x]-2)$

$[x] = 9 \;\therefore\; y = 2\times 9+3 = 21$

$\therefore \quad [x+y] = [x+21] = [x]+21 = 30$

71.(5) If $z = x+iy$



Required area = area of ΔABC

$= \dfrac{1}{2}\times(AB)(BC)$

$= \dfrac{1}{2}\times(2-1)\times\left(-1-\left(-\dfrac{3}{2}\right)\right)$

$= \dfrac{1}{4} = \dfrac{P}{Q} \Rightarrow P+Q = 5$

72.(5) $|A_k|^2 = A_k.\bar{A}_k = (x + y\alpha^k + p\alpha^{2k} + w\alpha^{3k} + f\alpha^{4k})\,(\bar{x}+\bar{y}\alpha^{-k}+\bar{p}\alpha^{-2k}+\bar{w}\alpha^{-3k}+\bar{f}\alpha^{-4k})$

$= x\bar{x}+y\bar{y}+p\bar{p}+w\bar{w}+f\bar{f}+x(\bar{y}\alpha^{-k}+\bar{p}\alpha^{-2k}+\omega\alpha^{-3k}+\bar{f}\alpha^{-4k})$

$+y\left(x\alpha^{k}+\bar{p}\alpha^{-k}+\bar{w}\alpha^{-2k}+\bar{f}\alpha^{-3k}\right)+\ldots\ldots+f(\bar{x}\alpha^{4k}+\bar{y}\alpha^{3k}+\bar{p}\alpha^{2k}+\bar{w}\alpha^{k})$

$\therefore \quad |A_0|^2+|A_1|^2+|A_2|^2+|A_3|^2+|A_4|^2 = \sum_{k=0}^{4}|A_n|^2 = 5(|x|^2+|y|^2+|p|^2+|w|^2+|f|^2)+x\sum_{k=0}^{4}(\bar{y}\alpha^{-k}+\ldots.+\bar{f}\alpha^{-4k})$

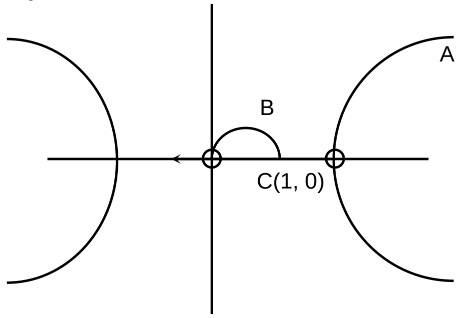
$+y\sum_{k=0}^{4}(\bar{x}\alpha^{k}+\bar{p}\alpha^{-k}+\ldots.+\bar{f}\alpha^{-3k})+\ldots.+f\sum_{k=0}^{4}\left(\bar{x}\alpha^{4k}+\ldots+\bar{w}\alpha^{k}\right)$

But $\sum_{k=0}^{4}\alpha^{ak} = 0$, if a is not divisible by 5

$\Rightarrow \quad |A_0|^2+|A_1|^2+\ldots\ldots+|A_4|^2 = 5(|x|^2+|y|^2+|p|^2+|w|^2+|f|^2)$

$= 5(5)$ ($\because$ x, y, p, w, f are points on $z\bar{z}=1$ i.e. $|z|^2=1$).

73.(0) A, B, C can be shown geometrically as below.



Clearly $A \cap B \cap C = \phi$.

74.(0) $\omega = \cos 12° + \cos 48° + i(\sin 48° + \sin 12°) = 2\cos 30° \cos 18° + 2i \sin 30° \cos 18° = 2\cos 18°\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$z = \omega^6 = 2^6 \cos^6 18°(\cos\pi + i\sin\pi)$

$z = -2^6(\cos^6 18°)$ $\quad\therefore\quad$ $I_m(z) = 0$

75.(3) $8iz^3 + 12z^2 - 18z + 27i = 0 \quad\Rightarrow\quad 4iz^2(2z - 3i) - 9(2z - 3i) = 0$

$(4iz^2 - 9)(2z - 3i) = 0 \quad\Rightarrow\quad |2z| = |3i|, \ |4iz^2| = 9$

$|z| = \frac{3}{2}, \ 4|z|^2 = 9 \Rightarrow |z| = \frac{3}{2} \Rightarrow 2|z| = 3$

76.(9) $\left|\frac{a^2x^2 + b^2y^2 + c^2z^2}{b^2x^2 + c^2y^2 + a^2z^2}\right| + \left(x_1^2 - 2y_1^2\right)\omega + \left(\left[x^2\right] + \left[y^2\right] + \left[z^2\right]\right)\omega^2 = 0 \quad \ldots\text{(i)}$

$1 + \omega + \omega^2 = 0$ ($\because$ ω is cube root of unity) …(ii)

Comparing (i) and (ii)

$$\left|\frac{a^2x^2 + b^2y^2 + c^2z^2}{b^2x^2 + c^2y^2 + a^2z^2}\right| = \left(x_1^2 - 2y_1^2\right) = \left[x^2\right] + \left[y^2\right] + \left[z^2\right]$$

a, b, c are cube roots of q $\Rightarrow$ $a = \lambda$, $b = \lambda\omega$, $c = \lambda\omega^2$

$$\left|\frac{a^2x^2 + b^2y^2 + c^2z^2}{b^2x^2 + c^2y^2 + a^2z^2}\right| = \left|\frac{\lambda^2x^2 + \lambda^2\omega^2y^2 + \lambda^2 \ \omega^4 \ z^2}{\lambda^2\omega^2x^2 + \lambda^2\omega^4y^2 + \lambda^2z^2}\right| = \left|\frac{1}{\omega^2}\right| = |\omega| = 1$$

$\therefore$ $x_1^2 - 2y_1^2 = 1 \Rightarrow x_1^2 = 1 + 2y_1^2 \Rightarrow x_1$ is odd.

Let $x_1 = 2n + 1$, then $y_1^2 = 2n(n + 1) \Rightarrow y_1$ is even

$\therefore$ y_1 is prime (given) $\Rightarrow y_1 = 2$

$\therefore$ $x_1 = 3 \quad \therefore [x^2] + [y^2] + [z^2] = 1$

$\therefore$ $[x^2 + x_1] + [y^2 + y_1] + [z^2 + x_1^2 + y_1^2] - 10$

$= [x^2] + [x_1] + [y^2] + [y_1] + [z^2] + [x_1^2] + [y_1^2] - 10 = 1 + 3 + 2 + 9 + 4 - 10 = 9.$

77.(1) $\left|\dfrac{2+3\omega+4\omega^2 z}{4\omega+3\omega^2 z+2z}\right| = \left|\dfrac{1}{z}\cdot\dfrac{2+3\omega+4\omega^2 z}{2+3\omega^2+4\omega z^{-1}}\right| = \left|\dfrac{1}{z}\right|\left|\dfrac{2+3\omega+4\omega^2 z}{2+3\omega^2+4\omega\bar{z}}\right| = \left|\dfrac{1}{z}\right|\left|\dfrac{2+3\omega+4\omega^2 z}{2+3\omega+4\omega^2 z}\right| = \left|\dfrac{1}{z}\right| = 1$

78.(0.414) $\left||z|-\dfrac{1}{|z|}\right| \le \left|z+\dfrac{1}{z}\right| = 2$

$\Rightarrow \quad -2 \le |z| - \dfrac{1}{|z|} \le 2 \quad \Rightarrow \quad |z|^2+2|z|-1\ge 0$ and $|z|^2-2|z|-1\le 0$

$\Rightarrow \quad |z| \ge \sqrt{2}-1,\ |z| \le \sqrt{2}+1 \quad \Rightarrow \quad |z|_{\min} = \sqrt{2}-1$

79.(0) $|z|=1$

Let $z\cos\theta + i\sin\theta$

$\Rightarrow \quad \dfrac{z^n}{z^{2n}+1} - \dfrac{\bar{z}^n}{\bar{z}^{2n}+1}$

$= \dfrac{\cos n\theta + i\sin n\theta}{1+\cos 2n\theta + i\sin 2n\theta} - \dfrac{\cos n\theta - i\sin n\theta}{1+\cos 2n\theta - i\sin 2n\theta}$

$= \dfrac{\cos n\theta + i\sin n\theta}{2\cos n\theta\left(\cos n\theta + i\sin n\theta\right)} - \dfrac{\cos n\theta - i\sin n\theta}{2\cos n\theta\left(\cos n\theta - i\sin n\theta\right)} = \dfrac{1}{2\cos n\theta} - \dfrac{1}{2\cos n\theta} = 0$

80.(6.25) $zz_1 = |z|^2 \Rightarrow z_1 = \bar{z}$

Now $|z-\bar{z}|+|z_1+\bar{z}_1| = 10$

$\Rightarrow \quad |z-\bar{z}|+|z+\bar{z}| = 10 \Rightarrow \quad |2iy|+|2x| = 10 \quad \Rightarrow \quad |x|+|y| = 5$

This represents a square and its area is $2\times 5^2 = 50$ units.

$\therefore \quad A = 50 \Rightarrow \dfrac{A}{8} = \dfrac{50}{8} = 6.25$

81.(1) $z = |z| + z^2$

Let $z = re^{i\theta}, r > 0 \quad \Rightarrow \quad re^{i\theta} = r + r^2 e^{2i\theta}$

Comparing real and imaginary parts, we get

$r\cos\theta = r + r^2\cos 2\theta$ and $r\sin\theta = r^2\sin 2\theta$

If $r=0;\ z=0$

If $\cos\theta = 1 + r\cos 2\theta$ and $\sin\theta = r\sin 2\theta$

$\Rightarrow \quad \sin\theta = 2r\sin\theta\cos\theta$

$\Rightarrow \quad \sin\theta = 0$ or $r\cos\theta = \dfrac{1}{2}$

$\Rightarrow \quad \theta = 0, \pi$ or $\cos\theta = 1 + \dfrac{2\cos^2\theta - 1}{2\cos\theta}$

$\Rightarrow \quad z = 0$ or $2\cos^2\theta = 2\cos\theta + 2\cos^2\theta - 1$

$\Rightarrow \quad z = 0$ or $\cos\theta = \dfrac{1}{2} \quad \Rightarrow \quad z = 0$ or $\theta = \pm\dfrac{\pi}{3}$

Thus, solutions are $z = 0,\ e^{ir/3}, e^{-ir/3}$

82.(0.20) $\left(e^{z^2}\right)=e^{\left(x^2-y^2\right)+2ixy} \Rightarrow \arg\left(e^{z^2}\right)=2xy$

Similarly, $\arg\left(e^{(z+i)}\right)=(y+1)$

Given $2xy=y+1 \Rightarrow y=\dfrac{1}{2x-1} \Rightarrow f(3)=\dfrac{1}{5}$

83.(10) Let $z^2=t$

$\Rightarrow \quad t^3+t^2-2=0 \Rightarrow (t-1)\left(t^2+2t+2\right)=0 \Rightarrow t=1$ or $t^2+2t+2=0 \Rightarrow z=\pm 1 or\ z^2=-1\pm i$

$\Rightarrow \quad |z|^2=\sqrt{2} \Rightarrow |z|^4=2 \Rightarrow \sum_{i=1}^{6}|z_i|^4=1+1+2+2+2+2=10$

84.(1) $z=e^{i2\theta}\sin\phi+\cos\phi$

$\Rightarrow \quad z=\cos\phi+\sin\phi(\cos 2\phi+i\sin 2\phi)$

$\Rightarrow \quad z=\cos\phi+\cos 2\phi\sin\phi+i\sin 2\phi\sin\phi$

$\Rightarrow \quad |z|=\sqrt{\cos^2\phi+\cos^2 2\phi\sin^2\phi+2\cos\phi\sin\phi\cos 2\phi+\sin^2 2\phi\sin^2\phi}$

$\Rightarrow \quad |z|=\sqrt{\cos^2\phi+\sin^2\phi+\sin 2\phi\cos 2\phi} \Rightarrow |z|=\sqrt{1+\dfrac{\sin 4\phi}{2}} \Rightarrow |z|_{\min}=\sqrt{\dfrac{1}{2}}$ and $|z|_{\max}=\sqrt{\dfrac{3}{2}}$

85.(1) Given $z=\alpha+i\beta$ is root of $z^5=1$

Also $|z|^5=1$

$\Rightarrow \quad |z|=1 \Rightarrow \bar{z}=\dfrac{1}{z}$

Now $4\alpha\left(\beta^4-\alpha^4\right)$

$= 4\alpha\left(\beta^2-\alpha^2\right)|z|^2 = -\dfrac{1}{2}(z+\bar{z})\left[(z-\bar{z})^2+(z+\bar{z})^2\right] = -(z+\bar{z})\left(z^2+\bar{z}^2\right)$

$= -\left(z^3+z+\dfrac{1}{z}+\dfrac{1}{z^3}\right) = -\dfrac{1}{z^3}\left(\dfrac{z^8-1}{z^2-1}\right) = -\dfrac{z^8-1}{z^5-z^3} = \dfrac{1-z^3}{1-z^3}=1 \qquad (\because z^5=1)$

86.(10) $\left|2+\dfrac{1}{z}\right|^2+|2-z|^2 = (2+\bar{z})^2+|2-z|^2 = 2\left[2^2+|z|^2\right]=10$

87.(0.50) $1+2|z|^2=|z^2+1|^2+2|z+1|^2$

$\Rightarrow \quad 1+2z\bar{z}=\left(z^2+1\right)\left(\bar{z}^2+1\right)+2(z+1)(\bar{z}+1) \Rightarrow (z\bar{z})^2+2(z+\bar{z})+z^2+\bar{z}^2+2=0$

$\Rightarrow \quad (z\bar{z})^2+2(z+\bar{z})+z^2+\bar{z}^2+2=0 \Rightarrow (z\bar{z}-1)^2+(z+\bar{z}+1)^2=0$

$\Rightarrow \quad z\bar{z}=1$ and $z+\bar{z}+1=0 \Rightarrow z+\dfrac{1}{z}+1=0$

$\Rightarrow \quad z^2+z=-1 \Rightarrow |z^2+z|=1$

88.(0) $z^2+|z|=\bar{z}^2$

Taking conjugate, we get $\bar{z}^2+|z|=z^2$

Adding (i) and (ii), we get

$\Rightarrow \quad 2|z|=0 \quad \Rightarrow \quad z=0$

89.(4) Z_1, Z_2, Z_3 are the roots of the equation $Z^3-Z^2+mZ-1=0$.

$\Rightarrow \quad Z_1+Z_2+Z_3=1$ and $Z_1Z_2Z_3=1$

Also, $|Z_1|=|Z_2|=|Z_3|=1$

Now, $|(Z_1+3)(Z_2+3)(Z_3+3)|$

$$=\left|Z_1Z_2Z_3+9(Z_1+Z_2+Z_3)+27+3\sum Z_1Z_2\right| = \left|1+9+27+3Z_1Z_2Z_3\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)\right|$$

$$=\left|1+9+27+3Z_1Z_2Z_3\left(\bar{Z}_1+\bar{Z}_2+\bar{Z}_3\right)\right| = \left|1+9+27+3Z_1Z_2Z_3\left(Z_1+Z_2+Z_3\right)\right| = \left|37+3(1)(1)\right|=40$$

90.(1) $|3z_1-2z_2-4|^2=|3z_1-1|^2+|2z_2+3|^2$

$\Rightarrow \quad |(3z_1-1)-(3z_2+3)|^2=|3z_1-1|^2+|2z_2+3|^2$

$\Rightarrow \quad A(3z_1-1),\ B(2z_2+3)$ and O(0) are situated as shown in the following figure.

Thus, OA and OB are perpendicular to each other.

$\Rightarrow \quad \arg\left(\frac{3z_1-1}{2z_2+3}\right)=\pm\frac{\pi}{2}$

So, $w=\frac{3z_1-1}{2z_2+3}$ is purely imaginary number.

$\Rightarrow \quad w=ki$

Hence, cube roots of w are $(ki)^{\frac{1}{3}}$

$\Rightarrow \quad w_1=k^{1/3}e^{i\pi/6},\ w_2=k^{1/3}e^{i5\pi/6}$ and $w_3=k^{1/3}e^{i3\pi/2} \quad \Rightarrow \quad \frac{w_2^2}{w_1w_3}=1$

91.(5) $z^3=\bar{z}$

$\Rightarrow \quad |z|^3=|\bar{z}|=|z| \quad \Rightarrow \quad |z|=0$ or $|z|^2=1 \quad \Rightarrow \quad z=0$ or $z^4=z\bar{z}=|z|^2=1$

Now $z^4=1$ has four roots. Thus, total number of roots is 5.

92.(4.88) $|\alpha+1+i|=1$

i.e., α lies on the circle having centre at $(-1,-i)$ and radius 1.

$\Rightarrow \quad |\alpha|_{\max}=\sqrt{2}+1$

Similarly for $|\beta-2-3i|=6$, we have

$|\beta|_{\max}=6+\sqrt{13}$

$\Rightarrow \quad 6|\alpha|_{\max}-|\beta|_{\max} = 6\sqrt{2}-\sqrt{13}=\sqrt{72}-\sqrt{13}$

93.(0.875) Let $z = r(\cos\theta + i\sin\theta)$

$$\left|2z + \frac{1}{z}\right| = 1$$

$$\Rightarrow \quad \left|\left(2r + \frac{1}{r}\right)\cos\theta + i\left(2r - \frac{1}{r}\right)\sin\theta\right| = 1$$

$$\Rightarrow \quad \left(2r + \frac{1}{r}\right)^2 \cos^2\theta + \left(2r - \frac{1}{r}\right)^2 \sin^2\theta = 1$$

$$\Rightarrow \quad \left(2r + \frac{1}{r}\right)^2 \left(2r + \frac{1}{r}\right)^2 \sin^2\theta + \left(2r - \frac{1}{r}\right)^2 \sin^2\theta = 1 \quad \Rightarrow 4r^2 + \frac{1}{r^2} + 4 - 8\sin^2\theta = 1$$

$$\Rightarrow \quad 4r^2 + \frac{1}{r^2} + 3 = 8\sin^2\theta$$

Now, $\dfrac{4r^2 + \dfrac{1}{r^2}}{2} \geq 2$ (using A.M. $\geq$ G.M)

$$\Rightarrow \quad 4r^2 + \frac{1}{r^2} \geq 4 \quad \Rightarrow \quad 8\sin^2\theta \geq 7$$

94.(18.75) Let PQ be the diameter perpendicular to AB.

Let P and Q be represented by z and $-z$.

$$\therefore \quad z_1 = ze^{i\alpha}, ze^{-i\alpha} \quad \text{and} \quad z_3 = (-z)e^{i\beta}, z_4 = (-z)e^{i\beta}$$

$$\therefore \quad z_1 z_2 = z_3 z_4 \quad \Rightarrow \quad k = 1 \quad \Rightarrow \quad \frac{75k}{4} = \frac{75}{4} = 18.75$$

95.(5) Let z be such a complex number, then according to the question

$$z^3 = \bar{z} \qquad \ldots(i)$$

$$\Rightarrow \quad |z| = 0 \text{ or } |z| = 1 \quad \Rightarrow \quad z = 0 \text{ or } z = \cos\theta + i\sin\theta$$

Putting $z = \cos\theta + i\sin\theta$ in (i), we get

$$\cos 3\theta = \cos\theta \text{ and } \sin 3\theta = -\sin\theta$$

$$\Rightarrow \quad \sin\theta = 0 \text{ or } \sin\theta = 1 \text{ or } \sin\theta = -1$$

$$\Rightarrow \quad \theta = 0 \text{ or } \pi, \text{ or } \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$$

$$\Rightarrow \quad z = 1, -1, i, -i \text{ or } z = 0$$

96.(6.25) Let $z_2 = 4 \quad z_1 = -3i$

$$z_3 = \frac{1}{1-i} = \frac{1+i}{2}$$

PERMUTATION & COMBINATION

1.(D) No. of ways = $\left(3^4 - {}^3C_1 2^4 + {}^3C_2 1^4\right)\times {}^4C_2 = 36\times 6 = 216$

2.(D) No. of ways = $3^n + {}^nC_2.3^{n-2} + {}^nC_4 3^{n-4} + ...$

$$= \frac{1}{2}\left[(3+1)^n + (3-1)^n\right] = \frac{4^n + 2^n}{2} = 2^{n-1}\left(2^n + 1\right)$$

3.(B) Total number of ways = $5! - 1 = 119$

4.(C)

$$(x_1 + x_2 + x_3)(y_1 + y_2) = 77$$

case I : $x_1 + x_2 + x_3 = 7$ and $y_1 + y_2 = 11$ case II: $x_1 + x_2 + x_3 = 11$ and $y_1 + y_2 = 7$

$= {}^6C_2 \times {}^{10}C_1 + {}^{10}C_2 \times {}^6C_1 = 150 + 270 = 420.$

5.(B) Case I : 3 alike, 1 distinct = $3\times 2\times \dfrac{\lfloor 4}{\lfloor 3} = 24$ Case II :2 alike, 2 distinct = $3\times 1\times \dfrac{\lfloor 4}{\lfloor 2} = 36$

Case III: 2 alike, 2 alike = ${}^3C_2 \times \dfrac{\lfloor 4}{\lfloor 2 \lfloor 2} = 18$

$m = 24 + 36 + 18 = 78$ and $n = 36$

$$\frac{m}{n} = \frac{78}{36} = \frac{13}{6}$$

6.(B) Case I : 2 alike, 1 distinct = $8\times 7 = 56$ Case II :3 alike = 8

Total no. of ways = 56 + 8 = 64

7.(A) Sum of unit place (S_1) = $27\times 45 = 1215$ Sum of 10th place (S_2) = $1+4+9 \ \times 81 = 1134$

Sum of 100th place (S_3) = $27\times 45 = 1215$ Total sum = $1215 + 1134\times 10 + 1215\times 100 = 134055$

8.(D) No. of ways = $3^6 - {}^3C_1(2)^6 + {}^3C_2(1)^6 = 540.$

9.(C) $L = \dfrac{\lfloor p+q}{\lfloor p \lfloor q}, M = \dfrac{\lfloor p+q}{\lfloor p \lfloor q}\times \lfloor 2, N = \dfrac{\lfloor p+q}{\lfloor p \lfloor q}$

$L = N = \dfrac{M}{2} \Rightarrow 2L = 2N = M$

10.(B) Case I : $\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 5 & \ & \ & \ \\ \hline\end{array} \rightarrow {}^5C_1 \times \lfloor 3 = 30$

Case II: $\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & 5 & \ & \ \\ \hline\end{array} \rightarrow {}^6C_1 \times {}^5C_4 \times \lfloor 3 = 180$

Case III : $\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & \ & 5 & \ \\ \hline\end{array} \rightarrow {}^7C_1 \times {}^6C_4 \times \lfloor 3 = 630$

Case IV : $\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & \ & \ & 5 \\ \hline\end{array} \rightarrow {}^8C_1 \times {}^7C_4 \times \lfloor 3 = 1680$

Total such numbers = 30 + 180 + 630 + 1680 = 2520

11.(B) Number of functions = $\dfrac{\text{total no. of functions}}{2} = \dfrac{2^{12}}{2} = 2^{11}$

12.(C) Total no. of non- negative integral solution of $x + y + z \leq n$

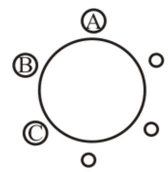
= total no. of non-negative integral solution of $x + y + z + w = n = {}^{n+4-1}C_{4-1} = {}^{n+3}C_3$

13.(A) Case I - 3 men and 2 women : No. of ways = ${}^4C_3 \times {}^6C_2 = 60$

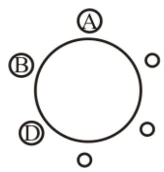
Case II – 4 men and 1 women : no. of ways = ${}^4C_4 \times {}^6C_1 = 6$

Total no. of ways = 60 + 6 = 66.

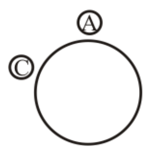
14.(D)



$\longrightarrow \lfloor 3 = 6$



$\longrightarrow \lfloor 3 = 6$



$\longrightarrow {}^3C_1 \times \lfloor 2 = 6$

$= 18$

15.(D) $xyz = 2^3 \times 3^1$

Number of non-negative integral solution $= {}^{3+3-1}C_{3-1} \times {}^{1+3-1}C_{3-1} = 30$

$\therefore$ Number of negative integral solutions = ${}^3C_2 \times 30 = 90$

Hence, total number of integral solutions = 120.

16.(B) Number of five-digit numbers = $\lfloor 3 \; 1+2+3+4 = 60$

17.(C) $\_T\_T\_T\_T\_T\_T\_T\_T\_$

No. of ways = ${}^9C_6 \times \dfrac{\lfloor 6}{\lfloor 5} = 504$

18.(B)

Case	P_1	P_2	P_3	No. of ways
1.	1	2	4	$\dfrac{\lfloor 7}{\lfloor 1 \lfloor 2 \lfloor 4} \times \lfloor 3 = 630$

19.(D)

	P_1	P_2	P_3	No. of ways
	1	3	4	$\dfrac{\lfloor 8}{\lfloor 1\lfloor 3\lfloor 4} \times \lfloor 3 = 1680$
	2	3	3	$\dfrac{\lfloor 8}{\lfloor 2\lfloor 3\lfloor 3} \times \dfrac{\lfloor 3}{\lfloor 2} = 1680$
	2	2	4	$\dfrac{\lfloor 8}{\lfloor 2\lfloor 2\lfloor 4} \times \dfrac{\lfloor 3}{\lfloor 2} = 1260$

$= 4620$

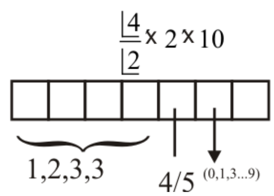
$K.{}^7P_3 = 4620$

$K = \dfrac{4620}{210} = 22$

20.(D) No of ways = ${}^{10}C_1.{}^8C_1.{}^6C_1.{}^4C_1 \times \dfrac{1}{\lfloor 4} = 80$

21.(A) No. of ways = ${}^4P_2 \times {}^6P_3 \times \lfloor 5 = 172800$

22.(B) No. of ways = $12 \times 2 \times 10 = 240$



23.(C) $\underline{S_1}\ \_\ \underline{S_2}\,\underline{G}\ \_\ \_\ \_ \rightarrow$ No. of ways $= 2 \times 2 \times \lfloor 4 = 96$ (both grandsons on same side of grandfather)

$\_\ \underline{S_1}\ \_\ \underline{G}\ \_\ \underline{S_2}\ \_ \rightarrow$ No.of ways $= 2 \times 3 \times 3 \times \lfloor 4 = 432$ (opposite side)

Total ways = 96 + 432 = 528

24.(C)

C_1	C_2	C_3	C_4	No. of ways
2	2	2	3	$\dfrac{\lfloor 9}{(\lfloor 2)^3 \lfloor 3}$

25.(A) Let D_1, D_2 D_3 be the districts that require 2, 3, 5 men respectively. Number of ways $= \dfrac{\lfloor 10}{\lfloor 3\lfloor 2\lfloor 5}$.

26.(B) Digit in units place in $6^n = 6$

Digit in units place in $9^m = \begin{cases} 1 & m = EVEN \\ 9 & m = ODD \end{cases}$

$\therefore$ Digit in units place in $6^n + 9^m = \begin{cases} 7 & m = EVEN \\ 5 & m = ODD \end{cases}$

Hence, for *m* being an ODD integer, the required expression $(6^n + 9^m)$ will be a multiple of 5.

$\Rightarrow$ The number of ordered pairs $= 50 \times 25 = 1250$

27.(C)

28.(ABD) **(A)** $^{n}C_r$

(B) $\dfrac{\lfloor n}{\lfloor r \lfloor n-r} = {}^{n}C_r\left(\underbrace{0,0,0,...0}_{n-r},\underbrace{1,1,...1}_{r}\right)$

(C) $^{n+r}C_r$

(D) $^{n-1}C_{r-1} + {}^{n-1}C_r = {}^{n}C_r$

29.(BC) **(A)** $E_2\ \lfloor 125 = \left[\dfrac{125}{2}\right]+\left[\dfrac{125}{2^2}\right]+\left[\dfrac{125}{2^3}\right]+....+\left[\dfrac{125}{2^7}\right] = 119$

$E_5\ \lfloor 125 = \left[\dfrac{125}{5}\right]+\left[\dfrac{125}{5^2}\right]+\left[\dfrac{125}{5^3}\right] = 31$

(B) $10^{10} - 1$

(C) $\boxed{4}\boxed{}\boxed{}\boxed{}\boxed{}$ Case I : 2 alike, 2 different $\rightarrow 1 \times {}^{3}C_1 \times \dfrac{\lfloor 4}{\lfloor 2}$

Case II : 4 distinct $\rightarrow \lfloor 4$

$\boxed{5}\boxed{}\boxed{}\boxed{}\boxed{}$ Case I : 2 alike, 2 alike - $\dfrac{\lfloor 4}{\lfloor 2 \lfloor 2}$

Case II : 2 alike, 2 distinct - $2 \times \dfrac{\lfloor 4}{\lfloor 2}$

Total numbers = 60 + 30 = 90

(D) $^{n}C_2 = 5050$

30.(BC) **(A)** $2^{10} - 2$; **(B)** $2^{10} - 1$; **(C)** $2^{10} - 1$; **(D)** 10

31.(BC) $2^n .1.3.5.7....$

$$^{2n}P_n = \frac{\lfloor 2n}{\lfloor n} = \frac{1.2.3.4.5.6....2n}{\lfloor n}$$

$$= \frac{2^n .1.2.3....n.1.3.5....}{\lfloor n}$$

$$= 2^n . \frac{\lfloor n .1.3.5....}{\lfloor n} = 2^n .1.3.5.....$$

32.(BCD) Number of ways = $3^5 - {}^{3}C_1 2^5 + {}^{3}C_2$ =150

(A) No. of ways = $^{5}P_3 = 60$

(B) No. of parallelograms = $^{6}C_2 \times {}^{5}C_2 = 150$

(C) No. of ways = $3^5 - {}^{3}C_1 2^5 + {}^{3}C_2 = 150$

(D) No. of ways = $3^5 - {}^{3}C_1 2^5 + {}^{3}C_2 = 150$

33.(ABCD) No. of permutation = $\dfrac{\lfloor 2n}{\lfloor n \lfloor n} = {}^{2n}C_n$ (where n a's and n b's)

34.(CD) Case I: n = 2m

If common diff = 1 ; no. of triplets = $2m-2$

common diff = 2 ; no. of triplets = $2m-4$

common diff = 3 ; no. of triplets = $2m-6$

.

.

.

common diff = m – 1 ; no. of triplets = 2

Total number of triplets = 2 + 4 + 6 + ….+ (2m – 2) = m(m – 1) = $\frac{n(n-2)}{4}$

Case II : n = 2m + 1 ;

If common difference = 1; no. of triplets = 2m – 1

common difference = 2; no. of triplets = 2m – 3

.

.

.

common diff = m ; no. of triplets = 1

Total number of triplets = 1 + 3 + 5 + ….+ (2m – 1) = $\frac{m}{2}\left[1+(2m-1)\right]$ = m^2 = $\left(\frac{n-1}{2}\right)^2$

35.(BD) (A) Number of ways = $\frac{\lfloor n-1}{\lfloor p}$ **(B)** Number of ways = ${}^{n-1}C_p$

(C) Number of ways = ${}^{n-1}C_{p-1}$ **(D)** Number of ways = ${}^{n-1}C_p$

36.(BD) Number of selections when x < y < z = ${}^{n}C_3$

Number of selections when x = y < z = ${}^{n}C_2$ (we have to select only two numbers out of n numbers)

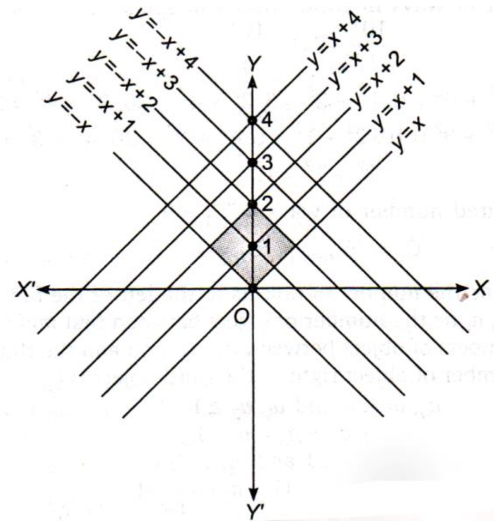
$\therefore$ Required number = ${}^{n}C_3 + {}^{n}C_2 = {}^{n+1}C_3$

37.(ABCD) **(A)** $3\times 7!$ **(B)** Total no. of arrangements of 'r' white and 15 – r black balls = $\frac{15!}{r!(15-r)!}$

(C) Total no. of combinations = $5\times 6\times 2^3 = 240$

(D) Total no. of selections = 35

38.(BD) Hence, number of squares whose diagonals length is 2 unit



$= 3 \times 3 = 9$

39.(ACD) Number of ways = $\frac{\lfloor 6}{\lfloor 2 \lfloor 2 \lfloor 2}$

(A) No. of ways = ${}^{2+3-1}C_{3-1} \times {}^{4+3-1}C_{3-1} = {}^{4}C_{2} \times {}^{6}C_{2} = \frac{\lfloor 4}{\lfloor 2 \lfloor 2} \times \frac{\lfloor 6}{\lfloor 4 \lfloor 2} = \frac{\lfloor 6}{\lfloor 2 \lfloor 2 \lfloor 2}$

(B) No. of ways = $\frac{\lfloor 6}{\lfloor 2 \lfloor 2 \lfloor 2} \times \frac{1}{\lfloor 3}$ **(C)** Coeff of $x^2\, y^2\, z^2$ = $\frac{\lfloor 6}{\lfloor 2 \lfloor 2 \lfloor 2}$ **(D)** No. of ways = $\frac{\lfloor 6}{\lfloor 2 \lfloor 2 \lfloor 2}$

40.(AC) p = 2520 = $2^3 \times 3^2 \times 5^1 \times 7^1$

x = $3 \times 2 \times 2 \times 2 = 24$

y = $4 \times 1 \times 2 \times 2 = 16$

41.(BD) Number of ways = $2^8 - 1$

42.(CD) Number of ways = $2^4 - 1$

43.(ABC) i.e., $0^2 + 0^2 = 0, 1^2 + 2^2 = 5, 2^2 + 4^2 = 20, 3^2 + 4^2 = 25$

$1^2 + 3^2 = 10, 2^2 + 1^2 = 5, 4^2 + 2^2 = 20, 4^2 + 3^2 = 25$,

$3^2 + 1^2 = 10$

$\therefore$ Required number of ways = ${}^{9}C_{1} = 9$ Also, ${}^{9}C_{8} = {}^{9}C_{9-8} = {}^{9}C_{1} = 9$

44.(ABC) Let a_0 be the number of objects to the left of the first object chosen, a_1 be the number of object between first and second, the numbers of object between the second and the third is a_2 and number of object right to the third object is a_3.

$\therefore$ $a_0, a_3 \geq 0$ and $a_1, a_2 \geq 1$

Also, $a_0 + a_1 + a_3 = n - 3$ ……(i)

$\because$ $1 + a_0, 1 + a_3 \geq 1$ and $a_1, a_2 \geq 1$ $\Rightarrow$ $1 + a_0 \;\; + a_1 + a_2 + \;\; 1 + a_3 \;\; = n - 1$

$\therefore$ Required number of ways = ${}^{n-1-1}C_{4-1} = {}^{n-2}C_{3} = \frac{n-2 \;\; n-3 \;\; n-4}{6}$

Also, ${}^{n-2}C_{3} = {}^{n-3}C_{3} + {}^{n-3}C_{2}$

45.(AB) Each selection of 4 points, two on one line and two on the other will give one point of intersection.

$\therefore$ Required number of intersection = ${}^{m}C_2 . {}^{n}C_2 = \frac{m\ m-1}{2}\frac{n\ n-1}{2} = \frac{mn\ m-1\ \ n-1}{4}$

46.(AD) Number of selections of 7 digits out of the digit 1, 2, 3, …., 9 = ${}^{9}C_7$

Number of digits out of these 7 selected digits excluding the greatest digit = 6

These 6 digits can be divided in two groups each having 3 digits = $\frac{6!}{3!3!2!} = {}^{6}C_3 \times \frac{1}{2!}$

But the 3 digits on one side can go on the other side

$\therefore$ Required number of ways = ${}^{9}C_7 . {}^{6}C_3 . \frac{1}{2!} . 2! = {}^{9}C_7 . {}^{6}C_3 = {}^{9}C_2 . {}^{6}C_3$

47.(ABCD) (A) ${}^{6}C_3\ {}^{4}C_2\ 5!\ 5! = (5!)^3$ (B) ${}^{6}C_1\ 9!$ (C) $(6+1)!\ 4!$ (D) ${}^{10}P_4$

48.(ABCD) We know that product of r consecutive integer is divisible by $r!$ we have $P = (n-10)(n-9)(n-8)...(n+10)$ is product of 21 consecutive integers.

Which is divisible by 21!

$[(n-10)(n-9)(n-8)...n][(n+1)(n+2)(n+3)...(n+10)]$

Which is divisible by 11! 10!

$[(n-10)...(n-6)]\times[(n-5)...(n-1)]\times[n(n+1)...(n+4)]\times[(n+5)...(n+9)]\times(n+10)$

Which is divisible by $(5!)^4$

Also, it can be shown that P is divisible by 2! 3! 4! 5! 6!

49.(ABD) The cases for $a_1 \{H, T\}$ i.e., $a_1 = 2$

The case for $a_2 : \{HT, TH, TT\}, a_2 = 3$

For $n \geq 3$, if the first outcome is *H*, then next just *T* and then a_{n-2}. If the first outcome is *T*, then a_{n-1} should follow.

So, $a_n = a_{n-1} + a_{n-2}$

So, $a_3 = a_1 + a_2 = 5,\ a_4 = 3 + 5 = 8$ and so on.

50.(ABCD) Number of five-digit numbers starting with $1 = {}^{8}C_4 = 70$

Number of five-digit numbers starting with $2 = {}^{7}C_4 = 35$

Total numbers = $70 + 35 = 105$

Thus, 105th five-digit number is 26789

51.(ABC) Consider square of 2×2, in which we have 2 pairs of squares which have common vertex.

We have such 7×7 squares of size 2×2.

So, number of ways of selecting two squares having one vertex common is $7 \times 7 \times 2 = 98$.

In each row or column, we have 7 pairs of squares having one side common.

So, number of ways of selecting two squares having one side common is $7 \times 8 \times 2 = 112$.

Therefore, number of ways of selecting two squares such that they neither have a common vertex nor have a common side is ${}^{64}C_2 - 98 - 112 = 1806$

52.(AB) Number of visits teacher makes = Number of ways of selecting 5 students from $25 = {}^{25}C_5$

Number visits a particular kid makes = Number of ways of selecting 4 students accompanying him/her $= {}^{24}C_4$

So, difference between the number of trips made by teacher and kid $= {}^{25}C_5 - {}^{24}C_4 = {}^{24}C_5$

53.(ABC) We can take the objects as:
0 identical and n distinct;
1 identical and $n-1$ distinct;
2 identical and $n-2$ distinct;
And so on…

$\therefore$ Total number of ways = $\sum_{r=0}^{n} {}^{n}C_r = 2^n$

Thus, (A) is true

Number of possible subsets of a set containing n elements is 2^n.

Also, ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n = 2^{2n}$

Thus, (B) and (C) are also true.

54.(AC) The required number of selections of 4 letters

= coeff. of x^4 in the expansion of $(x^0+x^1)(x^0+x^1)(x^0+x^1+x^2)(x^0+x^1+x^2)(x^0+x^1)$

= coeff. of x^4 in $(1+x)^3(1+x+x^2)^2 = 18$

55.(ABCD) In 1st round all the integers, which leaves the remainder 1 when divided by 15, will be marked. Last number of this category is 991. Next number to be marked is $(991+15-1000) = 6$.

So, second round of integers which leave the remainder 6 when divided by 15 will be marked.
Last number of this category is 996
Next number to be marked is $(996+15-1000) = 11$

Thus, third round of integers which leave the remainder 11 when divided by 15 will be marked.
Last number of this category is 986.
Next number to be marked is $(986+15-100) = 1$; which has already been marked.

56.(BC) Required number of teams $= \dfrac{{}^{10}C_4 . {}^{6}C_3 . {}^{3}C_3}{2!} = \dfrac{10!}{4! \times 6!} \times \dfrac{6!}{3! \times 3! \times 3!} \times 3 = {}^{10}C_4 \times {}^{5}C_2 = 2100$

57.(ABC) If the sides are a, a, b then the triangle will be formed only when $2a > b$. So, for any $a \in N, b$ can change from 1 to $2a-1$.
When $a \le 1004$ then

Number of triangles = $1+3+5+...+(2(1004)-1) = (1004)^2$

And when $1005 \le a \le 2008, b$ can take any value from 1 to 2008.

Since a has 1004 possibilities, Number of triangles = $1004 \times 2008 = 2(1004)^2$

$\therefore$ Total number of isosceles triangles = $3(1004)^2$

58.(BD) Since, S_1 and S_2 get 2 books each and S_3 and S_4 get 3 books each.

$\therefore$ Required No. of ways = $\dfrac{10!}{2!\,2!\,3!\,3!}$

59.(ABC) $\because$ $3^p = (4-1)^p = 4\lambda_1 + (-1)^p$

$5^q = (4+1)^q = 4\lambda_2 + 1$ and $7^r = (8-1)^r = 8\lambda_3 + (-1)^r$

Hence, both *p* and *r* must be odd, or both must be even. Thus, $p+r$ is always even. Also, $p+q+r$ can be odd or even.

60.(AD)

61.(BC) Let $x = \vec{r}.\hat{i}, y = \vec{r}.\hat{j}$ and $z = \vec{r}.\hat{k}$

where *x*, *y* and *z* are the components of $\vec{r}$ along *x*, *y* and z-axis resp. and they are given to be positive.

$\therefore$ $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Also $\vec{r}.\vec{a} \le 12$

$\Rightarrow$ $x+y+z \le 12$

$\therefore$ (the number of values of $\vec{r}(I)$) = (the number of positive integral solutions of ($x+y+z \le 12$))

$$\Rightarrow \quad I = \sum_{n=3}^{12} {}^{n-1}C_2 = {}^2C_2 + {}^3C_2 + ... + {}^{11}C_2 \quad \Rightarrow \quad I = {}^3C_0 + {}^3C_1 + {}^4C_2 + ... + {}^{11}C_9 \ \{\because {}^nC_r = {}^nC_{n-r}\}$$

$$\therefore \quad I = {}^4C_1 + {}^4C_2 + ... + {}^{11}C_9 = ... = {}^{12}C_9$$

62.(ABC) **63.(AD)** **64.(ABD)**

65. **[A-r]** **[B-s]** **[C-p]** **[D-q]**

(A) $5^7 = 78{,}125$ **(B)** ${}^5C_3 \times \left[3^7 - {}^3C_1 2^7 + {}^3C_2\right] = 10 \times [2187 - 384 + 3] = 18060$

(C) $5^7 - {}^5C_1 = 78{,}120$ **(D)** $5^7 - {}^5C_1 - {}^5C_2\left[2^7 - {}^2C_1\right] = 76{,}860$

66. **[A-r]** **[B-s]** **[C-p]** **[D-q]**

(A) $\lfloor 7 \left(1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + --- \frac{1}{\lfloor 7}\right) = 1854$

(B) ${}^7C_3 \times D_4 = 315$

(C) $\lfloor 7 - D_7 - {}^7C_1.D_6 = \lfloor 7 - 1854 - 265 \times 7 = 1331$

(D) $\lfloor 7 - D_7 - {}^7C_1 D_6 - {}^7C_2 D_5 = 407.$

67. **[A-q]** **[B-s]** **[C-p]** **[D-r]**

(A) $(6)^7 - {}^6C_1(5)^7 + {}^6C_2(4)^7 - {}^6C_3(3)^7 + {}^6C_4(2)^7 - {}^6C_5(1)^7$

$= 279936 - 468750 + 245760 - 43740 + 1920 - 6 = 15,\ 120$

(B) ${}^6C_3.\left[(3)^7 - {}^3C_1(2)^7 + {}^3C_2\right] = 20 \times [2187 - 384 + 3] = 36,\ 120$

(C) $6^7 - {}^6C_1 = 279930$

(D) ${}^6C_1 \times 5^7 - {}^6C_2(4)^7 + {}^6C_3(3)^7 - {}^6C_4(2)^7 + {}^6C_5(1)^7 = 264816$

68. **[A-s]** **[B-r]** **[C-p]** **[D-q]**

(A) $\lfloor 12$ **(B)** $\lfloor 11$ **(C)** $3 \times \lfloor 11$ **(D)** $6 \times \lfloor 11$

69. **[A-p, r, s]** **[B-p, r, s]** **[C-q, t]** **[D-q, t]**

Answers to the column-II as **p.** 32, **q.** 33, **r.**32, **s.** 32, **t.** 33

(A) Total number of ways of arrangement = 35!. So exponent of 2 in 35! = 32

(B) Total number of ways of arrangement = 9 × 35!. So exponent of 2 in 9 × 35! = 32

(C) Total number of ways of arrangement = 18 × 35!. So exponent of 2 in 18 × 35! = 33

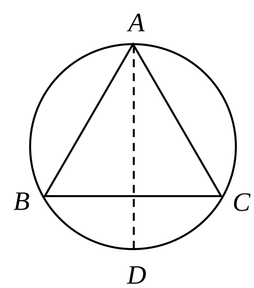
(D) Total number of ways of arrangement = 6 × 35!. So exponent of 2 in 6 × 35! = 33

70.(1) $x+2y=10,$ where x is the number of times he takes single steps, and y is the number of times he takes two steps

Cases	Total number of ways
1. $x=0,\ y=5$	$5!/5!=1$
2. $x=2,\ y=4$	$6!/2!4!=15$
3. $x=4,\ y=3$	$7!/4!3!=35$
4. $x=6,\ y=2$	$8!/2!6!=28$
5. $x=8,\ y=1$	$9!/8!=9$
6. $x=10,\ y=0$	10!/10! = 1

$\therefore$ $p=89$

71.(3) For every acute angle $\Delta ABC,$ we will



Get 3 obtuse angled triangles by shifting one of the vertices to the diametrically opposite point on the circum circle of the polygon.

72.(7) ${}^{n}C_2-n=14 \quad \Rightarrow \quad$ n = 7

73.(7) $f\ 1 < f\ 2 < f\ 3 < f\ 4 < f\ 5 < f\ 6$

Total no. of strictly increasing functions = ${}^{9}C_6=\dfrac{9\times 8\times 7}{6}=84$

74.(9) $f\ 1 \le f\ 2 \le f\ 3 \le f\ 4 \le f\ 5$

Total no. of non-decreasing functions = ${}^{13}C_5=1287$

$$\frac{m}{143}=9$$

75.(5) Number of ways = $3^5\times {}^{4+3-1}C_{3-1}=3^5\times {}^{6}C_2$

p = 5, q = 15; q – 2p = 15 – 10 = 5

76.(9) ____AB____

$m=\dfrac{7!\times 2!}{2}=5040$ Sum of digits = 9

77.(2) **Case I**: First row contains 1x, second row contains 4x and third row contain 1x

No. of ways = $2\times1\times2=4$

Case II: First row contains 2x, second row contains 3x, and third row contain 1x

No. of ways = $1\times{}^{4}C_{3}\times2=8$

Case III: First row contains 1x, second row contains 3x, and third row contain 2x

No. of ways = $2\times{}^{4}C_{3}\times1=8$

Case IV : First row contains 2x, second row contains 2x, and third row contain 2x

No. of ways = $1\times{}^{4}C_{2}\times1=6$

Total number of ways = 4 + 8 + 8 + 6 = 26

78.(6) Let $x_1, x_2, x_3, \ldots, x_7$ persons are entering into Gate A, B, C, D,…..G gates respectively

Now, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 200$

Total no. of ways to enter = ${}^{200+7-1}C_{7-1} = {}^{206}C_{6} = {}^{206}C_{200}$

Now the arrangement of these 200 persons will also count so

Total no. of ways = ${}^{206}C_{200}\times\underline{|200} = {}^{206}P_{200}$

n = 206, r = 200

n-r = 6 No. of different list = ${}^{206}P_{6}$

79.(9) $U_4 = \underline{|4}\left(1-\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}\right)=9$

80.(47) They have note of same denomination hence that denomination must be HCF of 6, 7 and 8 which is 1, so each of them have notes of $1.

Now let us assume that the number of notes that they have is x_1, x_2, and x_3. So, the numbers of ways in which they can denote Rs. 10 is the same as the number of solutions to the question

$x_1+x_2+x_3=10$

Subject to conditions

$0\le x_1\le 6,\ 0\le x_2\le 7,\ 0\le x_3\le 8$

Hence the required number of ways = coefficient of x^{10} in

$\left(1+x+x^2+\ldots+x^6\right)\left(1+x+x^2+\ldots x^7\right)\left(1+x+x^2+\ldots+x^8\right)$

= coefficient of x^{10} in $\left(1-x^7\right)\left(1-x^8\right)\left(1-x^9\right)\left(1-x\right)^{-3}$

= coefficient of x^{10} in $\left(1-x^7-x^8-x^9\right)\left(1+{}^{3}C_1x+{}^{4}C_2x^2+{}^{5}C_3x^3+\ldots{}^{12}C_{10}x^{10}\right)$

= ${}^{12}C_2-{}^{5}C_3-{}^{4}C_2-{}^{3}C_1 = 47$

81.(23) Each vertex can be coloured in 2 ways, so there are $2^8 = 256$ ways (when vertices are fixed under the action of the trivial symmetry) now consider symmetry under different rotations. There are $2^4 = 4$ (out of 256) that are fixed under a 90-degree rotation about the axis joining the mid-points of two opposite faces. There are $2^4 = 16$ (out of 256) that are fixed under a 120° rotation about the axis joining two diametrically opposite corners. (In this case rotation splits up the 8 vertices as $1+3+3+1=8$). There are $2^4 = 16$ (out of 256) that are fixed under a 180° rotation about the axis joining the mid-points of two opposite faces. (In this case rotation splits up the 8 vertices as $2+2+2+2=8$). There are $2^4 = 16$ (out of 256) that are fixed under a 180° rotation about the axis joining the mid-points of two opposite edges. (In this case rotation splits up the 8 vertices as $2+2+2+2=8$).

Hence the number of orbits is $\frac{1}{24}(1\times 256+6\times 4+8\times 16+3\times 16+6\times 16)=23$

82.(14)

83.(4) Ramesh has to 1st select in fruits out of 2*n* fruits, for this following case may arise.

Case (i) Number of distinct fruits 0 and number of identical fruits *n* then number of ways is ${}^nC_0$.

Case (ii) Number of distinct fruits 1 and number of identical fruits $n-1$ then number of ways is ${}^nC_1$.

Case (iii) Number of distinct fruits 2 and number of identical fruits $n-2$ then number of ways is ${}^nC_2$. Ans so on

Last case number of distinct fruits n and number of identical fruits 0 then number of ways in ${}^nC_n$

So total number of ways is ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n \left(2^n\right) = 16 \quad \Rightarrow \quad n = 4$.

84.(10) Let a particular customer buys ice creams Vanila, Strawberry, Chocolate and Butter Scotch of *a*, *b*, *c* and *d* numbers then from the given condition $0 < a+b+c+d \le 10$. Let us assume a dummy variable k such that $a+b+c+d+k=10$ then number of integral solution of this equation is ${}^{10+5-1}C_{5-1} = {}^{14}C_4$

$${}^{10+5-1}C_{5-1} = {}^{14}C_4 = \frac{14!}{(4!)(10!)} = \frac{14\times 13\times 12\times 11}{24} = 1001.$$

But there is one case when *k*=10 then $a+b+c+d=10$, this case is not applicable case hence required answer is $1001-1=1000 \Rightarrow \frac{k}{100}=10$

85.(3.150) Let $x_1, x_2, ..., x_6$ be the number that appears on the six dies. As per the given condition $x_1+x_2+x_3+...+x_6 \le 17$. Introducing a dummy variable $x_7\left(0 \le x_7\right)$ the inequality becomes an equation $x_1+x_2+...+x_6+x_7=17$. Here $1 \le x_i \le 6$ where *i* = 1, 2*m* … , 6 and $0 \le x_7$. So that No of solution = coefficient of x^{17} in $\left(x+x^2+...+x^6\right)^6\left(1+x+x^2+x^3+...\right)$ = coefficient of x^{17} in $\left(1-x^6\right)^6(1-x)^{-7}$

= coefficient of x^{17} in $\left(1-6x^6\right)(1-x)^{-7}$ = coefficient of x^{17} in $\left(1-6x^6\right)$

$\left(1+{}^7C_1x+...+{}^{10}C_4x^4+{}^{11}C_5x^5+...{}^{17}C_{11}x^{11}+...\right) = {}^{17}C_{11} - 6.{}^{11}C_5$

Since total number of outcomes is 6^6 Required number of ways to get a sum greater than

$17 = 6^6 - \left({}^{17}C_{11} - 6\,{}^{11}C_5\right) = 46656 - 12376 - 6\times 642 = 31508 \Rightarrow 3.150$

86.(243) Since it is given that $a+b+3c \le 30$, so assume a number $d \ge 0$ such that $a+b+3c+d=30$, a, b and c are the positive integers so $a=a_1+1$, $b=b_1+1$, and $c=c_1+1$ then the given equation we can write as $(a_1+1)+(b_1+1)+3(c_1+1)+d=30$. Or $a_1+b_1+3c_1+d=25$. Since coefficient of c_1 is 3 hence it can-not takes a value more than 8 if we form cases then we will have 9 cases:

Case (i) $c_1=0$ then $a_1+b_1+d=25$. Since coefficient of c_1 is 3 hence it can-not takes a value more than 8 if we form cases then we will have 9 cases:

Case (i) $c_1=0$ then $a_1+b_1+d=25$ and number of non-negative integral solution is

${}^{25+3-1}C_{3-1} = {}^{27}C_2 = 351$.

Case (ii) $c_1=1$ then $a_1+b_1+d=22$ and number of non-negative integral solution is

${}^{22+3-1}C_{3-1} = {}^{24}C_2 = 276$.

Case (iii) $c_1=1$ then $a_1+b_1+d=19$ and number of non-negative integral solution is

${}^{19+3-1}C_{3-1} = {}^{21}C_2 = 210$.

Case (iv) $c_1=3$ then $a_1+b_1+d=16$ and number of non-negative integral solution is ${}^{16+3-1}C_{3-1} = {}^{18}C_2 = 153$.

Case (v) $c_1=4$ then $a_1+b_1+d=13$ and number of non-negative integral solution is

${}^{13+3-1}C_{3-1} = {}^{15}C_2 = 105$.

Case (vi) $c_1=5$ then $a_1+b_1+d=10$ and number of non-negative integral solution is

${}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$.

Case (vii) $c_1=9$ then $a_1+b_1+d=7$ and number of non-negative integral solution is ${}^{7+3-1}C_{3-1} = {}^{9}C_2 = 36$.

Case (viii) $c_1=7$ then $a_1+b_1+d=4$ and number of non-negative integral solutions is

${}^{4+3-1}C_{3-1} = {}^{6}C_2 = 15$.

Case (ix) $c_1=8$ then $a_1+b_1+d=1$ and number of non-negative integral solutions is ${}^{1+3-1}C_{3-1} = {}^{3}C_2 = 3$.

So total number of integral solutions is $51+276+210+153+105+66+36+15+3 = \frac{1215}{5} = 243$.

87.(6) In the system now total m and another set of m parallel lines. From m lines two lines can be selected in ${}^{m}C_2$ ways. So total number of parallelograms is $({}^{m}C_2)^2 = 225$

88.(240) Since, $x \ge 1$ then $y \ge 2$ $\qquad [\because x < y]$

If $y = n$, then x takes values form 1 to $n-1$ and z can take the values from 0 to $n-1$ (i.e., n values)

Thus, for each values of $y(2 \le y \le 9)$, x and z take $n(n-1)$ values.

Hence, the 3-digit numbers are of the form xyz

$$= \sum_{n=2}^{9} n(n-1) = \sum_{n=1}^{9} n(n-1) \quad [\because \text{at } n=1, (n-1)=0]$$

$$= \sum_{n=1}^{9} n^2 - \sum_{n=1}^{9} n = \frac{9(9+1)(18+1)}{6} - \frac{9(9+1)}{2} = 240 = \lambda$$

89.(15.68) The first rook can be placed in 64 ways, and the second rook cannot be placed in the same row or the same column. So, it has 7 rows and 7 columns left for it. It can be placed in $7\times 7=49$ ways.

But the order in which the rooks are placed is not important. So, it will be divided by 2!

Total number of ways = $64\times 49/2=1568$. $\Rightarrow$ 15.68

90.(10) Total number of triangles formed is ${}^kC_3$, now consider the following cases

Case (i): Number of triangles having three sides common with the sides of polygon is 0.

Case (ii): Number of triangles having two sides common with the sides of polygon is k.

Case (iii): Number of triangles having one sides common with the sides of polygon = number of ways of selecting 3 points (vertices) out of which only two are consecutive is $K\left({}^{K-4}C_1\right)=K\left(K-4\right)$. So, number of required triangles is ${}^kC_3-k-k\left(k-4\right)=k\left(k-1\right)\left(k-2\right)/6-k-k\left(k-4\right)=\dfrac{k\left(k-4\right)\left(k-5\right)}{6}=50$

91.(729) In Physics 1st and 2nd prize can be given in $10\times 9=90$ ways and similarly prizes in other 2 subjects can be given in 90 ways, Hence total number of ways is $90\times 90\times 90=729000$. $\Rightarrow$ 729.

92.(315) Lets consider 5 points A, B, C, I and E. Consider a point A, number of straight lines that can be drawn through B, C, D and E is $\left({}^4C_2\right)=6$ straight lines, so from A we can draw perpendicular to these 6 straight lines,

similarly we will get 6 perpendiculars from other points as well so total number of perpendiculars is $6\times 5=30$.

The maximum number of intersections of these 30 straight lines is $\left({}^{30}C_2\right)=435$, but these 435 points are not distinct means out of 30 straight lines not all of them are non-concurrent. We have to consider the following cases.

Case (i) Consider one of the ten lines drawn from original 5 points, say this line AB, there are three perpendiculars are drawn on AB from points C, D and E, these 3 perpendiculars are parallel to each other hence they will not intersect each other these three point if not parallel could have intersected at 3 points hence we lost $3\times 10=30$ points.

Case (ii) Since three altitudes of a triangle intersect each other at a point (Called Ortho centre). From the original 5 points we will have $\left({}^{10}C_3\right)=10$ triangles. Consider one of these 10 triangles, perpendicular from vertex to opposite side (altitude) will intersect at one point instead of 3 points, hence from each triangle we lost 2 points so total number of points that we have to subtract is $2\times 10=20$ points.

Case (iii) Consider one of the original 5 points (say A) since from A we have drawn 6 perpendiculars and these 6 perpendiculars are concurrent hence they intersect each other at one point instead of $\left({}^6C_2\right)=15$ points so from one of this point we lost $15-1=14$ points, hence total we lost $14\times 5=70$ points.

So total number of points is $435-30-20-70=315$ points.

93.(2) Let's start with the minimum number of non-congruent rectangles and for that purpose with minimum area. If area is 1 sq. unit the we will get only 1 rectangle of 1×1
If area is 2 sq. unit then we will get only 1 rectangle of 1×2
If area is 3 sq. unit then we will get only 1 rectangle of 1×3
If area is 4 sq. unit then we will get only 2 rectangle of 1×4 or 2×2
If area is 5 sq. unit then we will get only 1 rectangle of 1×5
If area is 6 sq. unit then we will get 2 rectangle of 1×6 or 2×3
If area is 7 sq. unit then we will get only 1 rectangle of 1×7 but it is not possible in 6×6 square
Last rectangle will be of area 8 square unit
Here we have considered all the non-congruent rectangles and summation of these areas is
$1+2+3+4+4+5+6+6+8=39>36$ hence, we must have at least 2 congruent rectangles = 2

94.(7) Since it is given that $A\cap B=\{2, 3, 5, 7\}$ so remaining numbers $\{1, 4, 6, 8, 9, 10\}$ can be a member of either set A or set B or neither in set A nor in set B, so each member of set {1, 4, 6, 8, 9, 10} has 3 options so total number of ways/pairs is 3^6 but since $A\neq B$ hence required number of ways is 3^6-1.

95.(7) ${}^{N-2}C_3=10$

96.(2500) Since unit digit of 7^k ends in 7, 9, 3 or 1 (corresponding to $k=4x+1, 4x+2, 4x+3$ *and* $4x$ respectively.)
Thus, 7^p+7^q cannot end in 5 for any values of p, q. So, for 7^p+7^q to be divisible by 5, it should end in 0.
For 7^p+7^q to end in 0, the forms of p and q should be as follows:

	P	Q
1	$4x$ (Unit digit) 1	$4y+2$ (Unit digit 9)
2	$4x+1$ (Unit digit 4)	$4y+3$ (Unit digit 3)
3	$4x+2$ (Unit digit 9)	$4y$ (Unit digit 1)
4	$4x+3$ (Unit digit 3)	$4y+1$ (Unit digit 7)

Thus, for a given value of m there are just 25 values of n for which 7^p+7^q ends in 0.
[For instance, if $p = 4x$, then $q = 2, 6, 10, \ldots, 98$]
Thus there are $100\times25=2500$ ordered pairs (m, n) for which 7^p+7^q is divisible by 5.

97.(10) Let number of girls is 'g' then number of group having 4 boys and 1 girl = $\left({}^4C_4\right)\times\left({}^gC_1\right)=g$ and number 4 of groups having 3 boys and 2 girls = $\left({}^4C_3\right)\times\left({}^gC_2\right)=2g\left(g-1\right)$ thus, total number of tests is $g+2g\left(g-1\right)=66$
or $2g^2-g-66=0$ only integral value of g is 6
Hence total number of students is $4+6=10$

98.(4) Obviously N = 10!, therefore $\dfrac{N}{9!}=10$.

99.(5)

BINOMIAL THEOREM

1.(D) $(1+n)^{m+1}-n^{m+1}=1+{}^{m+1}C_1 n+{}^{m+1}C_2 n^2+{}^{m+1}C_3 n^3+....+{}^{m+1}C_m n^m$

Put $n=1,\ 2,\ 3,......n$ and add

$(1+n)^{m+1}-1^{m+1}=n+{}^{m+1}C_1S_1+{}^{m+1}C_2S_2+{}^{m+1}C_3S_3+....+{}^{m+1}C_mS_m$

$\Rightarrow \quad {}^{m+1}C_1S_1+{}^{m+1}C_2S_2+{}^{m+1}C_3S_3+....+{}^{m+1}C_mS_m=(n+1)^{m+1}-(n+1)$

2.(B) $\displaystyle\sum_{r=1}^{n}\frac{r(r+1)}{2r}\ {}^nC_r^2=\frac{1}{2}\sum_{r=1}^{n}(r\,{}^nC_r^2+{}^nC_r^2)=\frac{1}{2}\sum_{r=1}^{n}(n\ {}^{n-1}C_{r-1}{}^nC_{n-r}+{}^nC_r^2)=\frac{1}{2}(n\ {}^{2n-1}C_{n-1}+{}^{2n}C_n-1)$

3.(A) $\displaystyle S=\sum_{r=0}^{n}\frac{(-1)^r\,{}^nC_r}{(4r+2)}=\int_0^1 x(1-x^4)^n\,dx=\frac{1}{2}\int_0^1(1-x^2)^n dx$

$$I_n=\int_0^1(1-x^2)^n\,dx=x(1-x^2)^n\Big|_0^1+2n\int_0^1 x^2(1-x^2)^{n-1}dx=2nI_{n-1}-2nI_n$$

$$I_n=\frac{2n}{(2n+1)}I_{n-1}=\frac{2^2n(n-1)}{(2n+1)(2n-1)}I_{n-2}=.....$$

$$=\frac{2^n n(n-1)(n-2)....1}{(2n+1)(2n-1)(2n-3).....3}I_0\quad=\frac{2^n n!}{3\cdot5\cdot7.......(2n-1)(2n+1)}$$

4.(B) $\displaystyle S=\sum_{r=1}^{n}(-1)^{r-1}\left(1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{r}\right)\ {}^nC_r$

$$=\sum_{r=1}^{n}\left({}^nC_r(-1)^{r-1}\int_0^1(1+x+x^2+....+x^{r-1})\ dx\right)=\sum_{r=0}^{n}\int_0^1\left(\frac{1-x^r}{1-x}\right)\ {}^nC_r(-1)^{r-1}dx$$

$$=\int_0^1\left(\sum_{r-0}^{n}\frac{{}^nC_r(1-x^r)(-1)^r}{1-x}\right)dx=\int_0^1\left(\frac{0-(1-x)^n}{(1-x)}\ dx=\int_0^1(1-x)^{n-1}dx=\int_0^1x^{n-1}dx=\frac{1}{n}\right)dx$$

5.(B) $\displaystyle\sum_{r=0}^{n}{}^nC_r(1-x)^r(-1)^r=x^n$

$$\Rightarrow\quad \sum_{r=1}^{n}{}^nC_r(-1)^{r-1}(1-x)^{r-1}=\frac{1-x^n}{1-x}$$

$$\Rightarrow\quad \int_1^x\left(\sum_{r=1}^{n}{}^nC_r(-1)^{r-1}(1-x)^{r-1}\right)dx=\int_1^x(1+x+x^2+x^3+....+x^{n-1})\ dx$$

$$\Rightarrow\quad -\sum_{r=1}^{n}\frac{{}^nC_r(-1)^{r-1}(1-x)^r}{r}=\frac{x-1}{1}+\frac{x^2-1}{2}+\frac{x^3-1}{3}+....+\frac{x^n-1}{n}$$

$$\Rightarrow\quad \sum_{r=1}^{n}\frac{{}^nC_r(1-x)^r(-1)^{r-1}}{r}=\frac{1-x}{1}+\frac{1-x^2}{2}+\frac{1-x^3}{3}+....+\left(\frac{1-x^n}{n}\right)$$

6.(C) $\sum_{r=0}^{n}\frac{{}^nC_r}{r+1}a^{n-r}b^r = \frac{1}{n+1}\sum_{r=0}^{n}{}^{n+1}C_{r+1}a^{n-r}b^r = \frac{1}{(n+1)b}[(a+b)^{n+1}-a^{n+1}]$

Put $a = x^2,\ b = 2-x^2$

$\Rightarrow \quad \sum_{r=1}^{n}\frac{{}^nC_r}{r+1}x^{2n-2r}(2-x^2)^r = \frac{2^{n+1}-x^{2n+2}}{(n+1)(2-x^2)}$

7.(D) $n\sum_{r=1}^{n}r^2\ {}^{n-1}C_{r-1}p^r q^{n-r} = n\sum_{r=2}^{n}(r+1)(r-1)\ {}^{n-1}C_{r-1}\ p^r q^{n-r} + n\sum_{r=1}^{n}{}^{n-1}C_r\, p^r q^{n-r}$

$= n(n-1)\sum_{r=3}^{n}(r-2)^{n-2}C_{r-2}\,p^r q^{n-r} + 3n(n-1)\sum_{r=2}^{n}{}^{n-2}C_{r-2}\,p^r q^{n-r} + n\sum_{r=1}^{n}{}^{n-1}C_{r-1}p^r q^{n-r}$

$= n(n-1)(n-2)\sum_{r=3}^{n}{}^{n-3}C_{r-3}\,p^r q^{n-r} + 3n(n-1)p^2(q+p)^{n-2} + np(q+p)^{n-1}$

$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np = np((n-1)(n-2)p^2 + 3(n-1)p + 1$

8-10

8.(B) **9.(D)** **10.(C)**

$\sum_{r=0}^{n}\frac{r^2}{{}^nC_r} = a_n + \sum_{r=0}^{n}\frac{(r-1)(r+1)}{{}^nC_r} = a_n + (n+1)\sum_{r=0}^{n}\frac{r+2-3}{{}^{n+1}C_{r+1}}$

$= a_n + (n+1)(n+2)\sum_{r=0}^{n}\frac{1}{{}^{n+2}C_{r+2}} - 3(n+1)\sum_{r=0}^{n}\frac{1}{{}^{n+1}C_{r+1}}$

$= a_n + (n+1)(n+2)\left(a_{n+2} - 1 - \frac{1}{n+2}\right) - 3(n+1)(a_{n+1}-1)$

$= a_n + (n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} - n(n+1)$

11-13

11.(C) **12.(D)** **13.(A)**

Clearly, $A_k = 4$

$\therefore \quad 4 + B_k x + C_k x^2 + D_k x^3 = ... = (4 + B_{k-1}x + C_{k-1}x^2 + ... - 2)^2 = 4 + 4B_{k-1}x + (4C_{k-1} + B_{k-1}^2)x^2 + ...$

$\Rightarrow \quad B_k = 4B_{k-1}$ and $C_k = 4C_{k-1} + B_{k-1}^2$

$\therefore \quad B_k = 4B_{k-1} = 4^2 B_{k-2} = ... = 4^{k-1}B_1 = -4^k$

$C_k = B_{k-1}^2 + 4C_{k-1} = 4^{2k-2} + 4C_{k-1}$

$= 4^{2k-2} + 4(4^{2k-4}) + 4^2 C_{k-2} = 4^{2k-2} + 4^{2k-3} + 4^2(4^{2k-6} + 4C_{k-3}) = 4^{2k-2} + 4^{2k-3} + 4^{2k-4} + 4^3 C_{k-3}$

$= 4^{2k-2} + 4^{2k-3} + 4^{2k-4} + ... + 4^k + 4^{k-1}C_1 = 4^{2k-2} + 4^{2k-3} + 4^{2k-4} + ... + 4^k + 4^{k-1} = \frac{4^{k-1}(4^k-1)}{(4-1)} = \frac{4^{2k-1} - 4^{k-1}}{3}$

14.(AD) $\sum_{r=1}^{n}(r\ {}^nC_r)[(n-r)^n C_r] = n\sum_{r=1}^{n}\left[{}^{n-1}C_{r-1}\left((n-r)\ {}^nC_{n-r}\right)\right] = n^2\sum_{r=1}^{n}{}^{n-1}C_{r-1}\ {}^{n-1}C_{n-r-1} = n^2\ {}^{2n-2}C_{n-2} = n(n-1)\ {}^{2n-2}C_{n-1}$

$\left[\because n^2\ {}^{2n-2}C_{n-2} = n^2\frac{(2n-2)!}{n!(n-2)!} = \frac{n(n-1)(2n-2)!}{(n-1)!(n-1)!} = n(n-1)\ {}^{2n-2}C_{n-1}\right]$

15.(ABCD) $-\sum_{r=1}^{n}(-1)^r r\,{}^nC_r^2 = -n\sum_{r=1}^{n}(-1)^r\,{}^{n-1}C_{r-1}{}^nC_r = -n$ coefficient of x^n in $(1-x^2)^{n-1}(1-x)$

16.(BCD) $(x^3-1)^n = (x-1)^n(1+x+x^2)^n$

Coefficient of x^n in $(x^3-1)^n = {}^nC_0a_0 - {}^nC_1a_1 + {}^nC_2a_2 - {}^nC_3a_3 + \ldots + (-1)\,{}^nC_na_n$

$\Rightarrow \sum_{r=0}^{n}{}^nC_r(-1)^r a_r =$ coefficient of x^n in $(x^3-1)^n$

$$= \begin{cases} {}^nC_{\frac{2n}{3}}(-1)^{\frac{2n}{3}} = {}^nC_{\frac{n}{3}} & \text{if } n \text{ is multiple of } 3 \\ 0 & \text{if } n \text{ is not multiple of } 3 \end{cases}$$

17.(ABC) $\alpha = {}^nC_0\,x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \ldots$

$\beta = {}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 + \ldots$

$\alpha+\beta = (x+a)^n$ and $\alpha-\beta = (x-a)^n$

$(x+a)^{2n} - (x-a)^{2n} = (\alpha+\beta)^2 - (\alpha-\beta)^2 = 4\alpha\beta$

$(x+a)^{2n} + (x-a)^{2n} = (\alpha+\beta)^2 + (\alpha-\beta)^2 = 2(\alpha^2+\beta^2)$

$(x+a)^n(x-a)^n = (x^2-a^2)^n = (\alpha+\beta)(\alpha-\beta) = \alpha^2-\beta^2$

18.(BC) $T_{r+1} > T_r \quad \Rightarrow \quad {}^{24}C_rx^r > {}^{24}C_{r-1}x^{r-1}$

$\Rightarrow \quad \frac{(25-r)x}{r} > 1 \quad \Rightarrow \quad x > \frac{r}{25-r} \quad \Rightarrow \quad T_{r+1} > T_{r+2} \quad \Rightarrow \quad x < \frac{r+1}{24-r}$

Put $r = 12$

$\therefore \quad \frac{12}{13} < x < \frac{13}{12}$

19.(CD) $({}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \ldots + {}^mC_m) + 2({}^{n-1}C_m + {}^{n-2}C_m + \ldots + {}^mC_m)$

$+2({}^{n-2}C_m + {}^{n-3}C_m + \ldots + {}^mC_m) + \ldots + 2({}^{m+1}C_m + {}^mC_m) + 2({}^mC_m)$

$= {}^{n+1}C_{m+1} + 2({}^nC_{m+1} + {}^{n-1}C_{m+1} + {}^{n-2}C_{m+1} + \ldots + {}^{m+2}C_{m+1} + {}^{m+1}C_{m+1})$

$= {}^{n+1}C_{m+1} + 2\,{}^{n+1}C_{m+2} = {}^{n+1}C_{m+2} + {}^{n+2}C_{m+2}$

20.(ABCD) $a_r = a_{2n-r}\ \forall r \in \{0, 1, 2, \ldots, n\}$

$a_0 + a_1 + a_2 + \ldots + a_{n-1} + a_n + a_{n+1} + \ldots + a_{2n} = 3^n$

$\Rightarrow \quad a_0 + a_1 + a_2 + \ldots + a_{n-1} = \frac{(3^n - a_n)}{2}$

$(1-x+x^2)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} - a_3x^{2n-3} + \ldots + a_{2n}$

$\Rightarrow \quad a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2 =$ coefficient of x^{2n} in $(1+x^2+x^4)^n = a_n$

$\Rightarrow \quad a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + (-1)^{n-1} a_{n-1}^2 = \dfrac{a_n - (-1)^n a_n^2}{2}$

$$n(1+2x)\,(1+x+x^2)^{n-1} = \sum_{r=1}^{2n} r\, a_r\, x^{r-1}$$

$$n(1+2x)\,(1+x+x^2)^{n} = (1+x+x^2)\sum_{r=1}^{2n} r\, a_r\, x^{r-1}$$

Equating coefficient of x^r from both the sides

$n(a_r + 2a_{r-1}) = (r+1)a_{r+1} + ra_r + (r-1)a_{r-1} \Rightarrow (r+1)a_{r+1} = (n-r)a_r + (2n-r+1)a_{r-1}$

21.(BCD) $\quad ({}^nC_1\, {}^nC_2 ... {}^nC_{n-1})^{\frac{1}{n-1}} < \dfrac{{}^nC_1 + {}^nC_2 + ... + {}^nC_{n-1}}{n-1} \Rightarrow P < \left(\dfrac{2^n - 2}{n-1}\right)^{n-1}$

Also, $({}^nC_0\, {}^nC_1\, {}^nC_2 ... {}^nC_{n-1}\, {}^nC_n)^{\frac{1}{n+1}} < \dfrac{{}^nC_0 + {}^nC_1 + C_2 + ... + {}^nC_n}{n+1} \Rightarrow P < \left(\dfrac{2^n}{n+1}\right)^{n+1}$

22.(CD) ${}^{2n+1}C_0^2 + {}^{2n+1}C_1^2 + {}^{2n+1}C_2^2 + .. + {}^{2n+1}C_{2n+1}^2 = {}^{4n+2}C_{2n+1}$

${}^{2n+1}C_0^2 - {}^{2n+1}C_1^2 + {}^{2n+1}C_2^2 - {}^{2n+1}C_3^2 + ... - {}^{2n+1}C_{2n+1}^2 = 0$

$\Rightarrow \quad \displaystyle\sum_{r=0}^{n} {}^{2n+1}C_{2r}^2 = \sum_{r=0}^{n} {}^{2n+1}C_{2r+1} = \frac{1}{2}({}^{4n+2}C_{2n+1})$

23.(ABC) $\quad f(x) = (1+x^2-x^3)^{1000} = a_0 + a_1x + ... + a_{20}x^{20} + ...$

$f(-x) = (1+x^2+x^3)^{1000} = a_0 - a_1x + ... + a_{20}x^{20} + ...$

$a = d$

$g(x) = (1-x^2+x^3)^{1000} = b_0 + b_1x + ... + b_{20}x^{20} + ...$

$g(-x) = (1-x^2-x^3) = b_0 - b_1x + ... + b_{20}x^{20} + ...$

$\Rightarrow \quad b = c \qquad$ Clearly d is the largest.

24.(ACD) $\quad \beta_{100} = {}^{100}C_{100} + {}^{101}C_{100} + {}^{102}C_{100} + ...\, {}^{200}C_{100} = {}^{201}C_{100}$

$\beta_{99} = {}^{99}C_{99} + {}^{100}C_{99} + {}^{101}C_{99} + ...\, {}^{200}C_{99} = {}^{201}C_{100}$

25.(AC) $\left(5\sqrt{3}+8\right)^{2n+1} - \left(5\sqrt{3}-8\right)^{2n+1} = 2\left({}^{2n+1}C_1(5\sqrt{3})^{2n}8 + {}^{2n+1}C_3(5\sqrt{3})^{2n-2}8^3 + ...\right)$

$\Rightarrow [x] + \{x\} - N = 2k,\ k \in I$

$0 < \{x\} < 1$ and $0 < N < I$

$\Rightarrow -1 < \{x\} - N < 1$ and $\{x\} - N \in I \quad \Rightarrow \{x\} - N = 0 \quad \Rightarrow xN = x\{x\} = (11)^{2n+1}$

26.(CD) Put $x = i \Rightarrow i^n = (a_0 - a_2 + a_4 - a_6 + ...) + i(a_1 - a_3 + a_5 - a_7 + ...)$

$$\Rightarrow \quad a_1 - a_3 + a_5 - a_7 + ... = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n = 4k+1,\ k \in I \\ -1 & \text{if } n = 4k+3 \end{cases}$$

27.(ABC) $(13^n - 3^n) + 7^n = 10k + 7^n \Rightarrow$ last digit is 3 if $n = 4k+3,\ k \in I$

28.(BD) Coefficient of $x^t = {}^tC_t + {}^{t+1}C_t + {}^{t+2}C_t + ... + {}^nC_t = {}^{n+1}C_{t+1}$

Coefficient of $x^{t+1} = {}^{t+1}C_{t+1} + {}^{t+2}C_{t+1} + {}^{t+3}C_{t+1} + ... + {}^nC_{t+1} = {}^{n+1}C_{t+2}$

${}^{n+1}C_{t+1} = {}^{n+1}C_{t+2} \Rightarrow n - t = t + 2 \Rightarrow t = \dfrac{n-2}{2}$ their sum $= {}^{n+2}C_{t+2}$

29. **A-P, Q ; B-R, T ; C-P, Q ; D-P, Q, S**

(A) $\quad {}^nC_3\left(\dfrac{x}{a}\right)^{n-3}\left(\dfrac{1}{x}\right)^3 = \dfrac{5}{2} \Rightarrow n - 6 = 0$

$\Rightarrow \quad a^3 = 8 \quad \Rightarrow \quad a = 2$

(B) $\quad S = \displaystyle\sum_{p=1}^{4}\sum_{r=p}^{4}\frac{4!}{r!(4-r)!}\frac{r!}{(r-p)!\,p!} = \sum_{p=1}^{4}{}^4C_p\sum_{r=p}^{4}\frac{(4-p)!}{(r-p)!(4-r)!}$

$= \displaystyle\sum_{p=1}^{4}{}^4C_p\,2^{4-p} = 3^4 - 2^4 = 65$

(C) Coefficient of x^{13} in $(1-x)(1-x^4)^4 = -(-{}^4C_3) = 4$

(D) $\displaystyle\sum_{r=0}^{4}{}^4C_r(r-2)^2 = 16$

30. **A-T ; B-Q ; C-S ; D-R**

(A) Coefficient of $x^4 = \dfrac{6!}{2!\,4!}2^2 + \dfrac{6!}{3!\,2!}2^3\,3 + \dfrac{6!}{2!\,4!}2^4\,3^2 = 3660$

(B) ${}^{11}C_0\,{}^{22}C_{11} - {}^{11}C_1\,{}^{20}C_{11} + {}^{11}C_2\,{}^{18}C_{11} - {}^{11}C_3\,{}^{16}C_{11} + ...$

= coefficient of x^{11} in ${}^{11}C_0(1+x)^{22} - {}^{11}C_1(1+x)^{20} + ...$

= coefficient of x^{11} in $((1+x)^2 - 1)^{11}$ = coefficient of x^{11} in $x^{11}(2+x)^{11} = 2^{11} = 2048$

(C) ${}^5C_1\,{}^5C_5 - {}^5C_2\,{}^{10}C_5 + ... + {}^5C_5\,{}^{25}C_5$

= coefficient of x^5 in $[1-(1-(1+x)^5)^5]$

= coefficient of x^5 in $[1-(-x(1+(1+x)+(1+x)^2+(1+x)^3+(1+x)^4))^5]$

= coefficient of x^5 in $[1+x^5(1+(1+x)+(1+x)^2+(1+x)^3+(1+x)^4)^5] = 5^5 = 3125$

(D) Coefficient of x^4 in $= \dfrac{11!}{4!\,7!} + \dfrac{11!}{2!\,8!} + \dfrac{11!}{9!} + \dfrac{11!}{2!\,9!} = 990$

31.(2250000)

$S = {}^{50}C_6 - {}^5C_1\,{}^{40}C_6 + ... + {}^5C_4\,{}^{10}C_6$

= Coefficient of x^6 in ${}^5C_0(1+x)^{50} - {}^5C_1(1+x)^{40} + ... + {}^5C_4(1+x)^{10} - {}^5C_5(1+x)^0$

= Coefficient of x^6 in $((1+x)^{10} - 1)^5$

= Coefficient of x^6 in $x^5\ ({}^{10}C_1 + {}^{10}C_2x + ... + {}^{10}C_{10}x^9)^5$

$= {}^5C_4({}^{10}C_1)^4\left({}^{10}C_2\right) = 5 \times 10^4 \times 5 \times 9 = 2250000$

32.(21) $(7-1)^{2007}+(7+1)^{2007}=49k+2007\times 7\times 2=49\lambda+21$

33.(1024) ${}^{10}C_0{}^{20}C_{10}-{}^{10}C_1{}^{18}C_{10}+....{}^{10}C_5{}^{10}C_{10}$

$=$ Coefficient of x^{10} in ${}^{10}C_0(1+x)^{20}-{}^{10}C_1(1+x)^{18}+{}^{10}C_2(1+x)^{16}-{}^{10}C_3(1+x)^{14}+....+{}^{10}C_9(1+x)^2+{}^{10}C_{10}$

$=$ Coefficient of x^{10} in $((1+x)^2-1)^{10}$ = Coefficient of x^{10} in $x^{10}\ (2+x)^{10}=2^{10}$

34.(2252) Coefficient of x^5y^5 in

$$(1+x)^5(1+y)^5(x+y)^5=\sum_{r=0}^{5}{}^5C_r{}^5C_{5-r}{}^5C_{5-r}=\sum_{r=0}^{5}{}^5C_r^3$$

35.(13) Coefficient of $x^{\frac{n(n+1)}{2}-7}=-7+(1\cdot 6+2\cdot 5+3\cdot 4)-1\cdot 2\cdot 4)=13$

36.(1) Coefficient of x^n in ${}^nC_0(1+x)^{2n}-{}^nC_1(1+x)^{2n-1}+{}^nC_2(1+x)^{2n-2}-{}^nC_3(1+x)^{2n-3}+...+(-1)^n\ {}^nC_n(1+x)^n$

$=$ Coefficient of x^n in $(1+x)^n((1+x)-1)^n=1$

37.(1) $N=\dfrac{(7+18)^3}{(3+2)^6}=1$

38.(2) $\dfrac{14}{9}{}^mC_2{}^mC_4=\left({}^mC_3\right)^2\ \Rightarrow m=9$

$${}^9C_3 5^{-\log_{10}(6-\sqrt{8x})}\cdot 5^{\frac{\log_{10}(x-1)-2\log_{10}5}{2}}=\frac{84}{5}$$

$\Rightarrow\quad 6\sqrt{2x}-x-10=0$

$\Rightarrow\quad x-6\sqrt{2x}+10=0$

$\Rightarrow\quad \left(\sqrt{x}-5\sqrt{2}\right)\left(\sqrt{x}-\sqrt{2}\right)=0$

$\Rightarrow\quad x=2,\ 50$ (rejected)

Hence, $x=2$

39.(1) $(5+\sqrt{2})^n+(5-2\sqrt{6})^n=2k, k\in I$

$\Rightarrow\quad [x]+\{x\}+N=2k,\ 0<N<1,\ 0<\{x\}<1,\ \{x\}+N\in I$

$\Rightarrow\quad \{x\}+N=1$

$\therefore\quad x-x^2+x[x]=x-x(x-[x])=x(1-\{x\})$

$=xN=(5+2\sqrt{6})^n(5-2\sqrt{6})^n=1$

40.(3) ${}^mC_2+{}^nC_2-{}^mC_1\ {}^nC_1+{}^mC_1-{}^nC_1=-m$

$\Rightarrow\quad (m-n)^2+3(m-n)=0$

$\Rightarrow\quad n-m=3$

STRAIGHT LINE

1.(C) $\frac{1}{x_l} = a + (l-1)d$;

$\frac{1}{x_m} = a + (m-1)d; \frac{1}{x_n} = a + (n-1)d \quad \Rightarrow \quad \begin{vmatrix} a+(l-1)d & l & 1 \\ a+(m-1)d & m & 1 \\ a+(n-1)d & n & 1 \end{vmatrix} = 0$

2.(A) Area of ΔOPP_1 + area of ΔOPP_2 = area of ΔOP_1P_2

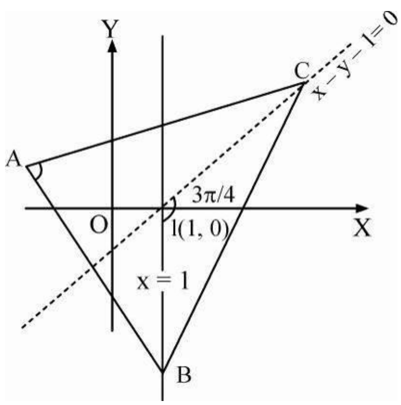
3.(D) Equation of CV is $y = \frac{3\beta}{(3-\alpha)}\left(1 - \frac{x}{2}\right)$. Hence fixed point is (2, 0).

4.(D) $a - 2\sqrt{bc} = b + c \quad \Rightarrow \quad \left(\sqrt{b} + \sqrt{c}\right)^2 - \left(\sqrt{a}\right)^2 = 0 \quad \Rightarrow \quad \sqrt{b} + \sqrt{c} - \sqrt{a} = 0$

Or $\quad \sqrt{b} + \sqrt{c} + \sqrt{a} = 0$ (rejected)

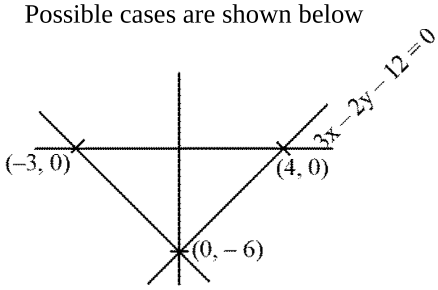
$\sqrt{b} + \sqrt{c} - \sqrt{a} = 0 \quad \Rightarrow \quad \sqrt{a}x + \sqrt{b}y + \sqrt{c} = 0$ Passes through fixed point (–1, 1)

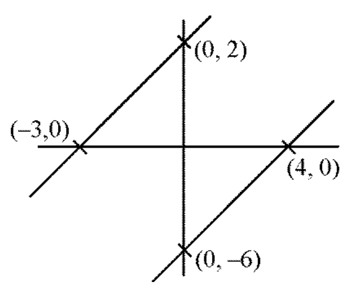
5.(C) $\angle BIC = \frac{3\pi}{4} = \frac{\pi}{2} + \frac{A}{2} \quad \Rightarrow \quad A = \frac{\pi}{2} \quad \left[\because \angle BIC = \frac{\pi}{2} + \frac{A}{2}\right]$



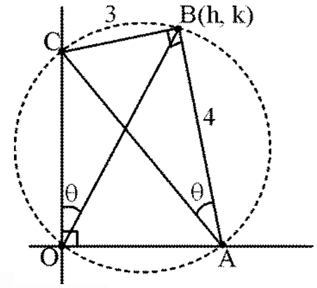
Hence, $\angle BAC = \frac{\pi}{2}$

6.(B) Possible cases are shown below





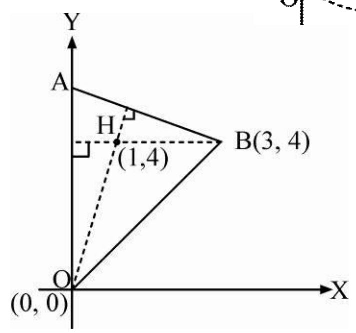
7.(B) $\tan\theta = \frac{3}{4}$ (from Fig.)



$\Rightarrow$ Slope of $OB = \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

$\Rightarrow$ Locus of (h, k) is $\frac{k}{h} = \frac{4}{3}$ $\Rightarrow$ $3y - 4x = 0$

8.(D) $BH \perp OA$ $\Rightarrow$ A lies on y-axis equation on AH is:

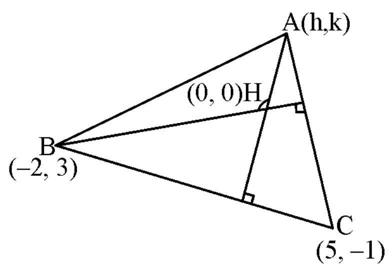


$y - 4 = -\frac{3}{4}(x-1)$

Put $x = 0$

$\Rightarrow$ $y = 4 + \frac{3}{4} = \frac{19}{4}$ $\Rightarrow$ $A = \left(0, \frac{19}{4}\right)$

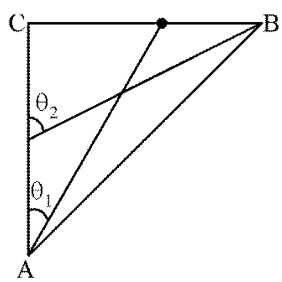
9.(C) Slope of $AH = \frac{k}{h} = \frac{7}{4}$



Slope of $AC = \frac{k+1}{h-5} = \frac{2}{3}$

So, $3\left(\frac{7}{4}h+1\right) = 2\ h-5$

10.(B) $\tan\theta_1 = \frac{BC}{2AC}$



$\tan\theta_2 = \frac{BC}{AC/2} = \frac{2BC}{AC}$

$\frac{\tan\theta_1}{\tan\theta_2} = \frac{1}{4}$

Case-I: If $m < 3$

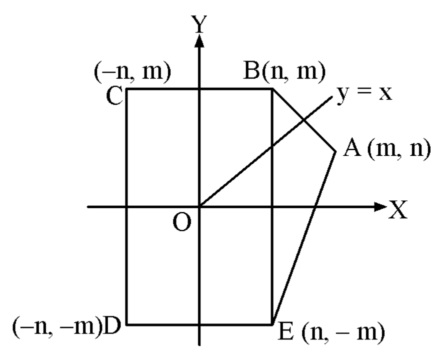
$\tan\theta_1 = \frac{1}{3}$ and $\tan\theta_2 = \frac{1}{m}$

$\frac{m}{3} = \frac{1}{4}$ $\Rightarrow$ $m = \frac{3}{4}$

Case-II: If $m > 3$,

$\tan\theta_1 = \frac{1}{m}$ and $\tan\theta_2 = \frac{1}{3}$ $\Rightarrow$ $\frac{3}{m} = \frac{1}{4}$ $\Rightarrow$ $m = 12$

11.(B) Area of rectangle $BCDE = 4mn$



Area of $\Delta ABE = \frac{2m\ \ m-n}{2}$

$= m^2 - mn$

$\therefore$ area of pentagon = $4mn + m^2 - mn = m^2 + 3mn$

Aliter :

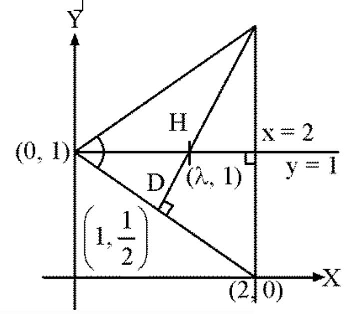
$$ar(ABCDE) = \frac{1}{2}\left\| \begin{matrix} m & n \\ n & m \end{matrix} \right| + \left| \begin{matrix} n & m \\ -n & m \end{matrix} \right| + \left| \begin{matrix} -n & m \\ -n & -m \end{matrix} \right| + \left| \begin{matrix} -n & -m \\ n & -m \end{matrix} \right| + \left| \begin{matrix} n & -m \\ m & n \end{matrix} \right\|$$

$$= \frac{1}{2}\left[m^2 - n^2 + mn + mn + mn + mn + mn + n^2 + m^2\right] = m^2 + 3mn$$

12.(B) Slope of $HD = 2$

$$\Rightarrow \qquad \frac{1-\frac{1}{2}}{\lambda - 1} = 2 \quad \Rightarrow \quad \lambda = \frac{5}{4}$$

$$\therefore \qquad H \equiv \left(\frac{5}{4}, 1\right)$$



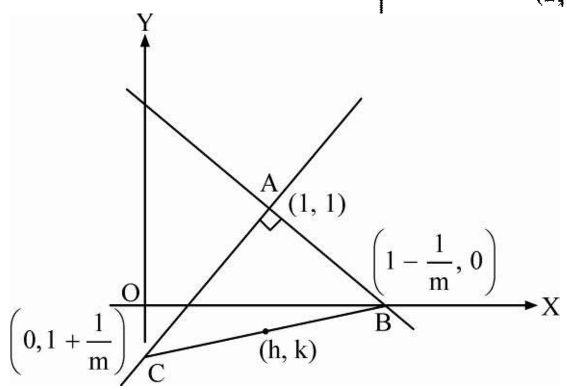
13.(A) $y - 1 = m(x-1)$

$$y - 1 = -\frac{1}{m}(x-1)$$

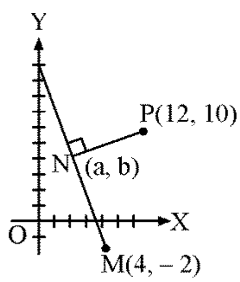
$$2h = 1 - \frac{1}{m}$$

$$2k = 1 + \frac{1}{m}$$

Locus is $x + y = 1$



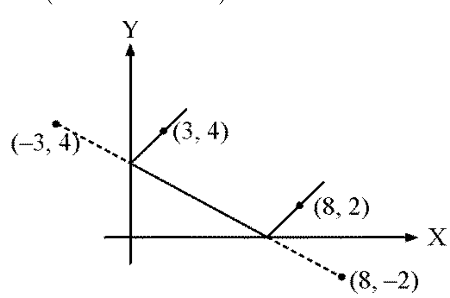
14.(B) (a, b) is the foot of $\perp$ar of (12, 10) on the line $y + 5x = 18$



$$\Rightarrow \qquad \frac{a-12}{5} = \frac{b-10}{1} = -\left(\frac{10+5\ 12\ -18}{26}\right) \qquad \Rightarrow \qquad a, b \ = \ 2, 8$$

15.(B) $\frac{0+2}{x-8} = \frac{4+2}{-3-8} = -\frac{6}{11}$

$$x = \frac{13}{3}$$

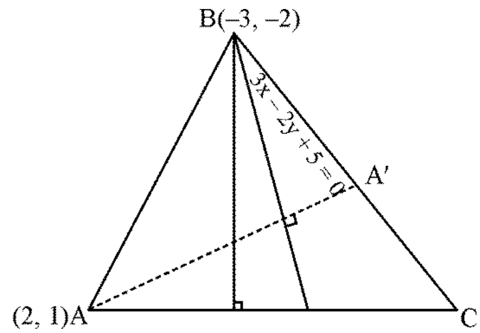


16.(B) Image of $A(2, -1)$ w.r.t.

$3x-2y+5=0, A'$ is given by

$$\frac{x-2}{3}=\frac{y+1}{-2}=-2\cdot\frac{(6+2+5)}{13}=-2$$

$$A'=(-4,3)$$



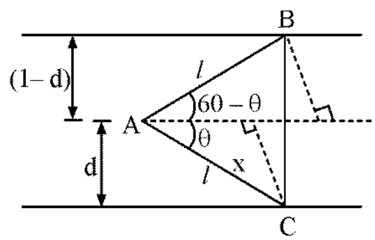
Equation of BC is

$$y-3=\frac{3+2}{-4+3}(x+4)$$

$$5x+y+17=0$$

17.(B) $l\sin(60-\theta)=l\frac{\sqrt{3}}{2}\cos\theta-\frac{l}{2}\sin\theta$

$$=1-d$$

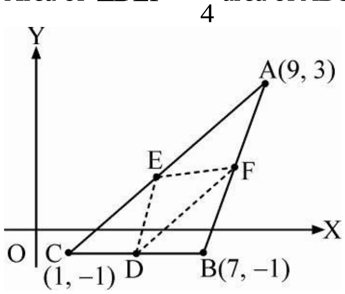


$\Rightarrow \quad l\cos\theta=\frac{2-d}{\sqrt{3}}$ & $\ell\sin\theta=d$

Squaring and adding Eqns. (1) and (2)

$\Rightarrow \quad l^2=d^2+\frac{4+d^2-4d}{3}=\frac{4}{3}\left(d^2-d+1\right) \quad \Rightarrow \quad l=2\sqrt{\frac{d^2-d+1}{3}}$

18.(D) Area of $\Delta DEF=\frac{1}{4}$ area of ABC

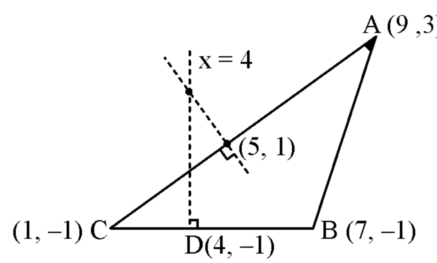


$$=\frac{1}{4}\times\frac{1}{2}\times6\times4=3$$

19.(B) Equation of perpendicular bisector of AC is

$$y-1=-2(x-5)$$

$\Rightarrow \quad y+2x=11$



Perpendicular bisector of BC is

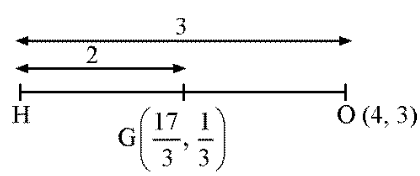
$\therefore \quad \text{Circumcentre}\equiv(4,3)$

$\Rightarrow \quad R=\sqrt{(4-1)^2+(3+1)^2}=5 \quad \Rightarrow \quad a+b+R=4+3+5=12$

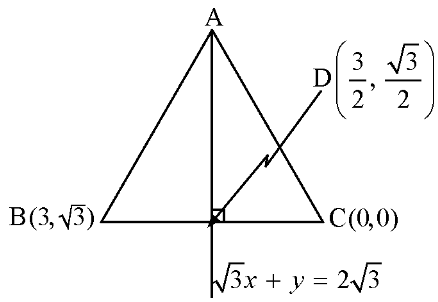
20.(C) $O\equiv$ circumcentre, $G\equiv$ centroid

$\Rightarrow \quad H\equiv\left(3\left(\frac{17}{3}\right)-2(4),3\left(\frac{1}{3}\right)-2(3)\right)$

$$H\equiv(9,-5)$$



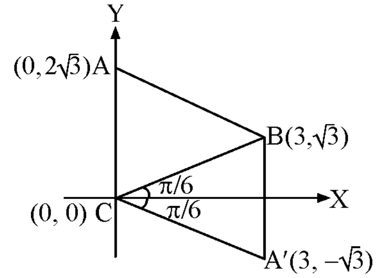
21-23.



Equation of BD is : $x - \sqrt{3}y = 0$

$\Rightarrow \quad D \equiv \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ and $C \equiv (0, 0)$

$AC = BC = 2\sqrt{3}$



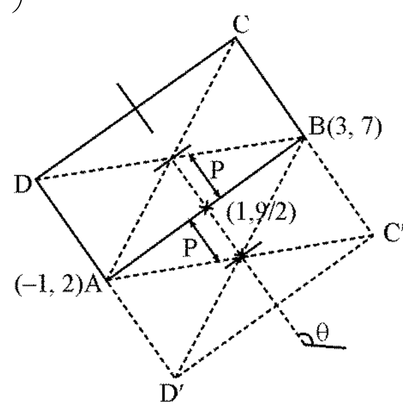
21.(B) Two triangles are possible

22.(D) $A \equiv (0, 2\sqrt{3})$ and $A'(3, -\sqrt{3})$

23.(A) Orthocentre $\equiv$ centroid of Δ

$H_1 \equiv \left(\frac{0+3+0}{3}, \frac{0+\sqrt{3}+2\sqrt{3}}{3}\right) \equiv (1, \sqrt{3})$ **;** $H_2 \equiv \left(\frac{0+3+3}{2}, \frac{0+\sqrt{3}-\sqrt{3}}{3}\right) \equiv (2, 0)$

24.(BD) $BC = \frac{3}{4}AB = \frac{3}{4}\sqrt{41}$



$P = \frac{1}{2}BC = \frac{3}{8}\sqrt{41}$; $\quad \tan\div = -\frac{4}{5}$

Intersection point of diagonals can be

$\left(1 + \frac{3}{8}\sqrt{41}\left(-\frac{5}{\sqrt{41}}\right), \frac{9}{2} + \frac{3}{8}\sqrt{41}\left(\frac{4}{\sqrt{41}}\right)\right)$

or $\left(1 - \frac{3}{8}\sqrt{41}\left(-\frac{5}{\sqrt{41}}\right), \frac{9}{2} - \frac{3}{8}\sqrt{41}\left(\frac{4}{\sqrt{41}}\right)\right)$ i.e., $\left(-\frac{7}{8}, 6\right) or \left(\frac{23}{8}, 3\right)$

25.(ABCD) $\frac{m_1}{m_2} = \frac{9}{2}, \frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{7}{9} \quad \Rightarrow \quad 9m_2^2 - 9m_2 + 2 = 0$

$9m_2^2 + 9m_2 + 2 = 0 \qquad m_2 = \frac{2}{3}, \frac{1}{3} m_2 = \frac{-2}{3}, \frac{-1}{3}$

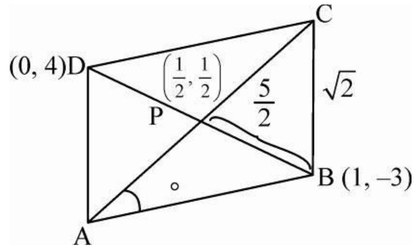
$m_2 = \frac{2}{3}, m_1 = 3 \quad \Rightarrow \quad y = 3x, 3y = 2x$

$m_2 = \frac{1}{3}, m_1 = \frac{3}{2} \quad \Rightarrow \quad 3y = x, 2y = 3x$

$m_2 = \frac{-2}{3}, m_1 = -3 \quad \Rightarrow \quad 3x + y = 0, 2x + 3y = 0$

$m_2 = -\frac{1}{3}, m_1 = \frac{-3}{2} \quad \Rightarrow \quad x + 3y = 0,\ 3x + 2y = 0$

26.(AB) $\tan\theta = \frac{1}{7}, \sin\theta \frac{1}{5\sqrt{2}}, \cos\theta = \frac{7}{5\sqrt{2}}$



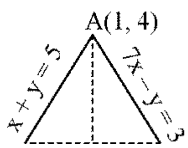
$$AP \equiv \frac{5}{2}\sqrt{2}\cot 30° = \frac{5}{2}\sqrt{2}\sqrt{3}$$

$$A \equiv \left(\frac{1}{2} - \frac{5\sqrt{2}\sqrt{3}}{2}\frac{7}{5\sqrt{2}}, \frac{1}{2} - \frac{5\sqrt{2}\sqrt{3}}{2}\frac{1}{5\sqrt{2}}\right)$$

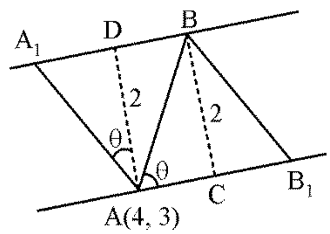
Or, $\left(\frac{1}{2} + \frac{5\sqrt{2}\sqrt{3}}{2}\frac{7}{5\sqrt{2}}, \frac{1}{2} + \frac{5\sqrt{2}\sqrt{3}}{2}\frac{1}{5\sqrt{2}}\right)$

$\Rightarrow \quad A = \left(\frac{1-7\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$ Or $\left(\frac{1+7\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$

27.(ABCD)



28.(AC) Area of parallelogram $AA_1BB_1 = \frac{8}{sin 2\theta}$



Which in least for $\theta = \frac{\pi}{4}$

So $1 = \left|\frac{m + \frac{3}{4}}{1 - \frac{3m}{4}}\right|$

29.(ABD) Taking B as origin
BC as x axis and let A (h, k)
Now $b + c > a$

$\Rightarrow \quad b + c > 6 \quad \Rightarrow \quad b + 2b > 6 \quad \Rightarrow \quad b > 2$.

Again $cos\, B = \frac{b^2 + c^2 - 36}{2bc} \Rightarrow \frac{b^2 + 4b^2 - 36}{4b^2} < 1 \quad \Rightarrow \quad b < 6 \quad \Rightarrow \quad 2 < b < 6 \;\; so \;\; 4 < c < 12.$

Since $b = 2c$.

Now $h^2 + k^2 = c^2$ and $(h-6)^2 + k^2 = b^2$

On eliminating b^2 $\Rightarrow \quad k^2 = 16 - (h-8)^2 \quad \Rightarrow \quad max\, k = 4$.

30.(BD) $\frac{c}{b} = 2 \quad \Rightarrow \quad c^2 = 4b^2$

$$c^2 = h^2 + k^2$$

$b^2 = (h-6)^2 + k^2 \quad \Rightarrow \quad h = \frac{b^2 + 12}{4}$

Now it in given that $k^2 > 10$

$\Rightarrow \quad c^2 - h^2 > 10 \quad \Rightarrow \quad 4b^2 - \left(\frac{b^2+12}{4}\right)^2 > 10$

$\Rightarrow \quad b^2 \in \left(20 - \sqrt{96}, 20 + \sqrt{96}\right) \quad \Rightarrow \quad 11 < b^2 < 30$

31.(BC) $(3x+4y+5)(x+2y+3)=0$

$$m_1=\frac{-3}{4}, m_2=-\frac{1}{2}\Rightarrow m_1m_2=\frac{3}{8}$$

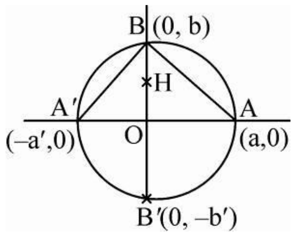
$$p=(1,-2)\quad ; \quad \tan\theta=\frac{-\frac{1}{2}+\frac{3}{4}}{1+\frac{3}{8}}=\frac{2}{11}\quad\Rightarrow\quad \sin\theta=\frac{2}{5\sqrt{5}}$$

32.(BC) H is the image of B'

$\Rightarrow \quad H\equiv(0,b')$

Applying power of 'O'

$$aa'=bb'\Rightarrow b'=\frac{aa'}{b}$$



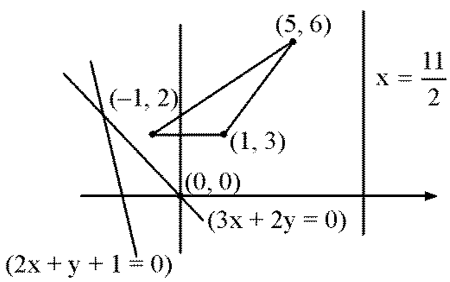
33.(AD) $m=\tan\theta_1, m'=\tan\theta_2$ inclination of given lines are $2\theta_1$ and $2\theta_2$

$\therefore$ Inclination of angle bisectors $\theta_1+\theta_2$ or $\theta_1+\theta_2+\frac{\pi}{2}$

$\therefore$ Slopes will be $\frac{m+m'}{1-mm'}, \frac{mm'-1}{m+m'}$

Eqn. of bisectors are $y-b=\frac{(m+m')}{(1-mm')}(x-a)$, $\quad y-b=\frac{(mm'-1)}{(m+m')}(x-a)$

34.(ABC)



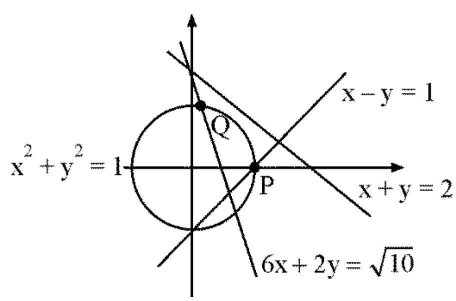
35.(ABC) For Q:

$$6\cos\theta+2\sin\theta=\sqrt{10}$$

$$\tan^{-1}3=\alpha, \sin(\theta+\alpha)=\frac{1}{2}$$

$$\theta+\alpha=\frac{5\pi}{6}$$

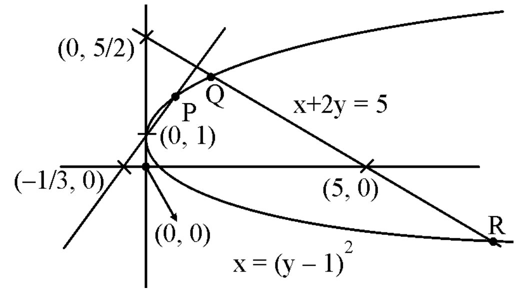
$$\theta=\frac{5\pi}{6}-\tan^{-1}3$$



36.(ABCD) Area = Mod of $\frac{1}{2}\begin{vmatrix}\alpha & 2\alpha+3 & 1\\ 1 & 2 & 1\\ 2 & 3 & 1\end{vmatrix}=\frac{1}{2}|-\alpha+2\alpha+3-1|$

$$\Rightarrow \frac{|\alpha+2|}{2}\in 2,3 \quad\Rightarrow\quad \alpha\in -8,-6\cup 2,4 \quad\Rightarrow\quad \alpha,\beta = -7,-11 \ , \ -6,-9 \ , \ 2,7 \ , \ 3,9$$

37.(BD) Intersection of $x=(y-1)^2$ with lines are

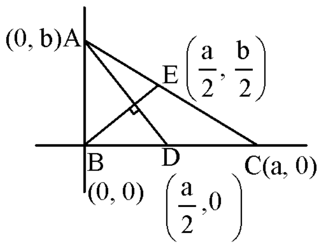


$(y-1)^2=\dfrac{y-1}{3} \Rightarrow y=1,\ y=\dfrac{4}{3} \Rightarrow P\equiv\left(\dfrac{1}{9},\dfrac{4}{3}\right)$

$(y-1)^2=5-2y \Rightarrow y^2=4 \Rightarrow y=\pm 2 \quad Q\equiv(1,2), R\equiv(9,-2)$

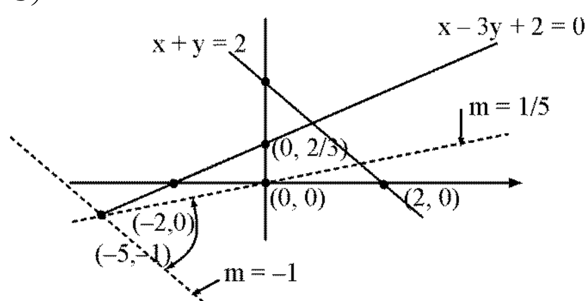
$\Rightarrow \quad a+1\in(-2,1)\cup\left(\dfrac{4}{3},2\right); \quad a\in(-3,0)\cup\left(\dfrac{1}{3},1\right)$

38.(AB) $AD\perp BE$



$\Rightarrow \quad \left(\dfrac{b}{a}\right)\dfrac{-b}{a/2}=-1 \quad \Rightarrow \quad 2b^2=a^2 \quad \Rightarrow \quad a=\pm\sqrt{2}b$

39.(BCD) $m\in\left(-1,\dfrac{1}{5}\right)$ for origin to lie inside the triangle

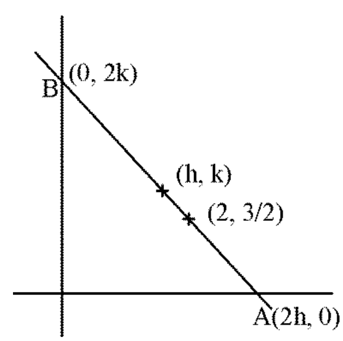


40.(AD) $D=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Applying $C_1\to C_1+C_2$, we get, $D=\begin{vmatrix} x_1+y_1 & y_1 & 1 \\ x_2+y_2 & y_2 & 1 \\ x_3+y_3 & y_3 & 1 \end{vmatrix}=\begin{vmatrix} -8 & y_1 & 1 \\ -8 & y_2 & 1 \\ -8 & y_3 & 1 \end{vmatrix}=0$

41.(ABC)

(A) Intersection point of lines is $\left(2,\dfrac{3}{2}\right)$,



Equation, of AB is $\dfrac{x}{2h}+\dfrac{y}{2k}=1$

Put $\left(2,\dfrac{3}{2}\right)\Rightarrow \dfrac{1}{h}+\dfrac{3}{4k}=1$

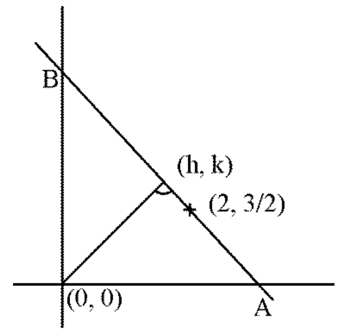
$\Rightarrow \quad 4k+3h=4hk \quad \Rightarrow \quad 3x+4y=4xy$

(B) Equation of AB is $hx+ky=h^2+k^2$

Put $\left(2,\frac{3}{2}\right)$

$\Rightarrow \quad 2h+\frac{3k}{2}=h^2+k^2$

$\Rightarrow \quad 2\left(x^2+y^2\right)-4x-3y=0$



(C) Equation of AB is $\frac{x}{3h}+\frac{y}{3k}=1$

Put $\left(2,\frac{3}{2}\right)$

42.(ABD)

Let $OA=r_1,\ OB=r_2,\ OP=r$

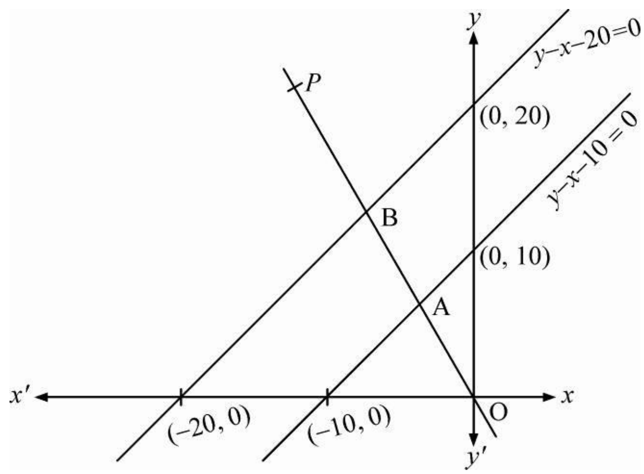
$\therefore \quad A \equiv \left(r_1\cos\theta,\ r_1\sin\theta\right)$

$B \equiv \left(r_2\cos\theta,\ r_2\sin\theta\right)$

$P \equiv \left(r\cos\theta,\ r\sin\theta\right)=\left(h,\ k\right)$

$\because \quad A$ lies on the line $y-x-10=0$

$\Rightarrow \quad r_1\sin\theta-r_1\cos\theta=10 \quad \Rightarrow \quad OA=\frac{10}{\sin\theta-\cos\theta}$



Similarly $OB=\frac{20}{\sin\theta-\cos\theta}$

(A) $\because \quad \frac{2}{OP}=\frac{1}{OA}+\frac{1}{OB} \quad \Rightarrow \quad \frac{2}{r}=\frac{1}{r_1}+\frac{1}{r_2} \quad \Rightarrow \quad \frac{2}{r}=\frac{\sin\theta-\cos\theta}{10}+\frac{\sin\theta-\cos\theta}{20}$

$\Rightarrow \quad \frac{40}{r}=3\sin\theta-3\cos\theta \quad \Rightarrow \quad 3\left(r\sin\theta\right)-3\left(r\cos\theta\right)=40 \quad \therefore \quad$ Locus of P is $3y-3x=40$

(B) $OP^2=\left(OA\right)\left(OB\right) \quad \Rightarrow \quad r^2=\frac{10}{\sin\theta-\cos\theta}\times\frac{20}{\sin\theta-\cos\theta}$

$\Rightarrow \quad \left(r\sin\theta-r\cos\theta\right)^2=200 \quad \therefore \quad$ Locus of P is $\left(y-x\right)^2=200$

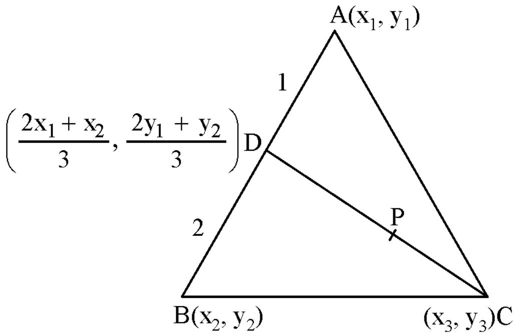
(C) $\therefore \quad \frac{1}{OP^2}=\frac{1}{OA^2}-\frac{1}{OB^2} \quad \Rightarrow \quad \frac{1}{r^2}=\frac{\left(\sin\theta-\cos\theta\right)^2}{10^2}-\frac{\left(\sin\theta-\cos\theta\right)^2}{20^2}$

$\because \quad$ Locus of P is $\left(y-x\right)^2=\frac{400}{3}$

(D) $\because \quad \frac{1}{OP^2}=\frac{1}{OA^2}+\frac{1}{OB^2} \quad \Rightarrow \quad \frac{1}{r^2}=\frac{\left(\sin\theta-\cos\theta\right)^2}{10^2}+\frac{\left(\sin\theta-\cos\theta\right)}{20^2}$

$\therefore \quad$ Locus of P is $\left(y-x\right)^2=80$

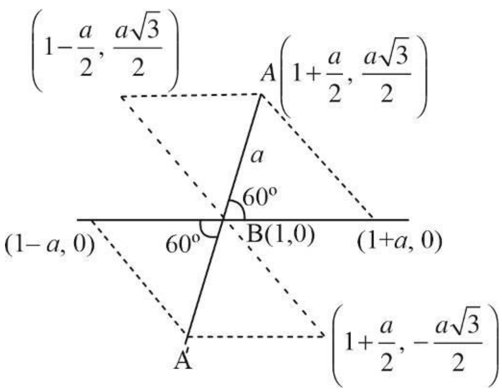
43.(AB)



$$P \equiv \left(\frac{\frac{2x_1+x_2}{3}+x_3}{2}, \frac{\frac{2y_1+y_2}{3}+y_3}{2} \right) \equiv \left(\frac{2x_1+x_2+3x_3}{6}, \frac{2y_1+y_2+3y_3}{6} \right)$$

$\therefore$ P lies inside the ΔABC $\Rightarrow$ Area of ΔPBC < area of ΔABC

44.(ABC)



45.(ABD)

$\alpha = tan^{-1}\sqrt{3}$

$tan\,\alpha = \sqrt{3}$

$tan\,\alpha = \dfrac{PC}{r/2}$

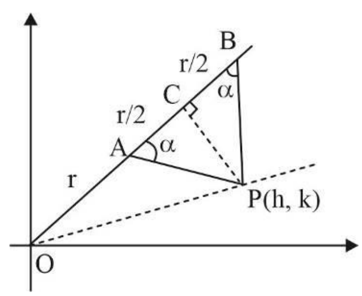
$PC = \dfrac{r}{2} tan\,\alpha$

Given $OA = AB = r$ in ΔOCP

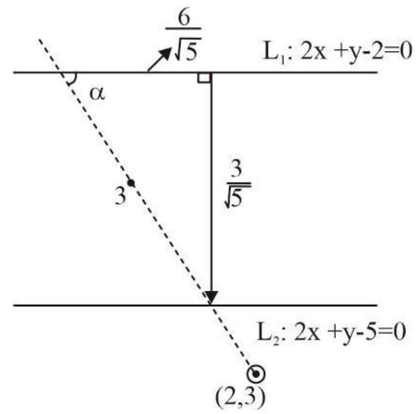
$(OP)^2 = (OC)^2 + (CP)^2$

$h^2 + k^2 = \left(\dfrac{3r}{2}\right)^2 + \left(\dfrac{r}{2} tan\,\alpha\right)^2$

Locus is $x^2 + y^2 = \left(\dfrac{9 + tan^2\,\alpha}{4}\right) r^2$



46.(BC) Let slope of line 'L' is m



$$tan\,\alpha = \left|\frac{m-(-2)}{1+m(-2)}\right| = \frac{1}{2}$$

$$\frac{m+2}{1-2m} = \pm\frac{1}{2}$$

$$2(m+2) = \pm(1-2m)$$

$+$ $\qquad$ $-$

$4m = -3$ $\quad$ or $\quad$ $4 = -1$ (*Not possible*)

$m = -\frac{3}{4}$ $\quad$ or $\quad$ line is perpendicular to the x-axis

47.(ABC) The equation $(2x+3y-5)cos\,\theta + (3x-5y+2)sin\,\theta = 0$

$(2x+3y-5) + tan\,\theta(3x-5y+2) = 0$

$L_1 + \lambda L_2 = 0$

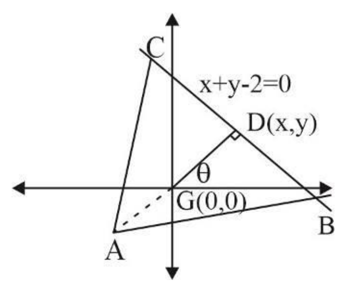
It represents the family of lines passing through the point of intersection of L_1 *and* L_2

Point of intersection of L_1 *and* L_2 is $(1,1)$

By reflection formula $\frac{x-1}{1} = \frac{y-1}{1} = \frac{-2\left(2-\sqrt{2}\right)}{2}$

Mirror image of $(1,1)$ is $\left(\sqrt{2}-1, \sqrt{2}-1\right)$

48.(ABCD) distance of $Point(0,0)$ from $x+y-2=0$; $GD = \sqrt{2}$

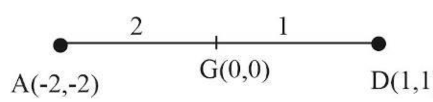


$$tan\,\theta = -\frac{1}{m_{CB}} = 1$$

$$cos\,\theta = \frac{1}{\sqrt{2}}, sin\,\theta = \frac{1}{\sqrt{2}}$$

$$x = 0 + \sqrt{2} \times \frac{1}{\sqrt{2}} = 1$$

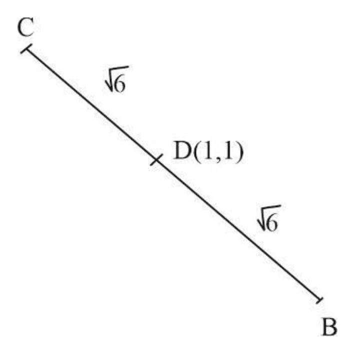
$$y = 0 + \sqrt{2} \times \frac{1}{\sqrt{2}} = 1$$



Point $D(1,1)$ and Point $A(-2,-2)$

$AD = 3\sqrt{2}$

As $AD = \frac{\sqrt{3}}{2}(side\,of\;\Delta ABC) = \frac{\sqrt{3}}{2}a = 3\sqrt{2}$



$a = 2\sqrt{6}$

$m_{BC} = -1$

$tan\,\alpha = -1$

$$cos\,\alpha = -\frac{1}{\sqrt{2}}, sin\,\alpha = \frac{1}{\sqrt{2}}$$

$Pt\ B\left(1+\frac{\sqrt{6}}{\sqrt{2}},1-\frac{\sqrt{6}}{\sqrt{2}}\right)$; $B\left(1+\sqrt{3},1-\sqrt{3}\right)$; $C\left(1-\sqrt{3},1+\sqrt{3}\right)$

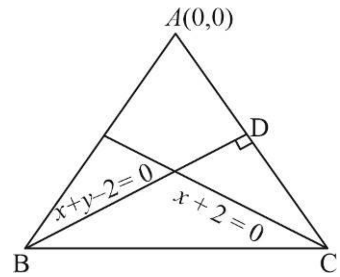
Area of $\Delta ABC=\frac{\sqrt{3}}{4}\times 4\times 6=6\sqrt{3}$ square units

49.(ABC) As C lies on $x+2=0$

x-coordinate of C is -2

$m_{AC}.m_{BD}=-1$

$m_{AC}=\frac{-1}{.m_{BD}}=1$



Equation of AC is $y=x$

Coordinate of C is $\left(-2,-2\right)$

D is $\left(1,1\right)$

If slope of AB is m then equation of AB is $y=mx$

$B\left(-4,-4m\right)$

As it lies on $x+y-2=0$

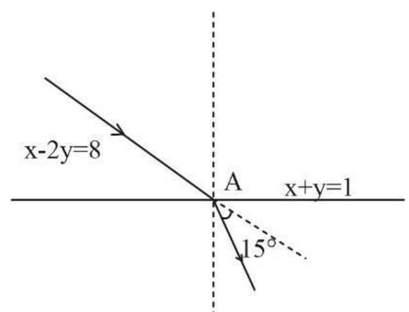
$-4-4m-2=0$; $m=\frac{-3}{2}$ Coordinate of B is $\left(-4,6\right)$

50.(AC) $x-2y=8$ ……(i)

$x+y=1$ ..(ii)

Solving (i), (ii) we get

$y=-7/3, x=\frac{10}{3}$; $A\left(\frac{10}{3},-\frac{7}{3}\right)$



Let the refracted ray have the slope $=m$

Then $\tan 15°=\left|\frac{m-\frac{1}{2}}{1+m.\frac{1}{2}}\right|=\left|\frac{2m-1}{m+2}\right|$ Or $\tan\left(45°-30°\right)=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\left|\frac{2m-1}{m+2}\right|$

Or $\left|\frac{2m-1}{m+2}\right|=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}$

$\frac{2m-1}{m+2}=2-\sqrt{3},-2+\sqrt{3}$

$2m-1=\pm\left(2-\sqrt{3}\right)\left(m+2\right)$

$\sqrt{3}m=5-2\sqrt{3}$, $\left(4-\sqrt{3}\right)m=2\sqrt{3}-3$

$m=\frac{5\sqrt{3}-6}{3},\frac{5\sqrt{3}-6}{13}$

Let the angle between $x+y=1$ and the line through $A\left(\frac{10}{3},-\frac{7}{3}\right)$ with the slope $\frac{5\sqrt{3}-6}{3}$ be ϕ. Then

$$tan\phi = \left|\frac{\frac{5\sqrt{3}-6}{3}-(-1)}{1+\frac{5\sqrt{3}-6}{3}(-1)}\right| = 5\sqrt{3}+8$$

If the angle between $x+y=1$ and the line through $A(10/3,-7/3)$ with the slope $\frac{5\sqrt{3}-6}{13}$ be Ψ then

$$tan\Psi = \left|\frac{\frac{5\sqrt{3}-6}{13}-(-1)}{1+\frac{5\sqrt{3}-6}{13}(-1)}\right| = \frac{7+5\sqrt{3}}{19-5\sqrt{3}}$$

$$= \frac{(7+5\sqrt{3})(19+5\sqrt{3})}{19^2-(5\sqrt{3})^2} = \frac{80+50\sqrt{3}}{11} > 0$$

$tan\phi > tan\Psi; \phi > \Psi$

The slope of the refracted ray $= \frac{5\sqrt{3}-6}{3}$

The equation of the refracted ray is $y+\frac{7}{3} = \frac{5\sqrt{3}-6}{3}\left(x-\frac{10}{3}\right)$

$3(5-2\sqrt{3})x-3\sqrt{3}y+13\sqrt{3}-50=0$

51.(ABCD) $12x^2+7xy-12y^2 = 12x^2+16xy-9xy-12y^2 = 4x(3x+4y)-3y(3x+4y) = (3x+4y)(4x-3y)$

The pair of lines given by the first equation has their separate equations $3x+4y=0, 4x-3y=0$ which are at right angle.

Again $12x^2+7xy-12y^2-x+7y-1=0=(3x+4y)(4x-3y)-x+7y-1=(3x+4y-1)(4x-3y+1)$

Thus the four lines are $3x+4y=0,\ 4x-3y=0,\ 3x+4y-1=0$ and $4x-3y+1=0$ of which the first and the third are parallel while the second and the fourth are also parallel. Moreover, the first and the second are at right angle. Hence the lines form a rectangle.

The distance between $3x+4y=0$ *and* $3x+4y-1=0$ is given by

$d =$ distance of $(0,0)$ *from* $3x+4y-1=0$

$\{(0,0)\ \textit{being a point on}\ 3x+4y=0\} = \left|\frac{-1}{\sqrt{3^2+4^2}}\right| = \frac{1}{5}$

The distance between $4x-3y=0$ *and* $4x-3y+1=0$ is given by $d' = \left|\frac{1}{\sqrt{3^2+4^2}}\right| = \frac{1}{5}\ \therefore d=d'$

Hence lines form a square.

52.(ABC) vertex $A \equiv (3,-1), B(x_1, y_1), C(x_2, y_2)$

Equation of median through B be $6x+10y-59=0$

Equation of angle bisector through C be $x-4y+10=0$

$x_2-4y_2+10=0$, $D=\left\{\frac{x_2+3}{2}, \frac{y_2-1}{2}\right\}$ the mid-point of AC is on median through B.

$x_2=10, y_2=5$ i.e., coordinate of $C \equiv (10,5)$

equation of $AC \equiv 6x-7y=25$

Let slope of BC be m, then BC and AC equally inclined to $x-4y+10=0$ we have

$$\frac{\frac{1}{4}-m}{1+\frac{1}{4}m}=\frac{\frac{6}{7}-\frac{1}{4}}{1+\frac{6}{7}\times\frac{1}{4}} \Rightarrow m=-2/9$$

So, equation of $BC = 2x+9y=65$

Also, B is on $2x+9y=65$ *and* $6x+10y-59=0$

By solving if $B \equiv \left\{-\frac{7}{2}, 8\right\}$ equation of $AB \equiv 18x+13y=41$

53.(AB) Here the origin remains fixed. Let the axes be turned about the fixed origin through an angle ϕ in the anticlockwise sense and new coordinates of the point (x, y) becomes (x', y').

Then the equation of transformation will be $x = x'\cos\phi - y'\sin\phi$; $y = x'\sin\phi + y'\cos\phi$

The changed equation will be

$$a(x'\cos\phi - y'\sin\phi)^2 + 2h(x'\cos\phi - y'\sin\phi)(x'\sin\phi + y'\cos\phi) + b(x'\sin\phi + y'\cos\phi)^2 = 0$$

$$\left(a\cos^2\phi + 2h\sin\phi\cos\phi + b\sin^2\phi\right)x^2 + (-a\sin 2\phi + 2h\cos 2\phi + b\sin 2\phi)x'y'$$

$$+\left(a\sin^2\phi - 2h\sin\phi\cos\phi + b\cos^2\phi\right)y^2 = 0$$

The transformed equation of the pair of lines for the new axes will be

$$\left(a\cos^2\phi + h\sin 2\phi + b\sin^2\phi\right)x^2 + \{(b-a)\sin 2\phi + 2h\cos 2\phi\}xy + \left(a\sin^2\phi - h\sin 2\phi + b\cos^2\phi\right)y^2 = 0$$

This equation will not contain xy if $(b-a)\sin 2\phi + 2h\cos 2\phi = 0$ *or* $\tan 2\phi = \frac{2h}{a-b}$

$\phi = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$ This is the required angle.

54. (ACD) $\sec^2(a+1)b + a^2 - 1 = 0$

$1+\tan^2(a+1)b + a^2 - 1 = 0$; $\tan^2(a+1)b + a^2 = 0$

$a = 0$ *and* $\tan(a+1)b = 0$; $a = 0$ *and* $b = n\pi, n \in I$

Equation of line passing through $(0, n\pi)$ and having slope $\frac{1}{2}$

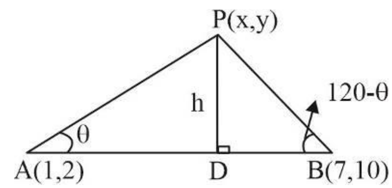
$y - n\pi = \frac{1}{2}(x-0)$; $x - 2y + 2n\pi = 0$

At $n=0$ $x-2y=0$

$n=1$ $x-2y+2\pi=0$

$n=-1$ $x-2y-2\pi=0$

55. (BC) As $\angle APB = 60°$



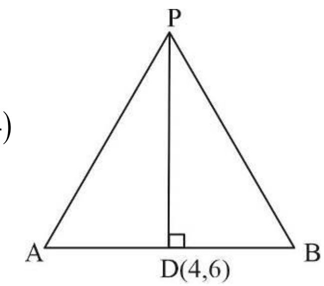
Area will be $\frac{1}{2} \times AP \times PB \times sin(60°) = \frac{1}{2} AB \times h$

$\frac{1}{2} \times (h\, cosec\theta)(h\, cosec(120-\theta)) \times \frac{\sqrt{3}}{2} = \frac{1}{2} \times 10 \times h$

$h = \frac{20}{\sqrt{3}} sin\theta\, sin(120-\theta) = \frac{10}{\sqrt{3}}\left[cos(2\theta - 120) - cos(120)\right]$

For max area h should be max.

So $2\theta - 120 = 0 \Rightarrow \theta = 60° \Rightarrow$ triangle is equilateral. So,

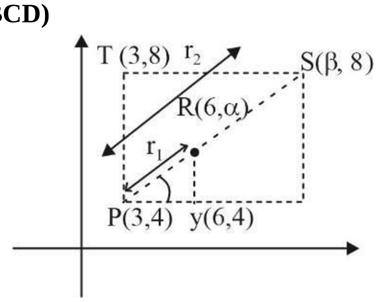


PD will be $\perp$ bisector as well as median equation of PD will be $(y-6) = \frac{-6}{8}(x-4)$

$4y - 24 = -3x + 12$

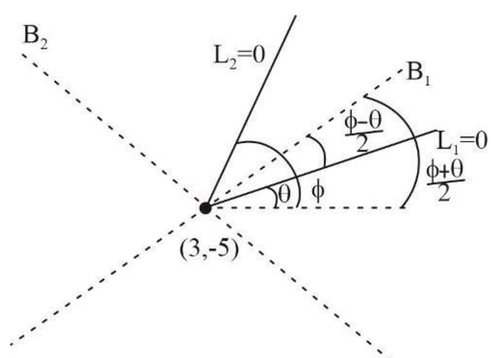
$3x + 4y = 36$ and P lies on right bisector of AB

56. (ABCD)



$PU = (PR)cos\theta \Rightarrow \frac{3}{cos\theta} = PR\, or\ PR = 3sec\theta$; $PT = (PS)sin\theta \Rightarrow \frac{4}{sin\theta} = PS\, or\ PS = 4cosec\theta$

57. (ABC)



Lines given will pass through $(3,-5)$ making angle θ *and* ϕ with x-axis.

Let B_2, B_1 be the angle bisector of lines L_1 *and* L_2.

Then B_1 will be inclined at an angle of $\frac{\theta+\phi}{2}$. So, its equation will be $\frac{x-3}{cos\left(\frac{\theta+\phi}{2}\right)} = \frac{y+5}{sin\left(\frac{\theta+\phi}{2}\right)}$

So, $\alpha = \frac{\theta+\phi}{2}$

Also, B_2 will be at 90° from B_1 so its angle of inclination will be $90+\left(\frac{\theta+\phi}{2}\right) or\, 90+\alpha$

So its equation will be $\frac{x-3}{cos(90+\alpha)}=\frac{y+5}{sin(90+\alpha)}$ *or* $\frac{x-3}{-sin\alpha}=\frac{y+5}{cos\alpha}$

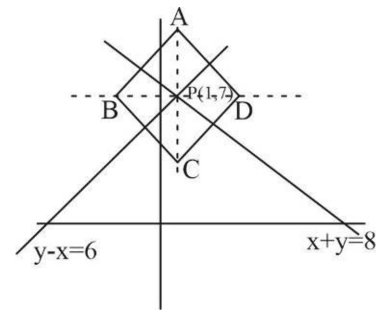
$\beta=-sin\alpha$ *and* $\gamma=cos\alpha$

58. (ABCD) There will be 4 points which will lie equidistant to both line and will be vertices of square will sides 200 and diagonal length $200\sqrt{2}$.

Length of $PC=100\sqrt{2}=AP=BP=PD$

So coordinate of $D\equiv\left[1+100\sqrt{2},7\right]$ $A\equiv\left[1,7+100\sqrt{2}\right]$

$B\equiv\left[1-100\sqrt{2},7\right]$ $C\equiv\left[1,7-100\sqrt{2}\right]$

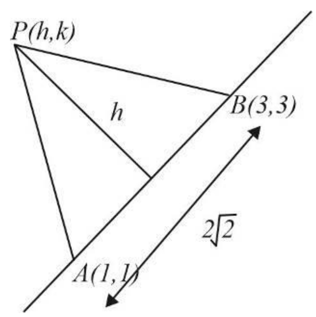


59.(AB) L must be angle bisector of L_1 *and* L_2

L is given by $\frac{3x+4y-1}{5}=\pm\frac{5x-12y+2}{13}$

$14x+112y-23=0$ (+ sign) ; $64x-8y-3=0$ $(-sign)$

60.(CD)



p will be on line parallel to $y=x$ at a perpendicular distance of $\frac{1}{\sqrt{2}}$

locus of P will be $y-x=1$ or $y-x=-1$

61. (ABC) Let $f(x,y)=3x+2y$

$f(1,3)>0, f(5,0)>0$ *and* $f(-1,2)>0$

All three vertices satisfy the inequality $3x+2y\geq 0$, thus inequality holds for all point inside the triangle

Let $f(x,y)=2x-3y-12$

$f(1,3)<0, f(5,0)<0$ *and* $f(-1,2)<0$

All three vertices satisfy the inequality $2x-3y-12\leq 0$, thus the inequality holds for all points lying inside the triangle.

Let $f(x,y)=x+y-6, then\ f(1,3)<0, f(5,0)<0\, and\ f(-1,2)<0$

So, the vertices $(1,3)$ *and* $(-1,2)$ do not satisfy the inequality $2x-y\geq 0$. Hence some points lying inside the triangle do not satisfy the inequality

62.(ABCD) $x_1 + y_1 = 4,\ x_2 + y_2 = 4$

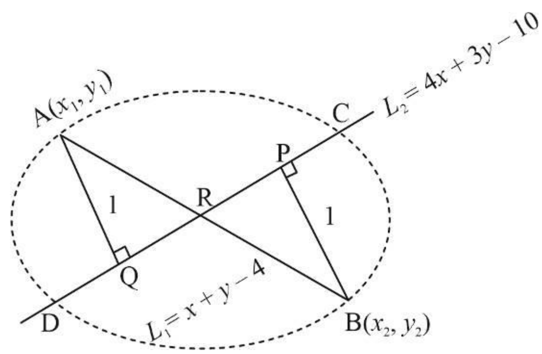
Also, $\left|\dfrac{4x_1 + 3y_1 - 10}{5}\right| = 1$, $\left|\dfrac{4x_2 + 3y_2 - 10}{5}\right| = 1$

$4x_1 + 3y_1 - 10 = \pm 5$

$4x_1 + 3y_1 - 10 = 5$ or $4x_2 + 3y_2 - 10 = -5$ (as (x_1, y_1) *and* (x_2, y_2) are similar)

$x_1 = 3;\ \ x_2 = -7;\ y_1 = 1;\ \ y_2 = 11$

So, $\dfrac{y_1 + y_2}{x_1 + x_2} = -3$



Also L_2 divides L_1 in ratio $-\left\{\dfrac{4(3) + 3(1) - 10}{4(-7) + 3(11) - 10}\right\} = 1$

$R \equiv \{-2, 6\}$

$AR = 5\sqrt{2} \Rightarrow QR = 7 = PR$

$PQ = 14$

$RD = 5\sqrt{2};\ QR = 7 \Rightarrow QD = 5\sqrt{2} - 7$

63.(ACD) Let the lines represented by the given equation be $y = x\tan\alpha$ *and* $y = x\tan\beta$

Then $\tan\alpha + \tan\beta = \dfrac{2\tan\theta}{\sin^2\theta} = \dfrac{2}{\sin\theta\cos\theta} = 4\operatorname{cosec}2\theta$

$$\tan\alpha\tan\beta = \frac{\tan^2\theta + \cos^2\theta}{\sin^2\theta} = \sec^2\theta + \cot^2\theta$$

$$\tan\alpha - \tan\beta = \pm\sqrt{\frac{4}{\sin^2\theta\cos^2\theta} - 4\left(\sec^2\theta + \cot^2\theta\right)} = \pm 2\sqrt{\frac{1 - \left(\sin^2\theta + \cos^4\theta\right)}{\sin^2\theta\cos^2\theta}}$$

$$= \pm 2\sqrt{\frac{\left(\cos^2\theta - \cos^4\theta\right)}{\sin^2\theta\cos^2\theta}} = \pm 2$$

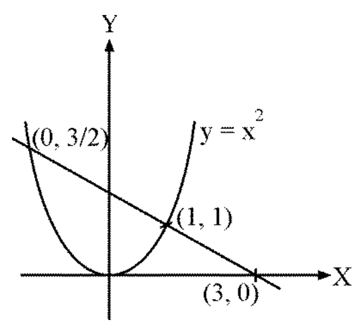
And $\dfrac{\tan\alpha}{\tan\beta} = \dfrac{4\operatorname{cosec}2\theta + 2}{4\operatorname{cosec}2\theta - 2} = \dfrac{2 + \sin 2\theta}{2 - \sin 2\theta}$

64. **[A-p] [B-s] [C-q] [D-s]**

(A) $2a^2 + a - 3 < 0$

$2a + 3 \quad a - 1 < 0$

$\Rightarrow \qquad a \in \left(\dfrac{-3}{2}, 1\right)$



But $a > 0$

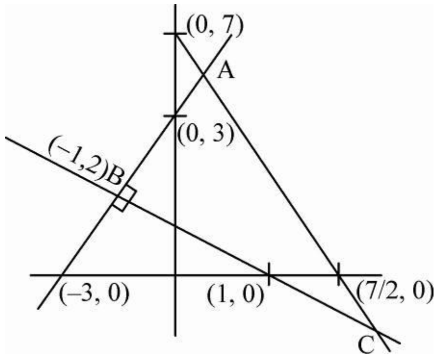
Hence no. of integral values of $a = 0$

(B) Perpendicular distance of $P\ \lambda, 4 - \lambda$ from $4x + 3y = 10$

$= \dfrac{|4\lambda + 3\ \ 4 - \lambda\ \ - 10|}{5} = 1 \quad \Rightarrow \qquad |\lambda + 2| = 5 \Rightarrow \lambda = -2 \pm 5 = 3, -7$

$$P \equiv \ 3,1\ ,\ -7,+11$$

(C) Point $B \equiv \ -1,2$



Aliter : The given triangle is right angled.

$\therefore$ Co-ordinates of orthocentre is $(-1,\ 2)$, by solving the equation $x+y=1$ and $x-y+3=0$

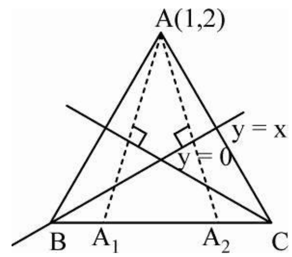
(D) Images of A w.r.t. $y = x$ and $y = 0$ lies on BC which are (2, 1), (1, –2)

$\therefore$ Equation of BC is $y = 3x-5$

Perpendicular distance of A from

$$BC = \frac{|3-2-5|}{\sqrt{10}}$$

$$d\ \ A, BC\ = \frac{4}{\sqrt{10}} \qquad \Rightarrow \qquad \sqrt{10}d\ \ A, BC\ = 4$$



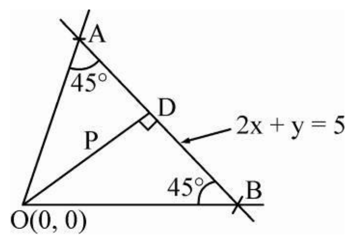
65. **[A-q] [B-p] [C-s] [D-r]**

(A) Since the triangle is isosceles, so
$OD = AD = BD$

$$OD = P = \frac{|5|}{\sqrt{5}}$$

Area of $\Delta OAB = \frac{1}{2}(2P)P = P^2$

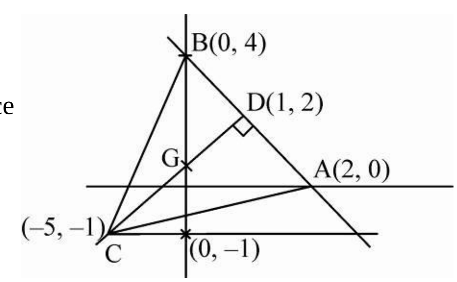
Area = 5



(B) Slope of CD is $\frac{1}{2}, C \equiv (-5,-1)$

Since G is centroid of triangle ABC,so Perpendicular distance from G to AB = $\frac{1}{3}$ (Perpendicular distance from C to AB)

$$= \frac{1}{3}\left(\frac{|-10-1-4|}{\sqrt{4}+1}\right) = \sqrt{5}$$

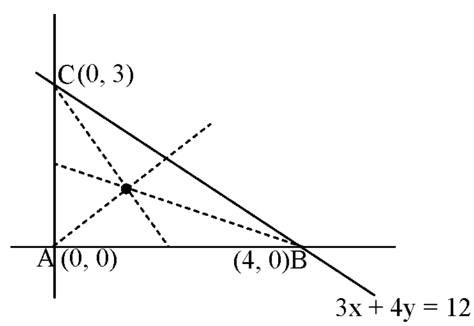


(C) Point P must lie on at least one of the angle bisectors.
Incentre,

$$I \equiv \left(\frac{4\ 0\ +3\ 4\ +5\ 0}{4+3+5}, \frac{4\ 3\ +3\ 0\ +5\ 0}{4+3+5}\right)$$

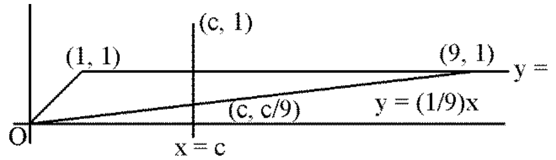
$I \equiv \ 1, 1$

$P \equiv \ 1, 1$ only



(D) $2\times\frac{1}{2}\left(1-\frac{c}{9}\right)(9-c)=\frac{1}{2}\times 8\times 1 \Rightarrow c=3$

Or $c = 15$ which is not possible



66. **[A-s] [B-p] [C-q] [D-r]**

(A) Let equation of line is $y-4=m(x-1)(m<0)$

$\therefore \quad OA = 1+\frac{4}{(-m)}$ and $OB = 4+(-m) \Rightarrow \quad OA+OB = 5+\frac{4}{(-m)}+(-m)$

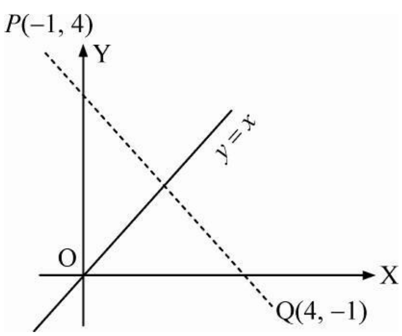
Apply AM$\geq$GM on $\frac{4}{(-m)}$ & $(-m)$

$$\therefore \quad \frac{\frac{4}{(-m)}+(-m)}{2} \geq \sqrt{\frac{4}{(-m)}\times(-m)}$$

$$\frac{4}{(-m)}+(-m)\geq 4 \quad \Rightarrow \quad 5+\frac{4}{(-m)}+(-m)\geq 9 \quad \Rightarrow \quad OA+OB\geq 9$$

(B) Image of point $Q(4,-1)$, w.r.t. to line mirror $y=x$ is $P(-1,4)$

$\therefore \quad PQ = \sqrt{50} = 4\sqrt{2}$



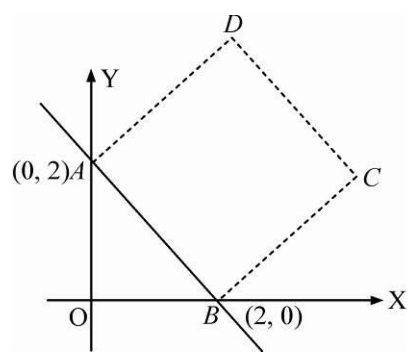
(C) From figure,

Perpendicular distance of O from $AB = \sqrt{2}$

Perpendicular distance of O from $CD = 3\sqrt{2}$

Perpendicular distance of O from $AD = \sqrt{2}$

Perpendicular distance of O from $BC = \sqrt{2}$



(D) Equation of line

$$x = 4+\frac{\lambda}{\sqrt{2}} \text{ \& } y = -1+\sqrt{2}\lambda \text{ can be written as } 2x = 8+\sqrt{2}\lambda \text{ \& } y=-1+\sqrt{2}\lambda$$

$\therefore$ Equation of line is $2x-y=9$

$\therefore$ Length of x-intercept $=\frac{9}{2}$

67. **[A-s] [B-p] [C-q] [D-r]**

(A) $6x^2 - axy - 3y^2 - 24x + 3y + \beta = 0$

$\therefore$ Intersection point of lines lies on x-axis.

Put $y = 0$.

$\Rightarrow \quad 6x^2 - 24x + \beta = 0$ must have equal real roots

$\Rightarrow \quad D = 0 \Rightarrow (24)^2 - 24\beta = 0 \;\Rightarrow\; \beta = 24$

Apply condition of pair of lines, i.e, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

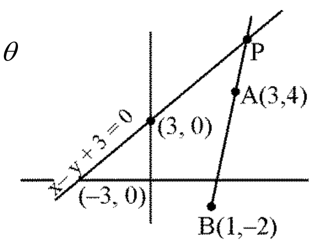
$$\Rightarrow \quad 6\;\; -3\;\; \beta + \;\; -\alpha\;\; -12\;\; \frac{3}{2} - 6\left(\frac{3}{2}\right)^2 - \;\; -3\;\; 12\;\;^2 - \beta\left(\frac{\alpha}{2}\right)^2 = 0$$

$$\Rightarrow \quad -18\;\; 24\;\; + 18\alpha - \frac{27}{2} + 3\;\; 144\;\; - 6\alpha^2 = 0 \;\;\Rightarrow\; 6\alpha^2 - 18\alpha + \frac{27}{2} = 0$$

$\Rightarrow \quad 4\alpha^2 - 12\alpha + 9 = 0$

$\alpha = 3/2$ Hence, $20\alpha - \beta = 6$

(B) $3x + 7y + 11\;\; \sec\theta + \;\; 5x - 3y - 11\;\; \operatorname{cosec}\theta = 0$ passes through(1, –2) $\forall$ permissible θ

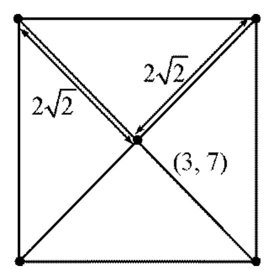


$B \equiv \;\; 1, -2$

It is clear that $|PA - PB| \le AB$

$\Rightarrow \quad |PA - PB|_{\max} = AB = \sqrt{6^2 + 2^2} \;=\; 2\sqrt{10} \Rightarrow n = 5$

(C) Vertices of square are



$$\left(3 \pm 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right), 7 \pm 2\sqrt{2}\,\frac{1}{\sqrt{2}}\right),$$

$$\left(3 \pm 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right), 7 \pm 2\sqrt{2}\,\frac{1}{\sqrt{2}}\right)$$

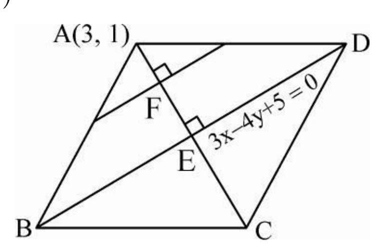
i.e., (5, 9), (1, 5), (1, 9), (5, 5)

$\therefore \quad \max\;\; y_1, y_2, y_3, y_4\;\; - \min\;\; x_1, x_2, x_3, x_4\;\; = 9 - 1 = 8$

(D) Point E

$$\frac{x-3}{3} = \frac{y-1}{-4} = -\frac{9-4+5}{25} = -\frac{2}{5} \Rightarrow E \equiv \left(\frac{9}{5}, \frac{13}{5}\right)$$

Line, L passing through midpoint of sides AB and AD passes through F.



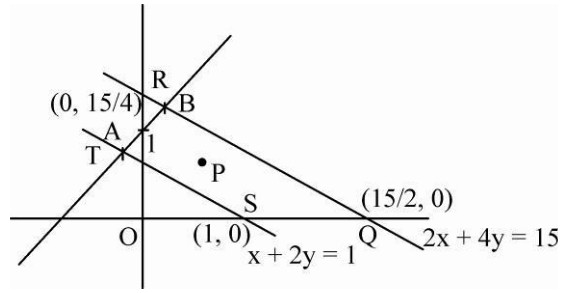
$$F \equiv \left(\frac{3 + \frac{9}{5}}{2}, \frac{\frac{13}{5} + 1}{2}\right) \equiv \left(\frac{12}{5}, \frac{9}{5}\right)$$

Equation of line L is $3x - 4y = \frac{36}{5} - \frac{36}{5} = 0$

$\Rightarrow \quad 3x - 4y = 0 \Rightarrow |a + b + c| = |3 - 4| = 1$

68. **[A-r]** **[B-p]** **[C-s]** **[D-q]**

(A) $P\left(1+\frac{t}{\sqrt{2}},\ 2+\frac{t}{\sqrt{2}}\right)$ point P lies on the line $y = x+1$



$$A = \left(\frac{-1}{3}, \frac{2}{3}\right), B = \left(\frac{11}{6}, \frac{17}{6}\right)$$

$$1+\frac{t}{\sqrt{2}} \in \left(\frac{-1}{3}, \frac{11}{6}\right)$$

$$t \in \left(\frac{-4\sqrt{2}}{3}, \frac{5\sqrt{2}}{6}\right)$$

Aliter : For point *P* lies in between parallel lies

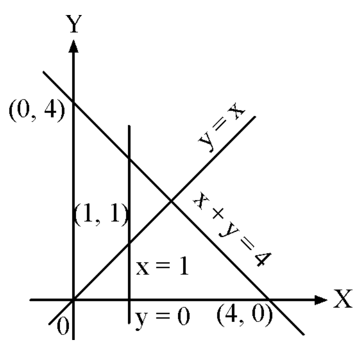
$\Rightarrow$ With respect to line QR, point *P* and origin are on same side and with respect to line *ST*, point *P* and origin are on opposite side

$$\Rightarrow \quad 2\left(1+\frac{t}{\sqrt{2}}\right)+4\left(2+\frac{t}{\sqrt{2}}\right)-15<0 \text{ and } \left(1+\frac{t}{\sqrt{2}}\right)+2\left(2+\frac{t}{\sqrt{2}}\right)-1>0 \ \Rightarrow\ t \in \left(\frac{-4\sqrt{2}}{3}, \frac{5\sqrt{2}}{6}\right)$$

(B) $P\left((2-t)x_1+(t-1)x_2,\ (2-t)y_1+(t-1)y_2\right)$ divides (x_1, y_1), (x_2, y_2) internally in ratio $(t-1) : (2-t)$

$\therefore \quad (t-1)(2-t)>0 \Rightarrow t \in (1, 2)$

(C) $t \in (0, 1)$ (See from figure)



(D) For A :

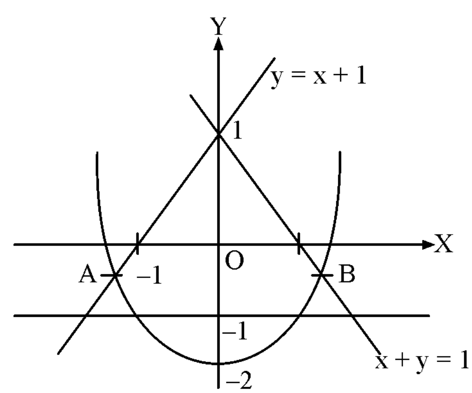
$x^2-2 = x+1$

$x^2-x-3 = 0$

$x = \frac{1\pm\sqrt{13}}{2}$; $\quad x = \frac{1-\sqrt{3}}{2}$

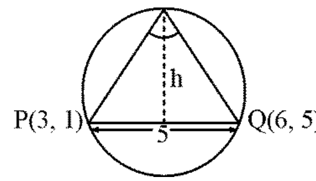
Similarly for *B*, $\quad x = \frac{\sqrt{13}-1}{2}$

$\therefore \quad t \in \left(\frac{1-\sqrt{13}}{2}, -1\right) \cup \left(1, \frac{\sqrt{13}-1}{2}\right)$



69. **[A-r] [B-s] [C-q] [D-p]**

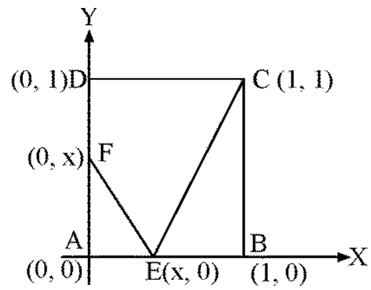
(A) $\Delta = \frac{1}{2} \times 5 \times h = 7 \Rightarrow h = \frac{14}{5}$



But h max = Radius of circle with diameter $PQ = \frac{5}{2} < \frac{14}{5}$

$\Rightarrow$ No such triangle exists.

(B) Area of *CDEF*,



$$A(x) = 1 - \frac{1}{2}x^2 - \frac{1}{2}(1-x) = \frac{1+x-x^2}{2}$$

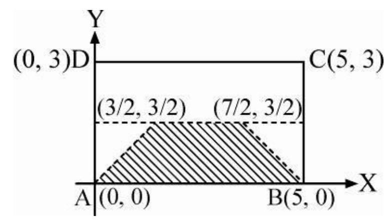
$$A_{\max} = A\left(\frac{1}{2}\right) = \frac{1+\frac{1}{2}-\frac{1}{4}}{2} = \frac{5}{8}$$

At $x = \frac{1}{2}$

(C) $d(P, AB) \le d(P < BC)$

$d(P, AB) \le d(P < CD)$

$d(P, AB) \le d(P < AD)$



$\Rightarrow$ P lies in region as shown $\quad Area = \frac{1}{2} \times \frac{3}{5} \times (5+2) = \frac{21}{4}$

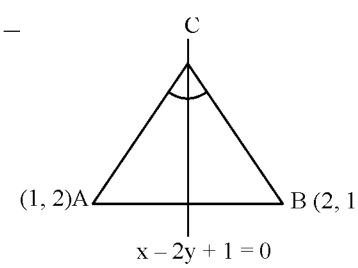
(D) $a + 2hm + bm^2 = 0$

Where $m = \frac{y}{x} \quad \Rightarrow \quad m_1 + m_1^2 = -\frac{2h}{b}; \quad m_1^3 = \frac{a}{b}$

$$\Rightarrow -\frac{8h^3}{b^3} = \left(m_1 + m_1^2\right)^3 = m_1^3 + m_1^6 + 3m_1^3\left(m_1 + m_1^2\right) = \frac{a}{b} + \frac{a^2}{b^2} + \frac{3a}{b}\left(-\frac{2h}{b}\right)$$

$$\Rightarrow -\frac{8h^3}{b^3} = \frac{ab + a^2 - 6ah}{b^2} \Rightarrow ab(a+b) - 6abh + 8h^3 = 0 \Rightarrow \alpha + \beta = 8 -$$

70.(2) Image of A say A' w.r.t. $x - 2y + 1 = 0$ lies on BC.



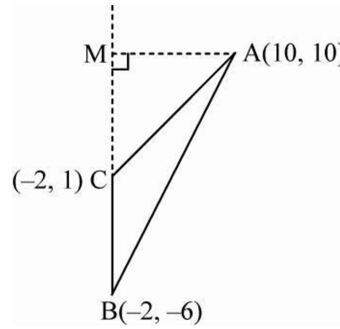
$$\frac{x-1}{1} = \frac{y-2}{-2} = -2\frac{(1-4+1)}{1+2^2} = \frac{4}{5}$$

$$A' \equiv \left(\frac{9}{5}, \frac{2}{5}\right)$$

Equation of BC joining $A'\left(\frac{9}{5}, \frac{2}{5}\right)$ and $B(2, 1)$ is $y - 1 = \frac{1-\frac{2}{5}}{2-\frac{9}{5}}(x-2) = \frac{3}{1}(x-2)$

$3x - y - 5 = 0 \Rightarrow a + b = 3 - 1 = 2$

71.(6) Let perpendicular bisector of AB is $3x+4y-20=0$ and perpendicular bisector of AC is $8x+6y-65=0$



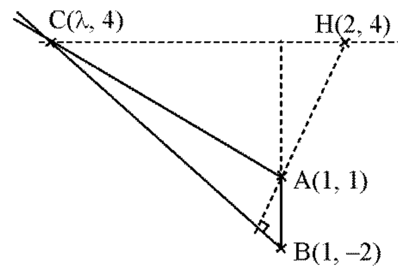
$\Rightarrow$ Image of A w.r.t. $3x+4y-20=0$ is B and image of A w.r.t $8x+6y-65=0$ is C.

For $B, \dfrac{x-10}{3}=\dfrac{y-10}{4}=-2\left(\dfrac{30+40-20}{25}\right) \quad \Rightarrow \quad B\equiv(-2,-6)$

For $C, \dfrac{x-10}{8}=\dfrac{y-10}{6}=-2\left(\dfrac{80+60-65}{100}\right) \quad \Rightarrow \quad C\equiv(-2,1)$

Area of $\Delta ABC=\dfrac{1}{2}(10+2)(1+6)=42 \quad \Rightarrow$ Required value $=6$

72.(2) $a+c-2b=0$



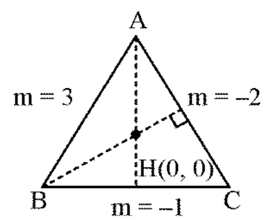
$\Rightarrow$ $ax+by+c=0$ passes through fixed point (1, –2) $\Rightarrow B\equiv\ 1,-2$

Slope of $BC=-\dfrac{2-1}{4-1}=\dfrac{4+2}{\lambda-1}$

$\lambda-1=-18 \quad \Rightarrow \quad \lambda=-17$

$C\equiv(-17,4)\equiv(h,k) \qquad \therefore \quad 2h+9k=-34+36=2$

73.(7) Let slopes of BC, CA, AB be –1, –2, 3 respectively.



$\because$ Slope of BC is $-1 \quad \Rightarrow \quad$ Slope of $AD=1$

$\therefore$ Equation of AD is $y=x \quad \Rightarrow$ co-ordinate of $A(x_1,\ x_1)$

Similarly co-ordinate of $B\left(x_2,\ \dfrac{x_2}{3}\right)$ & $C\left(x_3,\ -\dfrac{x_3}{3}\right)$

$\because$ Slope of $BC=\dfrac{\frac{x_2}{2}+\frac{x_3}{3}}{x_2-x_3}=\dfrac{3x_2+2x_3}{6(x_2-x_3)}=-1 \quad \Rightarrow \quad 9x_2=4x_3$

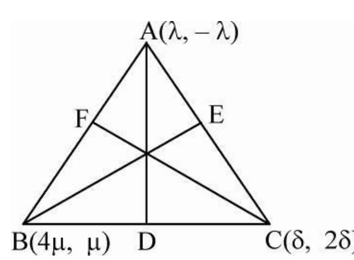
$\because$ Slope of $CA=\dfrac{x_1+\frac{x_3}{3}}{x_1-x_3}=-2=\dfrac{3x_1+x_3}{3\ x_1-x_3}\Rightarrow 9x_1=5x_3$

$$G\equiv\ h,k\ =\left(\frac{x_1+x_2+x_3}{3},\frac{x_1+\frac{x_2}{2}-\frac{x_3}{3}}{3}\right)$$

$3h=x_3+\dfrac{5x_3}{9}+\dfrac{4x_3}{9}=2x_3 \qquad$ **…(1)** $\quad$ and $\qquad 3k=\dfrac{5x_3}{9}+\dfrac{2x_3}{9}-\dfrac{x_3}{9}=\dfrac{4x_3}{9} \qquad$ **…(2)**

From eqs. (1) and (2), we get $\dfrac{k}{h}=\dfrac{2}{9}\Rightarrow y=\dfrac{2}{9}x;\ b-a=9-2=7$

74.(6) Equation of AD, $x+y=0$



Equation of line BE, $x-4y=0$

Equation of line CF, $2x-y=0$

Let centroid is $(x,\ \ y)$

$x=\dfrac{\lambda+4\mu+\delta}{3}$ …(i) and $\quad y=\dfrac{\mu-\lambda+2\delta}{3}$ …(ii)

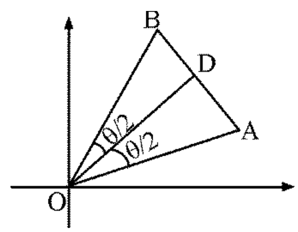
$$BC \perp AD;\ \frac{\mu-2\delta}{4\mu-\delta}=1 \quad \Rightarrow \quad \mu-2\delta=4\mu-\delta \Rightarrow 3\mu=-\delta$$

$$AC \perp BE;\ \frac{2\delta+\lambda}{\delta-\lambda}=-4 \quad \Rightarrow \quad 2\delta=\lambda$$

So we can say $\frac{\lambda}{6}=\frac{\delta}{3}=\frac{\mu}{-1}$ by using equation (i) and (ii),

$$\frac{x}{y}=\frac{6-4+3}{-1-6+6}=-5 \quad \Rightarrow \quad x=-5y \quad \Rightarrow \quad x+5y=0$$

75.(8) Slope of OA and OB are respectively $\frac{7}{17}$ and 1, OD is $\perp$ to AB.



Let equation of AB is $x\cos\alpha + y\sin\alpha = \sqrt{13}$,

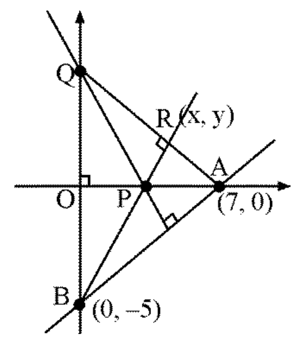
where OD makes α angle with x-axis.

So $\tan\theta = \dfrac{1-\frac{7}{17}}{1+\frac{7}{17}} = \frac{10}{24} = \frac{5}{12} \Rightarrow \tan\frac{\theta}{2} = \frac{1}{5}$

$$\tan\alpha = \tan\left(\frac{\theta}{2}+\tan^{-1}\left(\frac{7}{17}\right)\right) = \frac{\frac{1}{5}+\frac{7}{17}}{1-\frac{7}{5\ .\ 17}} = \frac{2}{3}$$

$$\sin\alpha = \frac{2}{\sqrt{13}},\quad \cos\alpha = \frac{3}{\sqrt{13}}$$

So equation of AB is $3x+2y=13$.



76.(2) As x and y-axis perpendicular the point P becomes orthocenter of triangle ABQ, so R is foot of 3[rd] altitude.

$$\Rightarrow \quad \left(\frac{y+5}{x}\right)\left(\frac{y}{x-7}\right)=-1 \quad \Rightarrow \quad x^2+y^2-7x+5y=0$$

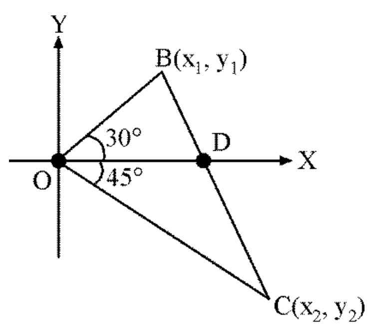
77.(2) Taking O as origin

And x –axis as the median

$$\frac{y_1}{x_1}=\tan 30^o=\frac{1}{\sqrt{3}}$$

$$\frac{y_2}{x_2}=\tan 135^o=-1$$

$$y_1=\frac{x_1}{\sqrt{3}},\ y_2=-x_2$$



Also $\dfrac{x_1+x_2}{2}=\left(11-6\sqrt{3}\right)^{\frac{-1}{2}}=p$

$$\frac{y_1+y_2}{2}=0$$

$$x_1=\frac{2\sqrt{3}p}{\sqrt{3}+1},\ x_2\,\frac{2p}{\sqrt{3}+1},\ y_1=\frac{2p}{\sqrt{3}+1},\ y_2=-\frac{2p}{\sqrt{3}+1}$$

$BC = 2$.

78.(3) Area of $\Delta ABC = 2k$

Since $AD \perp BE$, So

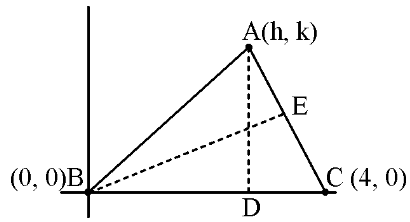
$$k^2 + (h+4)(h-2) = 0$$

And $AC = 3$, so

$$(h-4)^2 + k^2 = 9$$

Hence $k^2 = \dfrac{11}{4}$

$$\Delta = 3$$



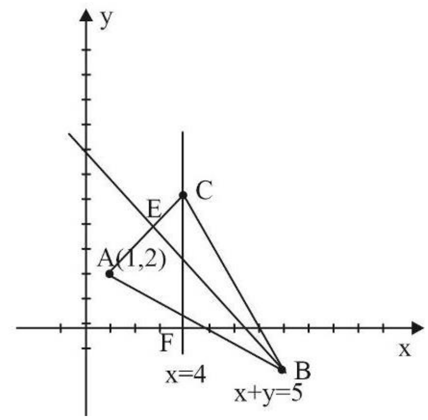
79.(13) Any point B on $x + y = 5$ is $(h, 5-h)$ and C on $x = 4$ is $(4, k)$

Midpoint of AB is $\left(\dfrac{1+h}{2}, \dfrac{7-h}{2}\right)$ which lies on $x = 4$, so $\dfrac{1+h}{2} = 4 \Rightarrow h = 7$

Thus, the coordinates of B are $(7, -2)$

Again, the midpoint of AC is $\left(\dfrac{5}{2}, \dfrac{2+k}{2}\right)$ which lies on $x + y = 5$, so $k = 3$.

Thus, the coordinates of C are $(4, 3)$



80.(3) Let the slope of the line be $tan\theta$. Any point on the line is $(-2 + r\cos\theta, -3 + r\sin\theta)$. If $AB = r_1$, then B has coordinates $(-2 + r_1\cos\theta, -3 + r_1\sin\theta)$, which lies ion $x + 3y = 9$

$$-2 + r_1\cos\theta - 9 + 3r_1\sin\theta - 9 = 0$$

$$r_1 = \frac{20}{\cos\theta + 3\sin\theta}$$

If $AC = r_2$ then C has coordinates $(-2 + r_2\cos\theta, -3 + r_2\sin\theta)$ which lies on $x + y + 1 = 0$

$$-2 + r_2\cos\theta - 3 + r_2\sin\theta + 1 = 0 \Rightarrow r_2 = \frac{4}{\cos\theta + \sin\theta}$$

Given $r_1 r_2 = 20$

$$\left|\left(\frac{20}{\cos\theta + 3\sin\theta}\right)\left(\frac{4}{\cos\theta + \sin\theta}\right)\right| = 20$$

$$|(\cos\theta + 3\sin\theta)(\cos\theta + \sin\theta)| = 4$$

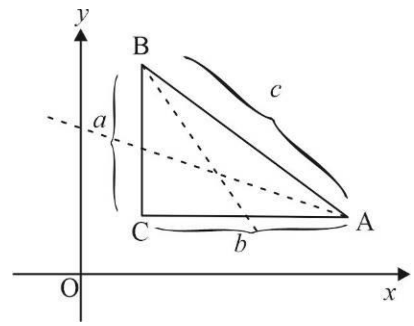
$$\cos^2\theta + 3\sin^2\theta + 4\cos\theta\sin\theta = \pm 4$$

$$1 + 3\tan^2\theta + 4\tan\theta = \pm 4\left(1 + \tan^2\theta\right)$$

$$7\tan^2\theta + 4\tan\theta + 5 = 0 \text{ or } \tan^2\theta - 4\tan\theta + 3 = 0$$

The first equation has no real roots, so $\tan^2\theta - 4\tan\theta + 3 = 0 \Rightarrow \tan\theta = 1 \text{ or } 3$

81. (9) If the slopes of the medians be $m_1 and\, m_2$, then

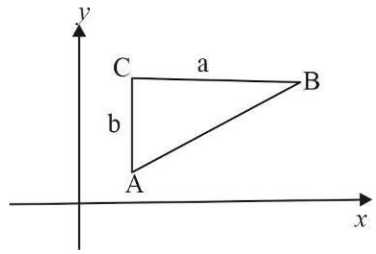


$$m_1 = \frac{a/2}{b} = 3,\ m_2 = \frac{a}{b/2}$$

(a and b are the length of the sides)

$$\frac{m_1}{m_2} = \frac{a \times b}{2b \times a \times 2} = \frac{1}{4}$$

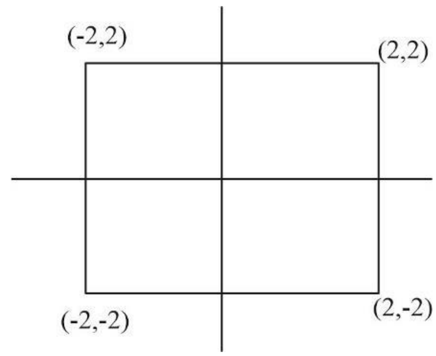
$$m_2 = 12$$



$$m_1 = \frac{b}{a/2} = 3;\ m_2 = \frac{b/2}{a}$$

$$\frac{m_1}{m_2} = \frac{2b \times 2a}{a \times b} = 4;\ m_2 = 3/4$$

82.(16) Point P is (x, y)



Given $max\{|x|,|y|\} = 2$

Case I: $|x| = 2$

$x = +2$, and $x = -2$

Then $|y| \le 2$; $-2 \le y \le 2$

Case II: $|y| = 2$

$y = 2 \text{ and } y = -2$

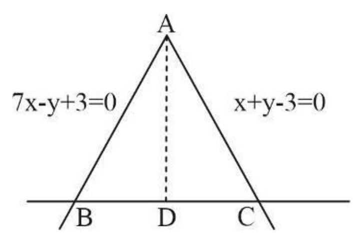
Then $|x| \le 2$; $-2 \le x \le 2$

Such curve will be a square of side 4 units and area will be 16 square units.

83. (3) $L_1 : 7x - y + 3 = 0$

$L_2 : -x - y + 3 = 0$

$a_1 a_2 + b_1 b_2 < 0$



Bisector with positive sign will give acute angle bisector

$$\frac{7x - y + 3}{\sqrt{50}} = +\left(\frac{-x - y + 3}{\sqrt{2}}\right) = 3x + y - 3 = 0$$

equation of line BC with slope $-\frac{1}{m_{AD}}$ and passing through $(1, -10)$ (from given relation)

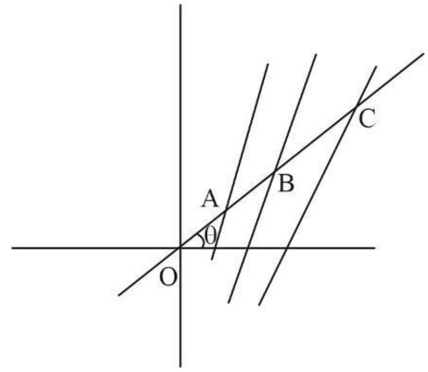
is $y + 10 = \frac{1}{3}(x - 1)$

$x - 3y - 31 = 0$

$a = 1, b = -3, c = -31$

$2a + 10b - c = 2 - 30 + 31 = 3$

84.(1.80) Let $OA = r_1, OB = r_2, OC = r_3$



And line 'L' makes angle θ with passing through origin $(y = mx\ form)$

Point $A(r_1 \cos\theta, r_1 \sin\theta); B(r_2 \cos\theta, r_2 \sin\theta); C(r_3 \cos\theta, r_3 \sin\theta)$

Now as point A, B and C lies on the lines L_1, L_2 and L_3 respectively,

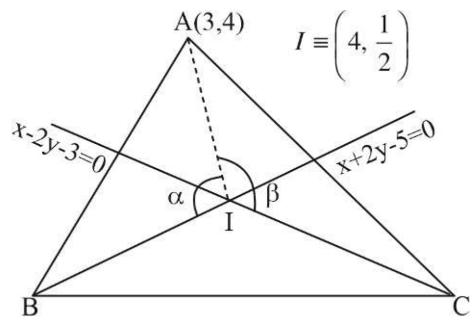
$2r_1 \cos\theta + 3r_1 \sin\theta - 5 = 0$

$\frac{1}{r_1} = \frac{2\cos\theta + 3\sin\theta}{5}, \quad \frac{1}{r_2} = \frac{\cos\theta + 2\sin\theta}{5}, \frac{1}{r_3} = \frac{6\cos\theta + 4\sin\theta}{5},$

Now, $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{9}{5}(\cos\theta + \sin\theta) = \frac{k(a+b)}{OP}$ $[(a, b) \equiv (r\cos\theta, r\sin\theta)$ where $OP = r]$

$k = \frac{9}{5}$

85.(4) $AI = \frac{\sqrt{53}}{2}, \ m_{AI} = \frac{4 - \frac{1}{2}}{-1} = \frac{-7}{2}$



$\tan\alpha = \left| \frac{\frac{-7}{2} + \frac{1}{2}}{1 + \frac{7}{4}} \right|$

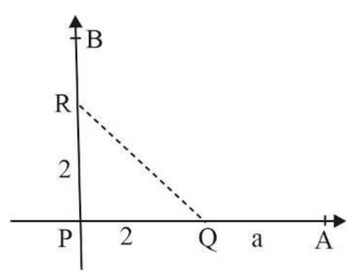
$\beta = 90 + \frac{B}{2}; \quad \tan\beta = \left| \frac{\frac{-7}{2} - \frac{1}{2}}{1 - \frac{7}{4}} \right| = \left| \frac{\frac{8}{2}}{\frac{3}{4}} \right| = 16/3$

$\tan\beta = \tan\left(90 + \frac{B}{2}\right) = \frac{16}{3} \quad \Rightarrow \quad \sin\frac{B}{2} = \frac{3}{\sqrt{265}}$

Now, $\frac{AI}{\sin\frac{B}{2}} = \frac{AB}{\sin\alpha} \quad \Rightarrow \quad \frac{\sqrt{53} \times \sqrt{265}}{2 \times 3} = \frac{AB}{12} \times \sqrt{265} \quad \Rightarrow \quad AB = 2\sqrt{53}$

Given $AB = k.AI \quad \Rightarrow \quad 2\sqrt{53} = k.\frac{\sqrt{53}}{2} \quad \Rightarrow \quad k = 4$

86.(3) Let $QA = a$



Given $QA.RB = PR^2 = 4$

$RB = \frac{4}{a}$

Point $A(2 + a, 0); B\left(0, 2 + \frac{4}{a}\right)$

Now equation of line AB; $y = \frac{\frac{4}{a} + 2}{-(2 + a)}(x - 2 - a)$

$-a(2 + a)y = (4 + 2a)(x - 2 - a)$

$-ay = 2(x-2-a)$

$2x-4+ay-2a=0$

$(x-2)+\frac{a}{2}(y-2)=0$

Always passes through $(2,2)$

If it satisfies $ax+by-6=0$ then $2a+2b-6=0$

$a+b=3$

87.(16) As rectangle $PQSR$ is a cyclic quadrilateral.

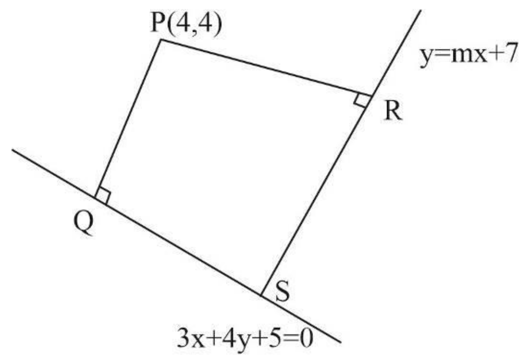
ΔPQR will have maximum area if $\angle QPR = \angle QSR = 90$

$m_{QS}.m_{SR}=-1$

$\left(-\frac{3}{4}\right)m=-1$

$m=\frac{4}{3}$

$12m = 12\times\frac{4}{3}=16$



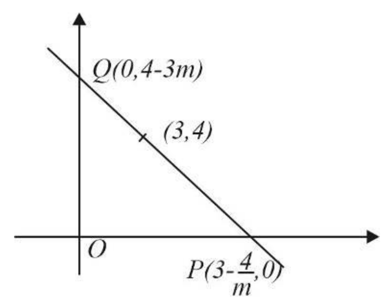
88.(24) Line passing through the point $(3,4)$ is $y-4=m(x-3)$

As slope m is negative here $ar(\Delta OPQ)$ $\Delta=\frac{1}{2}(4-3m)\left(3-\frac{4}{m}\right)$

$\Delta=\frac{1}{2}\left[24+\left(-9m-\frac{16}{m}\right)\right]$

By AM-GM inequality $\frac{-9m-\frac{16}{m}}{2}\geq\sqrt{9\times16}$

$\left(-9m-\frac{16}{m}\right)_{min}=24$, $\Delta_{min}=\frac{1}{2}\times48=24$



89.(6) Let m and m^2 be the slopes of the lines $m+m^2=-\frac{2h}{b}$ ………..(i) and $m^3=\frac{a}{b}$ ………..(ii)

Cubing equation (i)

$\left(m+m^2\right)^3=-\frac{8h^3}{b^3}$

$\Rightarrow\ m^3+m^6+3mm^2\left(m+m^2\right)=-\frac{8h^3}{b^3}$

$\Rightarrow\ \frac{a}{b}+\frac{a^2}{b^2}+\frac{3a}{b}\left(-\frac{2h}{b}\right)=-\frac{8h^3}{b^3}$

$\frac{a(a+b)}{b^2}+\frac{8h^3}{b^3}=\frac{6ah}{b^2}\ \Rightarrow\ \frac{(a+b)}{h}+\frac{8h^3}{ab}=6$

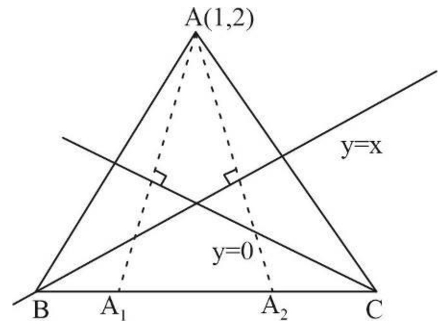
90.(4) Images of A w.r.t $y = x$ *and* $y = 0$ lies on BC which are $(2,1),(1,-2)$

Equation of BC is $y = 3x - 5$

Perpendicular distance of A from $BC = \frac{|3-2-5|}{\sqrt{10}}$

$d(A,BC) = \frac{4}{\sqrt{10}}$

$\sqrt{10}d(A,BC) = 4$

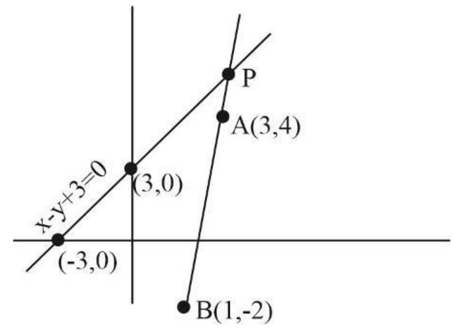


91.(5) $(3x+7y+11)\sec\theta + (5x-3y-11)\operatorname{cosec}\theta = 0$ passes through intersection of $3x+7y+11=0$ and $5x-3y-11=0$ for permissible values of θ given by $(1,-2)$

$B \equiv (1,-2)$

It is clear that $|PA - PB| \le AB$ $(Triangle\,inequality)$

$|PA-PB|_{max} = AB = \sqrt{6^2+2^2} = 2\sqrt{10} \Rightarrow n = 5$



92.(7) Let the slopes of BC, CA and AB be $-1,-2,3$ respectively and orthocenter be *H*.

Slope of $AH = 1 \Rightarrow A = (x_1, x_1)$

Slope of $BH = \frac{1}{2} \Rightarrow B = \left(x_2, \frac{1}{2}x_2\right)$

Slope of $CH = -\frac{1}{3} \Rightarrow C = \left(x_3, -\frac{x_3}{3}\right)$

Slope of $BC = \frac{\frac{x_2}{2}+\frac{x_3}{3}}{x_2 - x_3} = \frac{3x_2+2x_3}{6(x_2-x_3)} = -1 \Rightarrow 9x_2 = 4x_3$

Slope of $CA = \frac{x_1 + \frac{x_3}{3}}{x_1 - x_3} = -2 = \frac{3x_1+x_3}{3(x_1-x_3)} \Rightarrow 9x_1 = 5x_3$

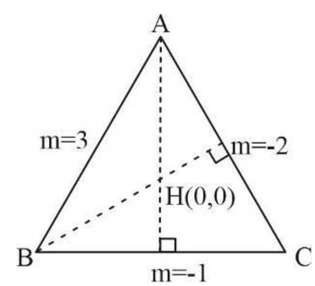
$G \equiv (h,k) = \left(\frac{x_1+x_2+x_3}{3}, \frac{x_1+\frac{x_2}{2}-\frac{x_3}{3}}{3}\right)$

$3h = x_3 + \frac{5x_3}{9} + \frac{4x_3}{9} = 2x_3$ ……….(i)

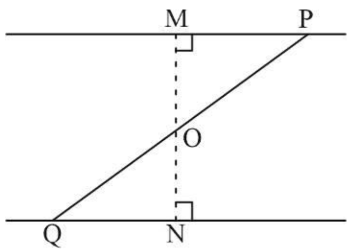
$3k = \frac{5x_3}{9} + \frac{2x_3}{9} - \frac{x_3}{9} = \frac{4x_3}{9}$ ……..(ii)

From equation (i) and (ii), we get

$\frac{k}{h} = \frac{2}{9} \Rightarrow y = \frac{2}{9}x; b - a = 9 - 2 = 7$

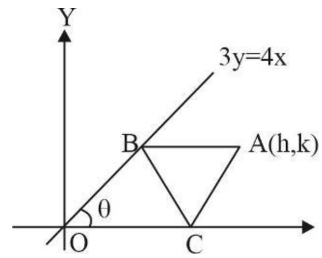


93.(4)



$$\frac{OQ}{OP}=\frac{OM}{ON}=\frac{6/5}{|c|/10}=\frac{3}{4} \Rightarrow |c|=16 \Rightarrow c=\pm 16 \Rightarrow k=4$$

94.(2) Let $\angle BOC=\theta$ then $tan\,\theta=\frac{4}{3}$



$sin\,\theta=\frac{4}{5}$ and $cos\,\theta=\frac{3}{5}$

If $OC=t$, then $OB=BA=AC=t$

Coordinates of C are $(t,0)$, coordinates of B are $(OB\,cos\,\theta, OB\,sin\,\theta)=\left(\frac{3}{5}t,\frac{4}{5}t\right)$

Equation of BC is $y-0=\frac{\frac{4}{5}t}{\frac{3}{5}t-t}(x-t) \Rightarrow y=-2(x-t)$

It passes through $\left(\frac{2}{3},\frac{2}{3}\right) \Rightarrow \frac{2}{3}=-2\left(\frac{2}{3}-t\right) \Rightarrow t=1$

Coordinates (h,k) of A are given by $h=\frac{3}{5}t+t=\frac{8}{5}t=\frac{8}{5}$ and $k=\frac{4}{5}t=\frac{4}{5} \Rightarrow \frac{h}{k}=2$

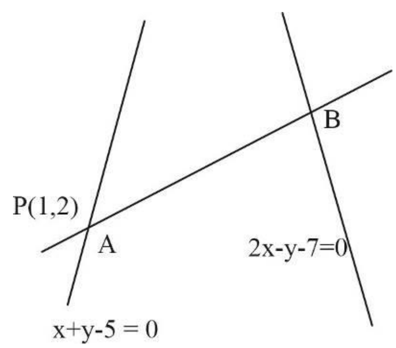
95.(5) Let the equation of the line passing through $P(1,2) be\; y-2=m(x-1)$ ....(i)

Let $PA=r_1, PB=r_2$ then $A=(1+r_1\,cos\,\theta, 2+r_1\,sin\,\theta)$; $B=(1+r_2\,cos\,\theta, 2+r_2\,sin\,\theta)$

Where $tan\,\theta=m$

As A is on the line $x+y-5=0, 1+r_1\,cos\,\theta+2+r_1\,sin\,\theta-5=0$ Or $r_1(cos\,\theta+sin\,\theta)=2$

$\frac{1}{r_1}=\frac{cos\,\theta+sin\,\theta}{2}$ ....(ii)



As B is on the line $2x-y-7=0$

$2(1+r_2\,cos\,\theta)-(2+r_2\,sin\,\theta)-7=0$

$r_2(2cos\,\theta-sin\,\theta)=7$

$\frac{1}{r_2}=\frac{(2cos\,\theta-sin\,\theta)}{7}$ ....(iii)

Now, HM of PA, PB is 10

$10=\frac{2PA.PB}{PA+PB}$; $\therefore 10=\frac{2r_1r_2}{r_1+r_2}$

$\frac{r_1+r_2}{r_1r_2}=\frac{1}{5}$; $\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{5}$ Or $\frac{2cos\,\theta-sin\,\theta}{7}+\frac{(sin\,\theta+cos\,\theta)}{2}=\frac{1}{5}$

From (ii), (iii) Or $10(2cos\,\theta-sin\,\theta)+35(cos\,\theta+sin\,\theta)=14$

$55cos\,\theta+25sin\,\theta=14$

$\sqrt{55^2+25^2}\,cos(\theta-\alpha)=14$ where $cos\,\alpha=\frac{55}{\sqrt{55^2+25^2}}$ $or\; cos(\theta-\alpha)=\frac{14}{5\sqrt{146}}$

$\theta = \alpha + cos^{-1}\dfrac{14}{5\sqrt{146}} = cos^{-1}\dfrac{11}{\sqrt{146}} + cos^{-1}\dfrac{14}{5\sqrt{146}}$

From (i) the equation of the line is $(y-2) = tan\left[cos^{-1}\left[\dfrac{11}{\sqrt{146}}\right] + cos^{-1}\left[\dfrac{14}{5\sqrt{146}}\right]\right](x-1)$

Also $cos^{-1}x + sin^{-1}x = \dfrac{\pi}{2}$

96.(12) The equation of any line passing through the given point $P(3,4)$ and making an angle $\pi/6$ with x-axis is

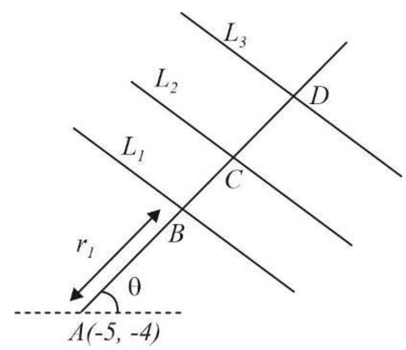
$\dfrac{x-3}{cos\,30°} = \dfrac{y-4}{sin\,30°} = r$ where r represents the distance of any point Q on this line from the given point $P(3,4)$.

Coordinates of any point Q on line are $(3 + r\,cos\,30°, 4 + r\,sin\,30°)$ which lies on the line $12x + 5y + 10 = 0$ then

$12\left(3 + \dfrac{r\sqrt{3}}{2}\right) + 5\left(4 + \dfrac{r}{2}\right) + 10 = 0 \Rightarrow r = \dfrac{132}{12\sqrt{3}+5}$

97.(27) Co-ordinate of B is $(-5 + r_1\,cos\,\theta, -4 + r_1\,sin\,\theta)$; $C(-5 + r_2\,cos\,\theta, -4 + r_2\,sin\,\theta)$; $D(-5 + r_3\,cos\,\theta, -4 + r_3\,sin\,\theta)$

As point B, C , D lies on L_1, L_2, L_3



$(-5 + r_1\,cos\,\theta) + 3(-4 + r_1\,sin\,\theta) + 2 = 0$

$r_1 = \dfrac{15}{cos\,\theta + 3\,sin\,\theta}$; $\dfrac{15}{r_1} = cos\,\theta + 3\,sin\,\theta = \dfrac{15}{AB}$ …………(i)

Similarly; $\dfrac{10}{r_2} = 2\,cos\,\theta + sin\,\theta = \dfrac{10}{AC}$ ………..(ii)

$\dfrac{6}{r_3} = cos\,\theta - sin\,\theta = \dfrac{6}{AD}$ ……(iii)

Put in given expression $(2\,cos\,\theta + 3\,sin\,\theta)^2 = 0 \Rightarrow tan\,\theta = -\dfrac{2}{3}$

$y + 4 = -\dfrac{2}{3}(x+5)$, $2x + 3y + 22 = 0$, $2 + b + c = 27$

98. (9) $6a^2 - 3b^2 - c + 7ab - ac + 4bc = 0$

$6a^2 + a[7b - c] + 4bc - 3b^2 - c^2 = 0$

$a = \dfrac{c - 7b \pm \sqrt{49b^2 + c^2 - 14bc - 24(4bc - 3b^2 - c^2)}}{12}$; $a = \dfrac{c - 7b \pm \sqrt{(11b)^2 + (5c)^2 - 110bc}}{12}$

$a = \dfrac{c - 7b \pm (11b - 5c)}{12}$; $a = \dfrac{c - 7b + (11b - 5c)}{12}$ or $a = \dfrac{c - 7b - (11b - 5c)}{12}$

$a = \dfrac{4b - 4c}{12}$ or $a = \dfrac{6c - 18b}{12}$; $a = \dfrac{b-c}{3}$ or $a = \dfrac{c - 3b}{2}$

So $3a - b + c = 0$ and $-2a - 3b + c = 0$ will concurrent at $(3,-1)$ and $(-2,-3)$

$\Rightarrow A = 3, B = -1, C = -2, D = -3$

CIRCLE

1.(A) $2\sqrt{rr_1}+2\sqrt{rr_2}=2\sqrt{r_1r_2} \Rightarrow \sqrt{r}\ 6+3\ =6\times 3 \Rightarrow$ r = 4 [Length of direct common tangent $\sqrt{d^2-(r_1-r_2)^2}$]

2.(B) For $\angle ACB$ to be maximum, circle passing through A, B will touch x-axis at C.
By power of point O

$$OC^2 = OA \quad OB$$

$$x^2=\alpha\beta \qquad x=\sqrt{\alpha\beta}$$

3.(A) Combined equation of pair of tangents is $xh+yk-1\ ^2 = \ x^2+y^2-1 \quad h^2+k^2-1$

Put $\quad x=1,\ h-1\ ^2+2ky\ \ h-1 \ \ = y^2\ \ h^2-1$

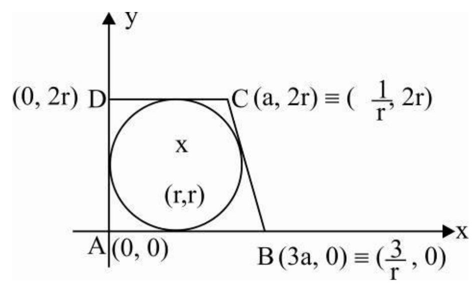
$$y^2\ \ h+1\ -2ky-\ \ h-1\ =0$$

$$AB=|y_1-y_2|=2 \quad \Rightarrow \quad 4=\frac{4k^2}{h+1\ ^2}+\frac{4\ \ h-1}{h+1}$$

$$(h+1)^2=k^2+(h^2-1)\Rightarrow k^2=2(h+1)$$

$$y^2=2\ \ x+1$$

4.(B) Area of trapezium $ABCD=\frac{1}{2}\ \ a+3a\ \ \ 2r\ \ =4 \Rightarrow ar=1$



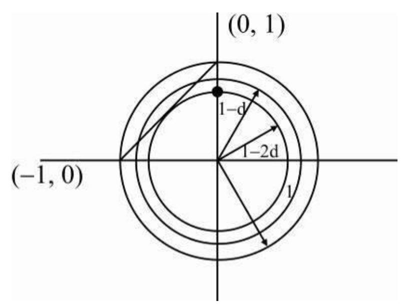
Equation of BC is $y=-r^2\left(x-\frac{3}{r}\right)$

$$y+r^2x-3r=0$$

$\because \quad BC$ is tangent to circle

$$\Rightarrow \quad \frac{|r+r^3-3r|}{\sqrt{1+r^4}}=r \Rightarrow r^4+4-4r^2=1+r^4 \Rightarrow r=\frac{\sqrt{3}}{2}$$

5.(B) For max common difference, the smaller circle just touches the line

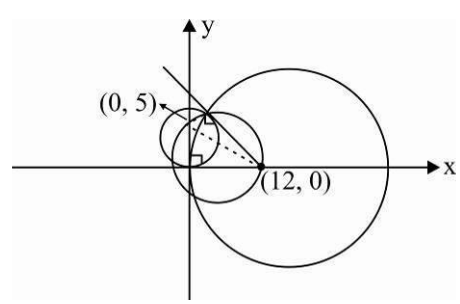


$$\Rightarrow \quad 1-2d=\frac{1}{\sqrt{2}}$$

$$d=\frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$\therefore \quad d\in\left(0,\frac{\sqrt{2}-1}{2\sqrt{2}}\right) \quad \text{i.e.,} \quad \left(0,\frac{2-\sqrt{2}}{4}\right)$$

6.(A) Equation of required circle is $\left(x^2+y^2-10y\right)+\lambda\left(x^2+y^2-24x\right)=0$



The circle process through the point (12, 0)
i.e. $144+\lambda(144-24\times 12)=0 \quad \therefore \quad \lambda=1$

$\therefore \quad$ Equation required circle is $x^2+y^2-12x-5y=0$

$\therefore \quad$ Radius $=\frac{13}{2}$

7.(A) $R + r + x = GB$ when $x = PB$

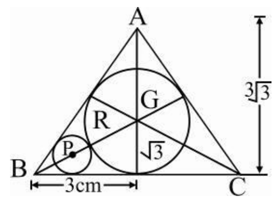
$\sqrt{3} + r + x = 2\sqrt{3}$

$$\left[R = 3\sqrt{3}\cdot\frac{1}{3} = \sqrt{3}\right]$$

$$x = \sqrt{3} - r = PB$$

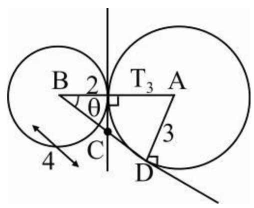
Now $\sin 30^\circ = \dfrac{r}{\sqrt{3} - r} = \dfrac{1}{2}$

$$2r = \sqrt{3} - r \Rightarrow r = \frac{1}{\sqrt{3}}$$



8.(B) $BC = \dfrac{2}{\cos\theta} = \dfrac{2}{\frac{4}{5}} = \dfrac{5}{2}$

$$CD = 4 - \frac{5}{2} = \frac{3}{2}$$



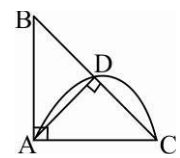
9.(D) $(AC)^2 = (AD)^2 + (CD)^2$

$AB^2 = BD \ \ BC = BD \ \ BD + DC$

$\Rightarrow \quad BD \ \ DC = AB^2 - BD^2 = AD^2 \Rightarrow DC = \dfrac{AD^2}{BC} = \dfrac{AD^2}{\sqrt{AB^2 - AD^2}}$

$\therefore \quad (AC)^2 = (AD)^2 + \dfrac{(AD)^4}{(AB)^2 - (AD)^2} = \dfrac{(AB)^2 (AD)^2}{(AB)^2 - (AD)^2}$

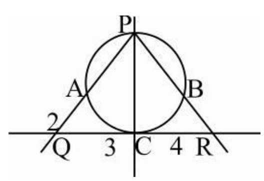
$$AC = \frac{AB \ \ AD}{\sqrt{AB^2 - AD^2}}$$



10.(B) $PQ \ \ AQ = QC^2 \quad \Rightarrow \quad PQ = \dfrac{9}{2}$

$\dfrac{QC}{RC} = \dfrac{PQ}{PR} \Rightarrow PR = 6$

$RC^2 = RB \ \ RP \Rightarrow RB = \dfrac{8}{3}$

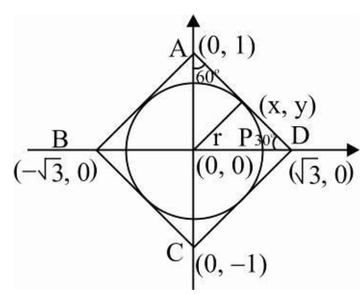


11.(B) $r = \sqrt{3}\sin 30^\circ = \dfrac{\sqrt{3}}{2}$

$PA^2 + PB^2 + PC^2 + PD^2$

$$= x - \sqrt{3}^{\,2} + y^2 + x^2 + y - 1^{\,2} + x + \sqrt{3}^{\,2} + y^2 + x^2 + y + 1^{\,2}$$

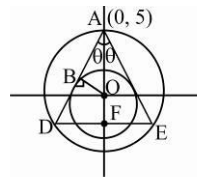
$$= 4 \ x^2 + y^2 + 2 \ = 4\left(\frac{3}{4} + 2\right) = 11$$



12.(A) $\sin\theta = \dfrac{3}{5}$

Slope of AB, AE are $\tan\left(\dfrac{\pi}{2} - \theta\right), \tan\left(\dfrac{\pi}{2} + \theta\right) = \dfrac{4}{3}, -\dfrac{4}{3}$

Equation of OB is $y=-\frac{3}{4}x$



$\Rightarrow \qquad 4y+3x=0$

D is image of A *w.r.t.* OB

$$\frac{x-0}{3}=\frac{y-5}{4}=-2\frac{4\ 5}{25}=-\frac{8}{5}$$

$$D\equiv\left(\frac{-24}{5},\frac{-7}{5}\right)\Rightarrow OF=\frac{7}{5}$$

Aliter : $\sin\theta=\frac{3}{5}$

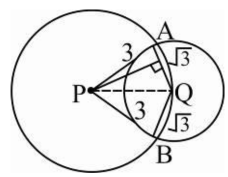
$\therefore \qquad OB\perp AD \qquad \Rightarrow \qquad AD=8$

Let $OF=x$

In ΔADF

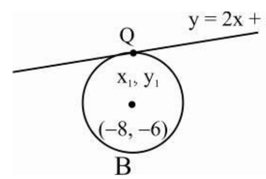
$$\cos\theta=\frac{5+2}{8} \quad\Rightarrow\quad \frac{4}{5}=\frac{5+x}{8}; \qquad x=\frac{7}{5}$$

13.(B) Area of $APBQ=2$ area ΔAPQ



$$=2\left(\frac{1}{2}\times\sqrt{3}\times\sqrt{3^2-\left(\frac{\sqrt{3}}{2}\right)^2}\right)=\sqrt{3}\cdot\frac{\sqrt{33}}{2}=\frac{\sqrt{99}}{2}$$

14.(A) Let the line touches the given circle at (x_1,y_1), then $y_1=2x_1+5$ and $\frac{y_1+6}{x_1+8}\times 2=-1$



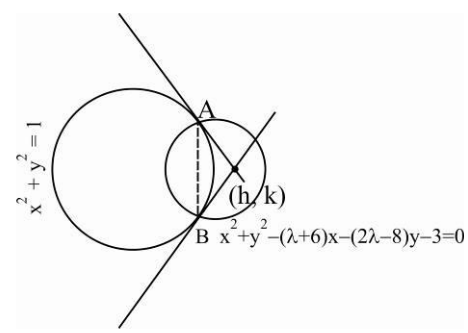
$\Rightarrow \qquad x_1=-6$ and $y_1=-7$

15.(A) Let both the circles intersects at A and B, then the equation of the line AB is

$$xh+yk=1 \qquad \ldots(1)$$

$$AB=S_1-S_2=0$$

$$\Rightarrow \qquad \lambda+6\ \ x+\ 2\lambda-8\ \ y+2=0 \qquad \ldots(2)$$



(1) & (2) represent the same line, so

$$\Rightarrow \qquad \frac{h}{\lambda+6}=\frac{k}{2\lambda-8}=\frac{-1}{2}$$

$$\Rightarrow \qquad \frac{2h}{2\lambda+12}=\frac{k}{2\lambda-8}=\frac{2h-k}{20} \quad =-\frac{1}{2}\left(as\,\frac{a}{b}=\frac{c}{b}=\frac{a-c}{b-d}\right)$$

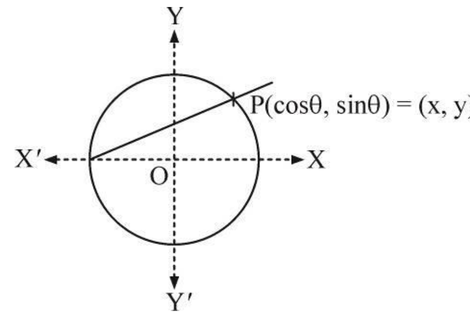
$$\Rightarrow \qquad 2x-y=-10$$

$$\Rightarrow \qquad 2x-y+10=0$$

16.(C) $\sin\theta=\frac{r}{3r}=\frac{1}{3}$ where angle EDC is θ

$$DE=2\ \ 2r\cos\theta\ \ =4\times 3\left(\frac{2\sqrt{2}}{3}\right)$$

$$DE=8\sqrt{2}$$

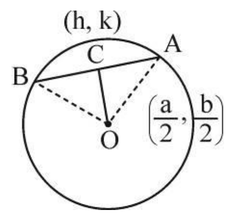


17.(D) $m = \dfrac{\sin\theta}{\cos\theta + 1} = \tan\left(\dfrac{\theta}{2}\right)$

$$P(x, y) = \left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}, \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)$$

$\therefore$ $x, y \in Q$, if $m \in Q \Rightarrow$ Infinite points are possible.

18.(C) In triangle OAC



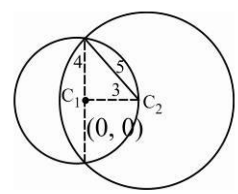
$$OA^2 = AC^2 + OC^2$$

$$\frac{a^2}{4} + \frac{b^2}{4} = 2\left[\left(h - \frac{a}{2}\right)^2 + \left(k - \frac{b}{2}\right)^2\right]$$

$$\Rightarrow \quad h^2 + k^2 - ah - bk + \frac{a^2}{8} + \frac{b^2}{8} = 0$$

Required Locus is $x^2 + y^2 - ax - by + \dfrac{a^2 + b^2}{8} = 0$

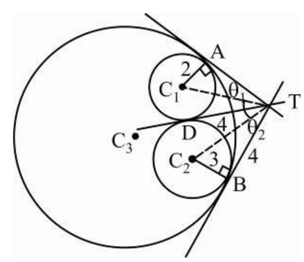
19.(B) Slope of $C_1C_2 = -\dfrac{4}{3} = \tan\theta$



$$\Rightarrow \quad C_2 \equiv \left(0 + 3\left(-\frac{3}{5}\right), 0 + 3\left(\frac{4}{5}\right)\right) \quad \text{Or} \quad \left(0 - 3\left(-\frac{3}{5}\right), 0 - 3\left(\frac{4}{5}\right)\right)$$

$$C_2 \equiv \left(\frac{9}{5}, -\frac{12}{5}\right) \text{or} \left(-\frac{9}{5}, \frac{12}{5}\right)$$

20-22. AT and BT are radical axis to C_3 and C_1 and C_3 & C_2 respectively.



$\therefore$ T is radical centre $\Rightarrow$ radical axis of C_1 and C_2 i.e., common tangent passes through T.

$TA = TB = TD = 4$

$$\tan\frac{\theta_1}{2} = \frac{2}{4} = \frac{1}{2}, \tan\frac{\theta_2}{2} = \frac{3}{4}, \ \angle ATD = \theta_1, \angle BTD = \theta_2$$

20.(D) $r_3 = TA\tan\left(\dfrac{\theta_1 + \theta_2}{2}\right) = 4\left(\dfrac{\frac{1}{2} + \frac{3}{4}}{1 - \frac{1}{2}\cdot\frac{3}{4}}\right) = 8$

21.(B) Circumcircle of ΔTAB will pass through C_3 has TC_3 as diameter

$$\Rightarrow \quad \text{Area} = \pi\left(\frac{TC_3}{2}\right)^2, TC_3 = \frac{TA}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{4}{1/\sqrt{5}} = 4\sqrt{5}$$

Area = $\pi\left(2\sqrt{5}\right)^2 = 20\pi$

22.(D) $C_3C_1 = r_3 - r_1$

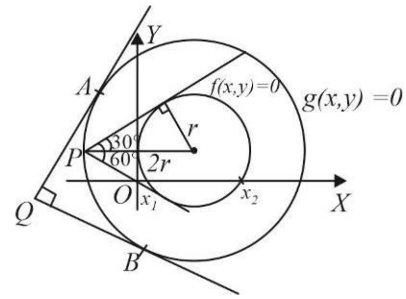
$$C_3C_2 = r_3 - r_2 \quad \Rightarrow \quad C_3C_1 - C_3C_2 = r_2 - r_1 = 1$$

23.-25 $g(x,y)=0$ represent a circle concentric with circle $f(x,y)=0$ with radius twice the radius of

$$f(x,y)=0, centre=\left(\frac{5}{2},2\right)$$

$$f(x,y)=x^2+y^2-5x-4y+4=0$$

$$g(x,y)=x^2+y^2-5x-4y-\frac{59}{4}=0$$



23.(D) Area of $\Delta QAB=\frac{1}{2}\times5\times5=\frac{25}{2}$

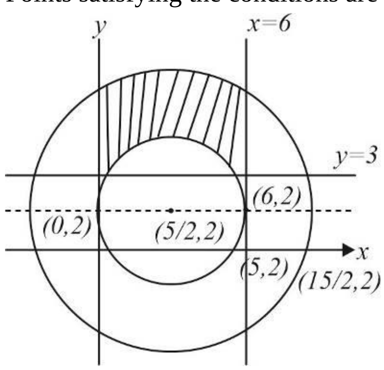
24.(D) $\theta=tan^{-1}\frac{3}{4}$

$$2\theta=tan^{-1}\left(\frac{2\left(\frac{3}{4}\right)}{1-\frac{9}{16}}\right)=tan^{-1}\left(\frac{24}{7}\right)$$

Area of region inside $f(x,y)=0$ above x-axis.

$$=\frac{1}{2}\left(\frac{5}{2}\right)^2\left(2\pi-tan^{-1}\left(\frac{24}{7}\right)\right)+\frac{1}{2}\times3\times2=3+\frac{25}{8}\left(2\pi-tan^{-1}\left(\frac{24}{7}\right)\right)$$

25.(D) Points satisfying the conditions are



$(1,5)(1,6)(2,5)(2,6)(3,5)(3,6)(4,5)(4,6)(5,4)(5,5)(5,6)$

26.(C) **27. (C) 28. (D)**

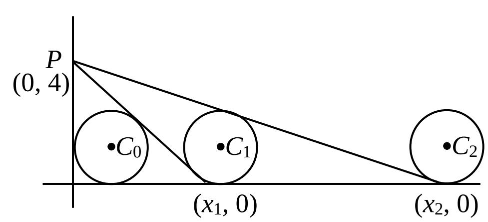
26.28 **Explanation:**

For C_n circle let centre be $(\alpha_n,1)$ and it touches line L_n and L_{n+1}.

i.e. $\frac{x}{x_n}+\frac{y}{4}=1, \frac{x}{x_{n+1}}+\frac{y}{4}=1$

$$\Rightarrow \frac{4\alpha_n+x_n-4x_n}{\sqrt{16+x_n^2}}=1 \qquad \ldots(i)$$

$$\Rightarrow \frac{4\alpha_n+x_{n+1}-4x_{n+1}}{\sqrt{16+x_{n+1}^2}}=-1 \qquad \ldots(ii)$$



$\Rightarrow \quad 3\left(x_{n+1}-x_n\right)=\sqrt{16+x_n^2}+\sqrt{16+x_{n+1}^2}$

Let $\quad x_n = 4cot\,\theta_n 0<\theta_n<\dfrac{\pi}{2} \qquad \Rightarrow \qquad tan\left(\dfrac{\theta_{n+1}}{2}\right)=\dfrac{1}{2}tan\left(\dfrac{\theta_n}{2}\right)$

As $\quad x_1 = 3 \quad \Rightarrow \quad tan\dfrac{\theta_1}{2}=\dfrac{1}{2} \qquad \Rightarrow \quad tan\left(\dfrac{\theta_n}{2}\right)=\dfrac{1}{2^n} \quad \Rightarrow \quad x_n=\left(\dfrac{2^{2n}-1}{2^{n-1}}\right) \qquad x_2=\dfrac{15}{2}$

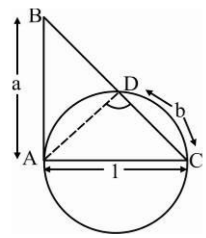
Putting is (i), we get $\alpha_2=\dfrac{31}{4}$.

$$\lim_{n\to\infty}\frac{x_n}{2^n}=\lim_{n\to\infty}\frac{2^{2n}-1}{2^{n-1}2^n}=2$$

29.(BC) In triangle ADC

$\tan C=\dfrac{AD}{b}=\dfrac{a}{1} \quad \Rightarrow \quad AD=ab<AC=1\Rightarrow ab<1$

In triangle *ADC*

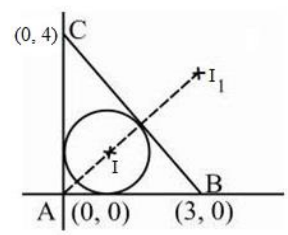


$$AC^2 = AD^2 + CD^2 = a^2b^2+b^2=1\text{, then } b^2=\frac{1}{a^2+1}$$

$$\frac{b^2}{a^2}=\frac{1}{a^4+a^2}>\frac{1}{a^4+a^2+\frac{1}{4}}$$

$$\frac{b}{a}>\frac{1}{a^2+\frac{1}{2}}$$

30.(AD) Circles satisfying the given condition are incircle and excircle of the ΔABC as shown



$$I\equiv\left(\frac{4\;3}{4+3+5},\frac{3\;4}{4+3+5}\right)\equiv\; 1,1$$

$$I_1\equiv\left(\frac{4\;3}{-5+4+3},\frac{4\;3}{-5+4+3}\right)\equiv\; 6,6$$

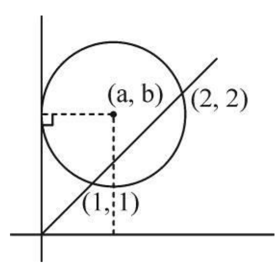
31.(AB) Equation of the circle passing through $(1, 1)$ and $(2, 2)$ is $\{(x-1)(x-2)+(y-1)(y-2)\}+\lambda(x-y)=0$

$\Rightarrow \qquad x^2+y^2-(3-\lambda)x-(3+\lambda)y+4=0 \qquad \Rightarrow \qquad \text{Radius} = \sqrt{\left(\dfrac{3-\lambda}{2}\right)^2+\left(\dfrac{3+\lambda}{2}\right)^2-4}$

Now let (a, b) be the centre of the circle.

Then we have $\quad$ (i) $\quad |a|=r,\,|b|>r \quad$ or $\quad$ (ii) $\quad |b|=r,\,|a|>r$

Case (i) $|a|=r \quad \Rightarrow \quad a^2=r^2$

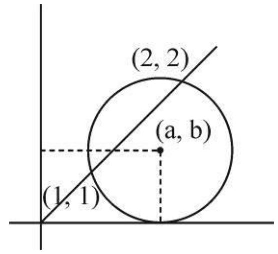


$\Rightarrow \qquad \left(\dfrac{3-\lambda}{2}\right)^2=\left(\dfrac{3-\lambda}{2}\right)^2+\left(\dfrac{3+\lambda}{2}\right)^2-4 \quad \Rightarrow \qquad \lambda=1 \text{ or } -7$

$\lambda=1 \qquad \Rightarrow \qquad |a|=1 \text{ and } |b|=2 \qquad \Rightarrow \qquad r=1$

$\lambda=-7 \qquad \Rightarrow \qquad |a|=5 \text{ and } |b|=2 \qquad \Rightarrow \qquad r=5$ (rejected)

$\therefore \qquad$ Equation of circle is $x^2+y^2-2x-4y+4=0$



Case(ii) $|b| = r \quad \Rightarrow \quad \left(\dfrac{3-\lambda}{2}\right)^2 = 4 \quad \Rightarrow \quad \lambda = -1$ or 7

If $\lambda = -1 \quad \Rightarrow \quad |a| = 2,\ r = 1$ so $|a| > r$

$\lambda = 7 \quad \Rightarrow \quad |a| = 2,\ r = 5$ so $|a| < r$

$\therefore\ \lambda = 7$, is invalid

$\therefore$ Equation of circle is $x^2 + y^2 - 4x - 2y + 4 = 0$

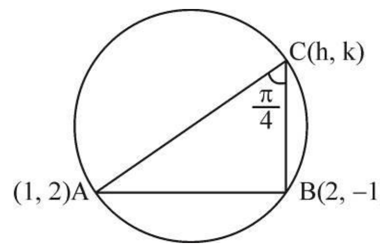
32.(ABD) $9l^2 + 6l + 1 = 5\ l^2 + m^2 \quad \Rightarrow \quad \dfrac{(3l+1)}{\sqrt{l^2+m^2}} = \sqrt{5}$

$\Rightarrow$ Perpendicular distance of (3, 0)

From $lx + my + 1 = 0$ is $\sqrt{5}$

$\Rightarrow$ Centre of circle = (3, 0)

Radius = $\sqrt{5}$ & Equation of circle is $(x-3)^2 + y^2 = 5$



33.(AB) $m(AC) = m_1 = \dfrac{k-2}{h-1}$ & $m(BC) = m_2 = \dfrac{k+1}{h-2}$

$\therefore \quad \tan\dfrac{\pi}{4} = \left|\dfrac{\dfrac{k-2}{h-1} - \dfrac{k+1}{h-2}}{1 + \dfrac{k-2}{h-1}\cdot\dfrac{k+1}{h-2}}\right| \quad \Rightarrow \quad \pm 1 = \dfrac{(k-2)(h-2)-(k+1)(h-1)}{(h-1)(h-2)+(k-2)(k+1)}$

Required equation of circle by taking and $-ve$ sign,

We get $x^2 + y^2 = 5$ & $x^2 + y^2 - 6x - 2y + 5 = 0$

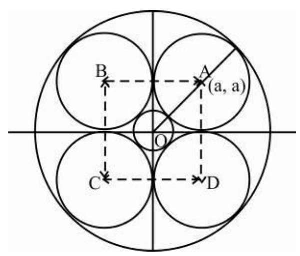
34.(ABCD) $OA = \sqrt{2}a$

Largest radius $= \sqrt{2}a + a$

Smallest radius $= \sqrt{2}a - a$

Area enclosed by circles

$= 2a^2 - 4\left(\dfrac{\pi}{4}a^2\right) = 4 - \pi\ a^2$

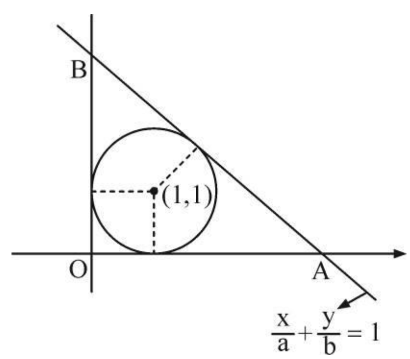


35.(AB) point (1, 1) & (0, 0) are of the same side of the line $\dfrac{x}{a} + \dfrac{y}{b} - 1 = 0$

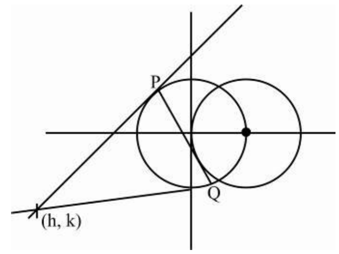
$\Rightarrow \quad \dfrac{1}{a} + \dfrac{1}{b} < 1$

Also, perpendicular distance from centre to the line is equal to radius

$\dfrac{-\dfrac{1}{a} - \dfrac{1}{b} + 1}{\sqrt{\dfrac{1}{a^2} + \dfrac{1}{b^2}}} = 1 \quad \Rightarrow \quad 1 - \dfrac{1}{a} - \dfrac{1}{b} = \sqrt{\dfrac{1}{a^2} + \dfrac{1}{b^2}}$



36.(AC) PQ is chord of contact of (h, k) *w.r.t.* $x^2+y^2=a^2$



$\Rightarrow$ Equation of PQ is $xh+yk=a^2$

$\because$ PQ is tangent to $(x-a)^2+y^2=a^2$

$\Rightarrow$ $\dfrac{|ah-a^2|}{\sqrt{h^2+k^2}}=a \quad \Rightarrow \quad (h-a)^2=h^2+k^2$

Locus is $(x-a)^2=x^2+y^2$ i.e., $y^2=a(a-2x)$

37.(AC) Equation of the circle touching the line $x+y-2=0$ at $(1,1)$ is $(x-1)^2+(y-1)^2+\lambda(x+y-2)=0$

i.e. $x^2+y^2+x(\lambda-2)+y(\lambda-2)+(2-2\lambda)=0$

Now equation of PQ is $x(\lambda-6)+y(\lambda-7)+(8-2\lambda)=0$

$\Rightarrow$ PQ can never be parallel to the line $x+y+20$ & PQ always passes through $(6,-4)$

38.(ACD)

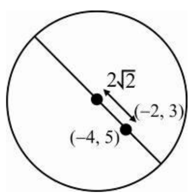
$a=9+2\sqrt{2}$

$b=9-2\sqrt{2}$

$ab=81-8=73$

$a+b=18$

$a-b=4\sqrt{2}$



39.(ACD)

$\because$ Sides of the square are parallel to co-ordinate axes

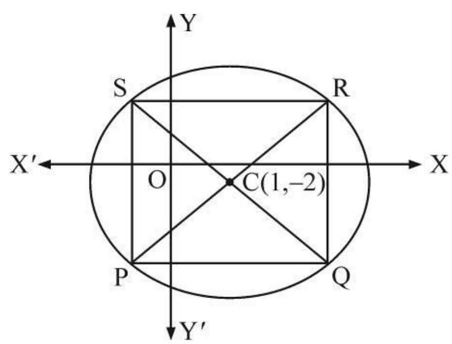
$\therefore$ Slope of PR and SQ are 1 and -1 respectively.

Co-ordinate of P & Q are

$\left(1\pm7\sqrt{2}\cdot\dfrac{1}{\sqrt{2}},\,-2\pm7\sqrt{2}\cdot\dfrac{1}{\sqrt{2}}\right)$ i.e. $(8,5)$ and $(-6,-9)$

Similarly co-ordinate of S & Q are

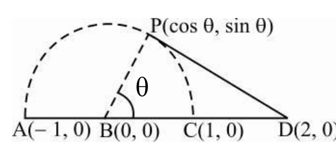
$\left(1\pm7\sqrt{2}\left(-\dfrac{1}{\sqrt{2}}\right),\,-2\pm7\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right)\right)$ i.e. $(-6,5)$ & $(8,-9)$



40.(AB) Area $=\dfrac{1}{2}(1)^2(\pi-\theta)+\dfrac{1}{2}(2)\sin\theta=\dfrac{\pi}{2}+\sin\theta-\dfrac{\theta}{2}$

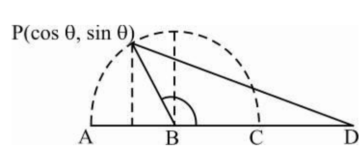
$A=\dfrac{\pi}{2}+\sin\theta-\dfrac{\theta}{2}$

$\dfrac{dA}{d\theta}=\cos\theta-\dfrac{1}{2} \quad \Rightarrow \quad \theta=\dfrac{\pi}{3}$ (for maximum or minimum)



A is max. for $\theta=\dfrac{\pi}{3}$

$\Rightarrow$ $A_{max}=\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{2}=\dfrac{2\pi+3\sqrt{3}}{6}$



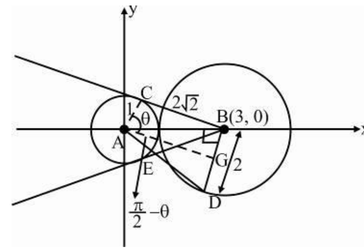
$L=3+(\pi-\theta)+\sqrt{(2-\cos\theta)^2+\sin^2\theta}=3+(\pi-\theta)+\sqrt{5-4\cos\theta}$

41.(ABC) In triangle ABC

$\tan\theta = 2\sqrt{2}$

$AG = 2\sqrt{2}$

$m_{BD} = 2\sqrt{2}$ where m_{BD} is the slope of BD



Then the coordinate of point $D \equiv \left(3-2\left(\frac{1}{3}\right), 0-2\left(\frac{2\sqrt{2}}{3}\right)\right) \equiv \left(\frac{7}{3}, \frac{-4\sqrt{2}}{3}\right)$

slope of line AE= Slope of line AD $= \frac{-4\sqrt{2}}{7}$

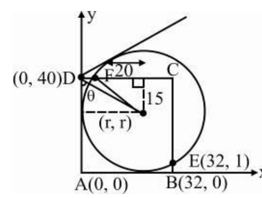
So coordinate of point $E = \left(0-1\left(-\frac{7}{9}\right), 0-1\left(\frac{4\sqrt{2}}{9}\right)\right) = \left(\frac{7}{9}, \frac{-4\sqrt{2}}{9}\right)$

Eqn. of BE is $y = \frac{\sqrt{2}}{5}(x-3)$, Solving intersection of line BE with circle S_1,

$x^2 + \frac{2}{25}(x-3)^2 = 1 \Rightarrow \quad 27x^2 - 12x - 7 = 0 \quad \Rightarrow \quad (9x-7)(3x+1) = 0 \quad \Rightarrow \quad F \equiv \left(-\frac{1}{3}, \frac{-2\sqrt{2}}{3}\right)$

42.(ABC) Equation of circle is $x^2 + y^2 - 2rx - 2ry + r^2 = 0$

Put (32, 1)



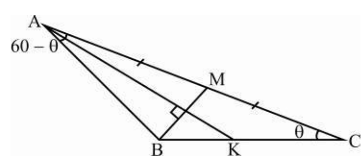
$\Rightarrow \quad r^2 - 66r + 1025 = 0 \quad \Rightarrow \quad r = 25, r = 41, r = 41$, is rejected

$\theta = \tan^{-1}\left(\frac{25}{15}\right) \quad \Rightarrow \quad 2\theta = 2\tan^{-1}\left(\frac{5}{3}\right) = \pi - \tan^{-1}\left(\frac{15}{8}\right)$

Area of $AFCB = \frac{1}{2}(40)(32+27) = 1180$

43.(ABCD) ΔAMB is isosceles

$\because$ angle bisector of $\angle A$ is $\perp$ to MB



$\frac{c}{\sin\theta} = \frac{2c}{\sin 60^\circ} \Rightarrow \sin\theta = \frac{\sqrt{3}}{4}$

$a^2 = c^2 + 4c^2 - 4c^2\cos(60^\circ - \theta)$

$= 5c^2 - 4c^2\left(\frac{\sqrt{13}}{4}\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{4}\right)$

$a = \frac{\sqrt{7-\sqrt{13}}}{\sqrt{2}}c = \frac{\sqrt{13}-1}{2}c$

(A) $\frac{a}{c} = \frac{\sqrt{13}-1}{2}$

(B) $R = \frac{abc}{4\Delta} = \frac{ac(2c)}{2ac\frac{\sqrt{3}}{2}} = \frac{2c}{\sqrt{3}}$

(C) $\frac{\Delta}{\pi R^2} = \frac{\frac{1}{2}\left(ac\frac{\sqrt{3}}{2}\right)}{\pi\frac{4}{3}c^2} = \frac{3\sqrt{3}}{16\pi}\frac{a}{c} = \frac{3\sqrt{3}}{32\pi}\left(\sqrt{13}-1\right)$

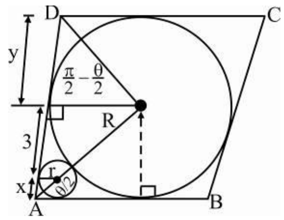
(D) $AB = c, AC = 2c, \frac{AB}{AC} = \frac{1}{2}$.

44.(ABC) $2\sqrt{Rr}=3\Rightarrow r=\frac{9}{4\times 3}=\frac{3}{4}$

$$\frac{r}{R}=\frac{x}{x+3}=\frac{1}{4}\Rightarrow x=1$$

$$\tan\frac{\theta}{2}=\frac{3}{4}$$

$$y=R\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=3\tan\frac{\theta}{2}=\frac{9}{4}$$



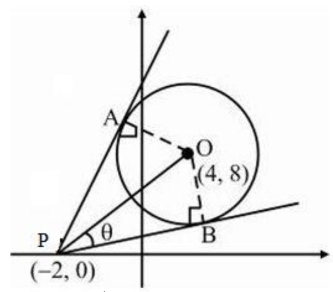
$ABCD$ is a rhombus with side $=4+\frac{9}{4}=\frac{25}{4}$ Area of $ABCD=\frac{25}{4}\times\frac{25}{4}\sin\theta=\frac{25}{4}\times\frac{25}{4}\frac{2\left(\frac{3}{4}\right)}{1+\frac{9}{16}}$

$$A=\frac{75}{2}=\frac{1}{2}d_1d_2\ \Rightarrow\ d_1d_2=75$$

45.(BC) $\sin\theta=\frac{2\sqrt5}{10}=\frac{1}{\sqrt5}$

Slopes of PA and PB are $\tan\ \alpha\pm\theta$

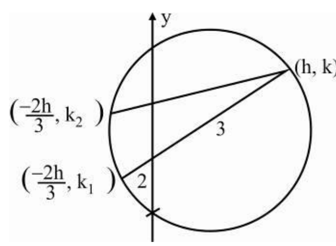
Where $\tan\alpha=\frac{8}{6}=\frac{4}{3}=\frac{\frac{4}{3}+\frac{1}{2}}{1-\frac{4}{3},\frac{1}{2}},\frac{\frac{4}{3}-\frac{1}{2}}{1+\frac{4}{3},\frac{1}{2}}=\frac{11}{2},\frac{5}{10}$

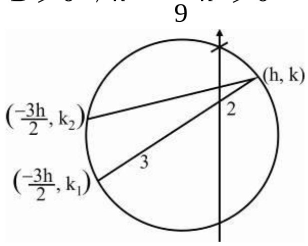


$\therefore$ $A,B\equiv\left(4+2\sqrt5\left(\frac{-11}{5\sqrt5}\right),8+2\sqrt5\left(\frac{2}{5\sqrt5}\right)\right)$ $\left(4-2\sqrt5\left(\frac{-1}{\sqrt5}\right)8-2\sqrt5\left(\frac{2}{\sqrt5}\right)\right)\equiv\left(\frac{-2}{5},\frac{44}{5}\right)$, (6, 4)

46.(ABD) Put $x=\frac{-2h}{3},y^2-ky+\frac{10h^2}{9}=0$

$$D>0\Rightarrow k^2-\frac{40}{9}h^2>0$$





Put $x=\frac{-3h}{2}\ \Rightarrow\ y^2-ky+\frac{15h^2}{4}=0$

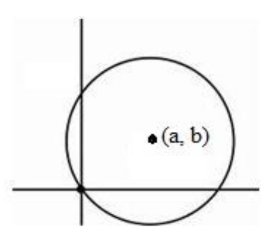
$$D>0\Rightarrow k^2-15h^2>0$$

47.(ABD) Eqn. of circle is $x\pm a^2\ +\ y\pm b^{\ 2}=a^2+b^2$

or $x\pm b^{\ 2}+\ y\pm a^{\ 2}=a^2+b^2$

$x^2+y^2\pm 2ax\pm 2by=0$ or $x^2+y^2\pm 2bx\pm 2ay=0$

Equation of tangent origin is $ax\pm by=0$ Or $bx\pm ay=0$



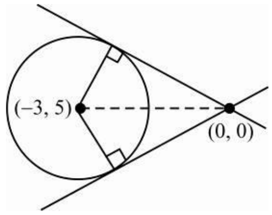
48.(AD) Area of quadrilateral

$$15=\sqrt{34-C}\sqrt{C}$$

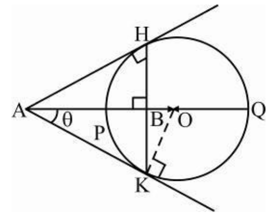
$$C^2-34C+225=0$$

$$C-25\quad C-9\ =0$$

$$C=9,25$$



49.(ABC) $\dfrac{AP+AQ}{2}=\dfrac{OA-r\ +\ OA+r}{2}=OA$



$\cos\theta=\dfrac{AK}{OA}=\dfrac{AB}{AK}\quad\Rightarrow\quad AK\ ^2\ =\ OA\quad AB$

Also $\ AK\ ^2\ =\ AP\quad AQ$

$\therefore\quad OA\quad AB\ =\ AP\quad AQ\quad\Rightarrow\left(\dfrac{AP+AQ}{2}\right)AB=\ AP\quad AQ\quad\Rightarrow\quad AB=\dfrac{2\ AP\quad AQ}{AP+AQ}$

50.(BCD) If the mid-point of the chord be $P(h,k)$, then the equation of the chord is

$hx+ky=h^2+k^2$ ……..(i)

Homogenise the equation $y^2-4x-4y=0$ with the help of (i), Thus,

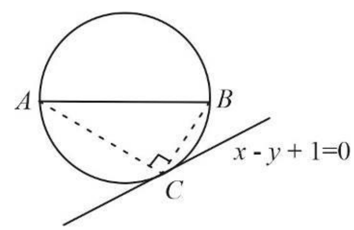
$y^2=4(x+y)\left(\dfrac{hx+ky}{h^2+k^2}\right)$ or $4hx^2+\left(4k-h^2-k^2\right)y^2+4(h+k)xy=0$ ………(ii)

It represents a pair of perpendicular lines if $4h+4k-h^2-k^2=0\ or\ h^2+k^2-4h-4k=0$

Hence, the locus of P is $x^2+y^2-4x-4y=0$ ……….(iii)

The chord of contact of circle (iii) with the given circle is $4x+4y=9$, which is also the chord of contact of tangents drawn from the point (4, 4)

51.(CD) The line segment *AB* subtend the greatest angle at *C* on the line $x-y+1=0$, so the circle drawn through *A* and *B* must touch the line $x-y+1=0$ at *C*. The equation of the circle can be assumed as



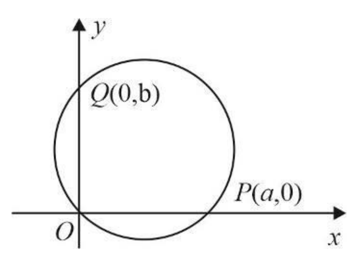
$(x-1)^2+(y-2)^2+\lambda(x-y+1)=0$

$x^2+y^2+(\lambda-2)x-(\lambda+4)y+5+\lambda=0$ ……..(i)

Again $\angle ACB=\dfrac{\pi}{2}$, so the line *AB* is diameter of circle.

Thus the radius of the circle (i) is $\sqrt{2}$. $\left(\dfrac{\lambda-2}{2}\right)^2+\left(\dfrac{\lambda+4}{2}\right)^2-(5+\lambda)=2\Rightarrow\lambda=\pm2$

52.(CD) Le the equation of the variable circle be $x^2+y^2-ax-by=0$ …….(i)



It cuts X-axis at $P(a,0)$ and Y-axis at $Q(0,b)$. Thus,

$OP=a\ and\ OQ=b$

Given, $\ m.OP+n.OQ=1$

$ma+nb=1\ or\ b=\dfrac{1-ma}{n}$

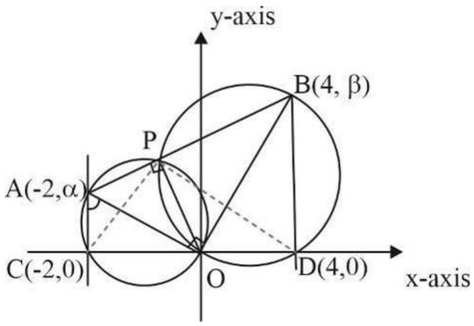
Putting the value of b in the equation of the circle (i), we have

$$\left(x^2+y^2-\frac{1}{n}y\right)-a\left(x-\frac{m}{n}y\right)=0 \text{ ....(ii)}$$

Since, 'm' and 'n' are constants, so for all values of 'a' . The above circle given by (ii) passes through the points of intersection of the circle $x^2+y^2-\frac{1}{n}y=0$ and line $x-\frac{m}{n}y=0$

The circle always passes through the point $\left(\frac{m}{m^2+n^2},\frac{n}{m^2+n^2}\right)$.

53.(AD)



Locus is the circle $(x+2)(x-4)+y^2=0$

54.(BC) $QR=2\sqrt{r_1r_2}$

Similarly, $RC=2\sqrt{rr_2}$, $CS=2\sqrt{rr_3}$, $RS=2\sqrt{r_3r_2}$, $QS=2\sqrt{r_1r_3}$ and $QC=2\sqrt{r_1r}$

55. (BCD) Since XY, AM and ND are concurrent $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$

$$a\left(bc-a^2\right)-b\left(b^2-ac\right)+c\left(ab-c^2\right)=0$$

$$abc-a^3-b^3+abc+abc-c^3$$

$$3abc-a^3-b^3-c^3=0$$

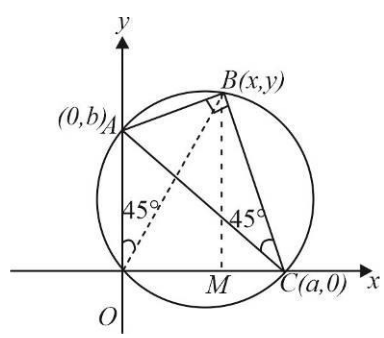
$$a^3+b^3+c^3=3abc=(a+b+c)\left(a+b\omega+c\omega^2\right)\left(a+b\omega^2+c\omega\right)=0$$

56.(AB) From the figure is obvious that $y-x=0$

$a^2+b^2=2$

Circumcentre of the triangle ABC $(a/2, b/2)$ so the locus of the circumcnetre of the triangle ABC is $x^2+y^2=\frac{1}{2}$

Since triangle is isosceles



57.(ABCD) $x^2+y^2+2gx+2fy+c=0$ ....(i)

be the equation of the circle

Point $(a,t),(t,a)$ and (t,t) lie on the given circle

After getting the value of g, f and c from equation (i) $\left(x^2+y^2-ax-ay\right)-t(x+y-2a)=0$

The circle $x^2+y^2-ax-ay=0$ always touches the line $x+y-2a=0$ at (a, a)

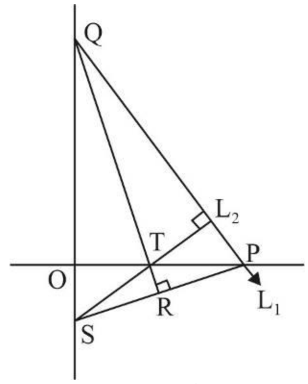
58.(ABC) T is the orthocenter of the triangle QPS so the

$\angle QRP$ and $\angle QOP$ are $90°$ and the points Q

and P are fixed.

So locus of R is a circle having PQ as diameter.

Point O will also lie on the circle



59.(BC) The given circles are $x^2 + y^2 - 2x = 0,\ x > 0$

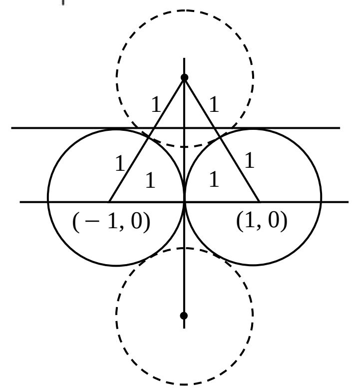
and $\quad x^2 + y^2 + 2x = 0,\ x < 0$

From the figure, the centres of the required circles will be

$\left(0, \sqrt{3}\right)$ and $\left(0, -\sqrt{3}\right)$

$\therefore$ Equations of circles will be

$$(x-0)^2 + \left(y \mp \sqrt{3}\right)^2 = 1^2$$



60.(BD) For two chords PT and QR of the circle, which intersect at S, we have

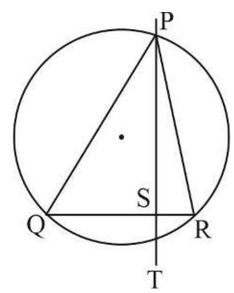
$PS \times ST = QS \times SR$

Now, $\dfrac{\dfrac{1}{PS} + \dfrac{1}{ST}}{2} > \sqrt{\dfrac{1}{PS} \times \dfrac{1}{ST}}$

$\dfrac{1}{PS} + \dfrac{1}{ST} > \dfrac{2}{\sqrt{QS \times SR}}$

Again, $\dfrac{QS + SR}{2} > \sqrt{QS \times SR}$ Or $\dfrac{QR}{2} > \sqrt{QS \times SR}$ *or* $\dfrac{1}{\sqrt{QS \times SR}} > \dfrac{2}{QR}$

$\dfrac{1}{PS} + \dfrac{1}{ST} > \dfrac{4}{QR}$



61.(BC) Given circle is $x^2 + y^2 - x + 3y = 0$ **. . . .(i)**

Given line is $x + y - 1 = 0$ **. . . .(ii)**

Equation of any line through the origin is $y - mx = 0$ **. . . .(iii)**

Let H be the centre of circle (i), then $H = \left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$

As equal chords of a circle are equidistant from the centre, we have L_1 *and* L_2 are equidistant from H.

$$\Rightarrow \left|\frac{\frac{1}{2} - \frac{3}{2} - 1}{\sqrt{2}}\right| = \left|\frac{-\frac{3}{2} - \frac{m}{2}}{\sqrt{1+m^2}}\right| \Rightarrow \sqrt{2} = \frac{3+m}{2\sqrt{1+m^2}} \Rightarrow 7m^2 - 6m - 1 = 0 \Rightarrow m = 1, -\frac{1}{7}$$

Therefore, equation of L_1, $x - y = 0$ or $x + 7y = 0$

62.(AB) The equations of two given circles are $S_1 : (x-a)^2 + y^2 = r_1^2$ and $S_2 : (x+a)^2 + y^2 = r_2^2$

Any point on the circles S_1 is $(a + r_1 \cos\theta, r_1 \sin\theta)$. Tangent to S_1 at this point is

$x\cos\theta + y\sin\theta = r_1 + a\cos\theta$ ……(i)

The line (i) touches the circle S_2 if $|r_1 + a\cos\theta + a\cos\theta| = r_2$

Or $2a\cos\theta + r_1 = \pm r_2$ ………..(ii)

Now, if the point of contact be represented by (h,k), then $h = a + r_1\cos\theta$ and $k = r_1\sin\theta$

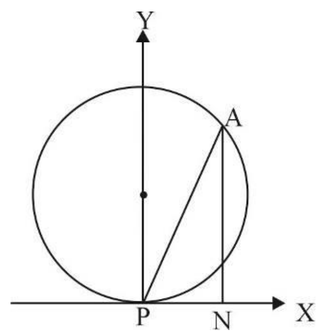
$h^2 + k^2 = a^2 + r_1^2 + 2ar_1\cos\theta = a^2 + r_1(\pm r_2)$

$h^2 + k^2 = a^2 \pm r_1 r_2$

Thus, (h,k) lies on the circle $x^2 + y^2 = a^2 \pm r_1 r_2$

63.(ABD) Let the lines L be the X-axis and the circle S be $x^2 + (y-1)^2 = 1$, which touches the line L at $P(0,0)$.

Let any point on the circle S be $A(\cos\theta, 1+\sin\theta)$



Area of $\Delta PAN = \frac{1}{2}|\cos\theta(1+\sin\theta)| = \frac{1}{2}|f(\theta)|$

Where, $f(\theta) = \cos\theta(1+\sin\theta)$

$f'(\theta) = \cos\theta(\cos\theta) - \sin\theta(1+\sin\theta) = \cos 2\theta - \sin\theta$

$f'(\theta) = 0 \Rightarrow \cos 2\theta - \sin\theta = 0$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

$f''(\theta) = -2\sin 2\theta - \cos\theta$

$f''(\theta) < 0$ if $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$

$f\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6}\left(1+\sin\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{4}$ and $f\left(\frac{5\pi}{6}\right) = \cos\frac{5\pi}{6}\left(1+\sin\frac{5\pi}{6}\right) = -\frac{3\sqrt{3}}{4}$

Maximum area of $\Delta PAN = \frac{3\sqrt{3}}{8}$ $\quad ar(\Delta PAN) \le \frac{3\sqrt{3}}{8}$

64.(ACD) Coordinates of O are (5, 3) and radius = 2

Equation of tangent at A(7, 3) is $7x + 3y - 5(x+7) - 3(y+3) + 30 = 0$

i.e. $2x - 14 = 0$ i.e. $x = 7$

Equation of tangent at B(5, 1) is $5x + y - 5(x+5) - 3(y+1) + 30 = 0$

i.e. $-2y + 2 = 0$ i.e. $y = 1$

$\therefore$ coordinates of C are (7, 1)

$\therefore$ area of OACB = 4 . . . $\because$ angle between AC & BC $90°$, therefore quadrilateral OACB is a square.

Equation of AB is $x - y = 4$ (radical axis)

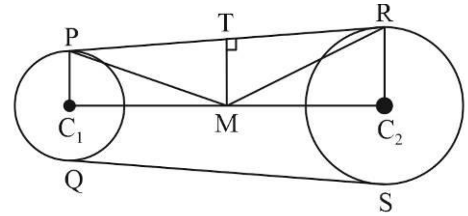
$\Rightarrow$ equation of smallest circle is $(x-7)(x-5) + (y-3)(y-1) = 0$

$x^2 + y^2 - 12x - 4y + 38 = 0$

65.(AC) Let T be the midpoint of PR, perpendicular from T to PR meets C_1C_2 at M

Also $MP^2 = MR^2 = MQ^2 = MS^2 = MT^2 + TR^2$

Hence M is centre of required circle.



$$PR = \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$$

$$= \sqrt{100-4} = \sqrt{96}$$

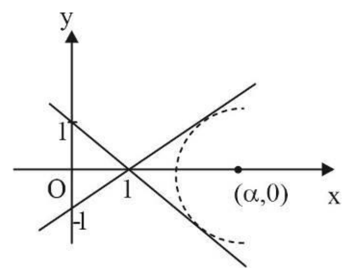
$$\lambda^2 = MT^2 + TR^2 = \left(\frac{2+4}{2}\right)^2 + TR^2 = 9 + \frac{96}{4} = 9 + 24 = 33$$

66.(AC) We know, angle bisector of angle between two tangents from a point (outside the circle) should pass through the centre of circle.

As, angle bisector of given two tangents is x-axis.

$\Rightarrow$ centre of circle will lie on x-axis.

So, Let centre $\equiv (\alpha, 0)$

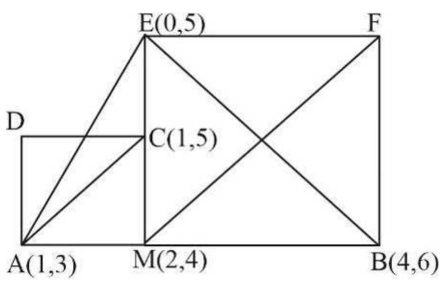


$$\Rightarrow \quad \frac{|\alpha + 0 - 1|}{\sqrt{2}} = \sqrt{(\alpha - 3)^2 + \left(0 - \sqrt{7/2}\right)^2}$$

$$\Rightarrow \quad (\alpha - 1)^2 = 2\left[(\alpha - 3)^2 + \frac{7}{2}\right] \Rightarrow \alpha = 6, 4$$

$\Rightarrow$ centre of circle $\equiv (6, 0)$ or $(4, 0)$

67.(BC)



Shown in the figure since $C(1,5)$ is the orthocentre of triangle of triangle ΔAEB .

Similarly for the other side the coordinates of $C(3,3)$

68.(ABC) The curve through the intersection of S_1 and S_2 is given by $S_1 = \lambda S_2 = 0$

Or $$x^2(\sin^2\theta + \lambda\cos^2\theta) + 2(h\tan\theta + 2h'\cot\theta)xy - (\cos^2\theta + \lambda\sin^2\theta)y^2$$
$$+(32 + 16\lambda)x + (16 + 32\lambda)y + 19(1+\lambda) = 0$$

The above equation will represent a circle if $\sin^2\theta + \lambda\cos^2\theta = \cos^2\theta + \lambda\sin^2\theta$

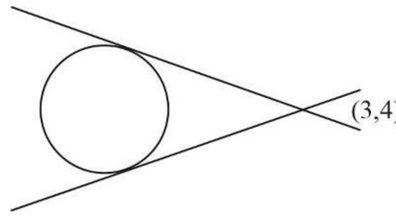
$\Rightarrow \quad (1-\lambda)\sin^2\theta = (1-\lambda)\cos^2\theta \quad \Rightarrow \quad (1-\lambda)(\sin^2\theta - \cos^2\theta) = 0$ i.e., $\lambda = 1$ or $\theta = \frac{\pi}{4}$

Also, $h\tan\theta - \lambda h'\cot\theta = 0$ [$\because$ Coefficient of xy is zero]

Or $h\tan\theta = \lambda h'\cot\theta$

Which is satisfied if $\lambda = 1$ and $\theta = \frac{\pi}{4}$ i.e., $h = h'$ or $\lambda = -1, \theta = \frac{\pi}{4}$ and, hence, $h + h' = 0$

69.(AC) Given circle $(x-1)^2+(y-3)^2=1$



Let of tangent to circle is $y-3=m(x-1)+1\sqrt{1+m^2}$

$(3,4)$ lies on axis $1=2m+\sqrt{1+m^2}$

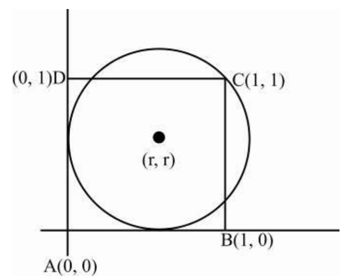
$(1-2m)^2=1+m^2$

$4m^2-4m+1=1+m^2$

$3m^2-4m=0$; $m=0,\frac{4}{3}$; $\frac{y-4}{n-3}=m=0$ *or* $\frac{4}{3}$

Smallest is 0 and largest $=\frac{4}{3}$

70. **[A-q]** **[B-p]** **[C-t]** **[D-r]**



(A) Line does not intersect the circle

(B) Line is tangent to circle, then $\frac{1-c}{\sqrt{2}}=\sqrt{2}$

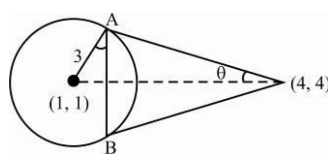
$[\because\ -1,0$ lies above the line $y-x-c=0]$ $\Rightarrow\ c=-1$

(C) Equation of given circle is $x^2+y^2-2rx-2ry+r^2=0$

It passes through (1, 1), so $r^2-4r+2=0, r=2\pm\sqrt{2}$

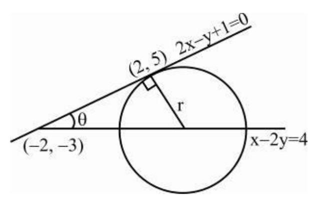
$r=2-\sqrt{2}$ $\quad\because r<1$

(D) $\sin\theta=\frac{3}{3\sqrt{2}}=\frac{1}{\sqrt{2}}$



$\theta=\frac{\pi}{4}$ $\quad AB=2r\cos\theta=2\ 3\ \frac{1}{\sqrt{2}}=3\sqrt{2}$

71. **[A-r]** **[B-p]** **[C-q]** **[D-q]**



(A) $\tan\theta=\frac{2-\frac{1}{2}}{1+2\left(\frac{1}{2}\right)}=\frac{3}{4}$

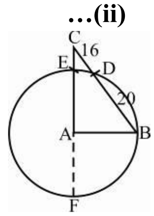
$\frac{r}{\sqrt{8^2+4^2}}=\tan\theta=\frac{3}{4}=\frac{r}{4\sqrt{5}}$ $\quad\Rightarrow\quad r=3\sqrt{5}$

(B) $AC^{\ 2}+r^2=\ 36^{\ 2}$ **…(i)**

$(CE)(CA)=(CD)(BC)$

$\Rightarrow$ $AC-r\quad AC+r\ =16\times 36$ **…(ii)**

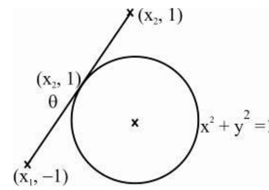
On adding (1) and (2), we get :



$\Rightarrow$ $2\ AC^{\ 2}=36\times 52$

$\Rightarrow$ $AC=6\sqrt{26}$

(C) Let the equation of tangent is $x\cos\theta + y\sin\theta = 1$



$x_1, -1$ lies on tangent

$\Rightarrow \quad x_1\cos\theta - \sin\theta = 1$

$x_1\cos\theta = 1 + \sin\theta$ **…(i)**

Now $x_2, 1$ lies on tangent

$\Rightarrow \quad x_2\cos\theta = 1 - \sin\theta$ **…(ii)**

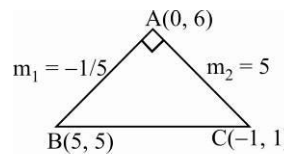
1×2, $x_1x_2\cos^2\theta = 1 - \sin^2\theta = \cos^2\theta \quad \Rightarrow \quad x_1x_2 = 1$

(D) Let circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ Put $\left(t, \frac{1}{t}\right)$

$t^4 + 2gt^3 + ct^2 + 2ft + 1 = 0$ *a*, *b*, *c*, *d* are the roots of this equation. So, $abcd = 1$

72. **[A-s]** **[B-q]** **[C-t]** **[D-r]**

(A) Note that the *ABC* is right angled at *A*



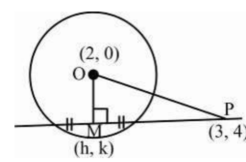
$\therefore$ Equation of circle $x+1 \;\; x-5 + \;\; y-1 \;\; y+5 \;\; = 0$

$x^2 + y^2 - 4x - 6y - 5 + 5 = 0$

$x^2 + y^2 - 4x - 6y = 0$

Hence the circle passes through the origin $\quad\therefore\quad$ tangent at (0, 0) is $4x + 6y = 0$

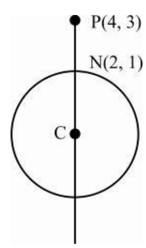
(B) Locus is Arc of the circle with *OP* as diameter intercepted by the given circle, *i.e.*,



$x-2 \;\; x-3 + y \;\; y-4 \;\; = 0$

(C) Diameter of the Circle Is Given As

$2x - y - 2 = 0$ **…(1)**



Slope of *PN*: $\dfrac{3-1}{4-2} = 1$

Equation of normal through *PN* is

$y - 1 = \;\; x - 2$

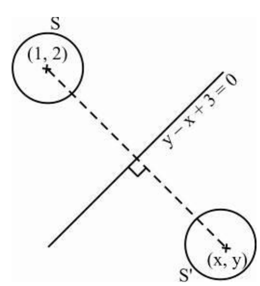
$x - y - 1 = 0$ **…(2)**

Solving (1) and (2), centre is (1, 0)

Hence equation of the circle is $x-1 \;\;^2 + y^2 = \;\; 2-1 \;\;^2 + 1$

$x^2 + y^2 - 2x - 1 = 0$

(D) $\dfrac{x-1}{-1} = \dfrac{y-2}{1} = -2\dfrac{2-1+3}{2}$



$\Rightarrow \quad x, \mathrm{y} \;\; = \;\; 5, -2$

Eqn. of *S'* is $x-5 \;\;^2 + \;\; 4+2 \;\;^2 = 1$

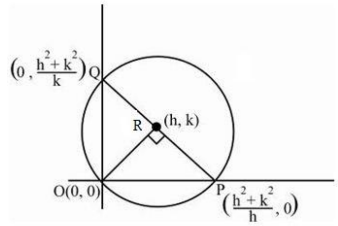
$x^2 + y^2 - 10x + 4y + 28 = 0$

73. **[A-r, B-s, C-p, D-q]**

(A) Equation of PQ is

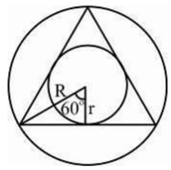
$$hx+ky=h^2+k^2$$

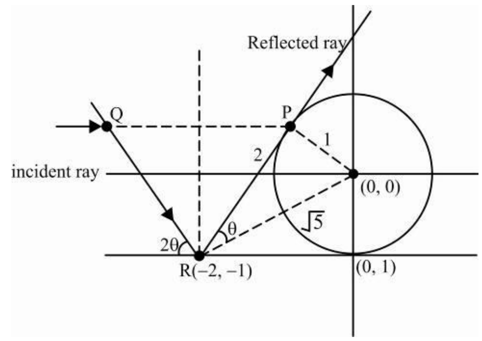
$$PQ^2=4a^2\ ;\quad \left(h^2+k^2\right)^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)=4a^2$$



(B) $r=R\cos 60^\circ=\dfrac{R}{2}$

(C) $\tan 2\theta=\dfrac{2\left(\frac{1}{2}\right)}{1-\frac{1}{4}}=\dfrac{4}{3}$





Slope of $QR=-\dfrac{4}{3}$

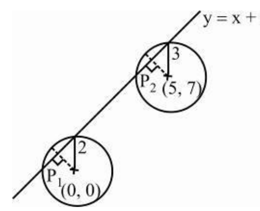
$\therefore$ Equation of incident ray QR is $y+1=-\dfrac{4}{3}(x+2)$

$$3y+4x+11=0$$

(D) $r_1^2-P_1^2=r_2^2-P_2^2$

$\Rightarrow\quad 4-\dfrac{C^2}{2}=9-\dfrac{(2-C)^2}{2}\quad\Rightarrow\quad C=-\dfrac{3}{2}$

$\Rightarrow$ Equation of line $y=x-\dfrac{3}{2}$



74. **[A-r, B-p, C-s, D-p]**

(A) Let the circle be $(x-r)^2+(y-r)^2=r^2$

$$x^2+y^2-2rx-2ry+r^2=0\ldots\ldots(r_1,\ r_2)$$

Orthogonally $\Rightarrow 2r_1r_2+2r_1r_2=r_1^2+r_2^2 \Rightarrow 6r_1r_2=(r_1+r_2)^2$

Circle passes through (a,b)

$\Rightarrow\quad r^2-2(a+b)r+(a^2+b^2)=0$

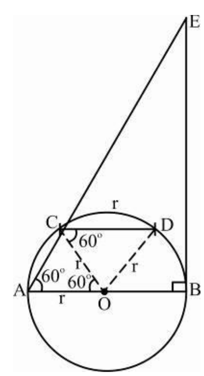
From (1), $6(a^2+b^2)=4(a+b)^2 \Rightarrow a^2+b^2-4ab=0$

(B) Put $\left(h,-\dfrac{b}{2}\right)$ to equation of circle $h^2+\dfrac{b^2}{4}-ah+\dfrac{b^2}{2}=0$

$h^2-ah+\dfrac{3b^2}{4}=0$ must get two distinct real roots $\Rightarrow D>0\Rightarrow a^2-3b^2>0$

(C) From figure it is clear that

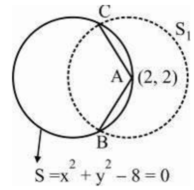
$\Rightarrow\quad AE=\dfrac{AB}{\cos 60^\circ}=2(AB)$



(D) 2r = perpendicular distance between two parallel tangents

$$\Rightarrow\quad 2r=\left|\frac{4+\frac{7}{2}}{5}\right|=\frac{15}{2\times 5}=\frac{3}{2}$$

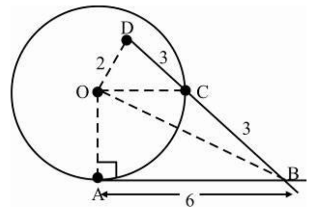
75.(8) So equation of circle having centre (2, 2) and radius 1 unit is $S_1 \equiv (x-2)^2 + (y-2)^2 = 1$



Equation of chord AB is $S_1 - S_2 = 0$

$4x + 4y = 15$

76.(4) $r^2 = \dfrac{2\ OD^2 + OB^2\ - BD^2}{4}$



$r^2 = \dfrac{2\ 4 + \ OB\ ^2\ - 6^2}{4}$ **…(1)**

Also, $OB\ ^2 = \ OA\ ^2 + \ AB\ ^2 = r^2 + 36$ **…(2)**

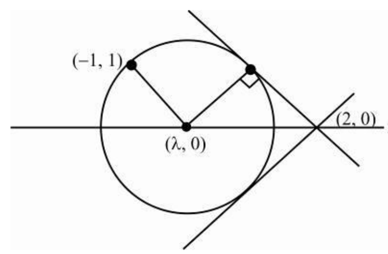
From (1) and (2), we get

$\therefore \quad 4r^2 = 2\ r^2 + 36\ - 28$

$r^2 = 22 \Rightarrow r = \sqrt{22} \qquad \Rightarrow \qquad r\ = 4$

77.(6) It is obvious that the centre of the circle will lie on x axis let it is $(\lambda, 0)$, then

$$r^2 = (\lambda+1)^2 + 1 = \left(\frac{|\lambda - 2|}{\sqrt{2}}\right)^2$$



$\lambda^2 + 8\lambda = 0 \qquad \Rightarrow \qquad \lambda = 0, -8$

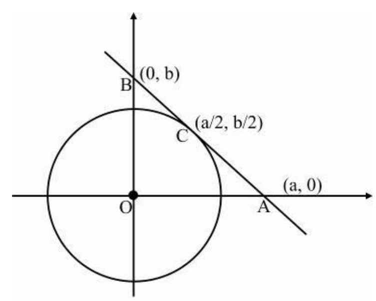
$\Rightarrow$ Equation of circle is $x^2 + y^2 = 2$

and $\quad x+8\ ^2 + y^2 = 50$ i.e., $x^2 + y^2 - 2 = 0$

and $\quad x^2 + y^2 + 16x + 14 = 0$

$r_1 + r_2 = \sqrt{2} + \sqrt{50} = 6\sqrt{2}$

78.(4) Let equation of circle is



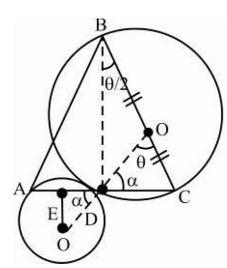
$\left(x - \dfrac{a}{2}\right)^2 + \left(y - \dfrac{b}{2}\right)^2 + \lambda\,(bx + ay - ab) = 0$ it passes through the point $\left(\dfrac{a}{2}, 0\right)$, then we get $\lambda = \dfrac{b}{2a}$

So, equation of circle will be

$$x^2 + \frac{a^2}{4} + y^2 + \frac{b^2}{4} - ax - by + \frac{b^2}{2a}x + \frac{b}{2}y - \frac{b^2}{2} = 0$$

$x^2 + y^2 + x\left(-a + \dfrac{b^2}{2a}\right) + y\left(\dfrac{b}{2} - b\right) + \dfrac{a^2}{4} - \dfrac{b^2}{4} = 0$. Radius is $\dfrac{b}{4a}\sqrt{a^2 + b^2}$

79.(1) $\cos\alpha = \dfrac{1/2}{2} = \dfrac{1}{4}$ [from ΔOED]



Then $\quad \theta = \pi - 2\alpha$

$BD = 2\cot\dfrac{\theta}{2} = 2\cot\left(\dfrac{\pi}{2} - \alpha\right) = 2\tan\alpha$ [from ΔBDC]

$= 2\sqrt{15}$

$\therefore \quad$ Area of $\Delta ABC = \dfrac{1}{2} \times\ AC\ \ BD\ = \dfrac{1}{2} \times 3 \times 2\sqrt{15} = 3\sqrt{15} = \sqrt{135}$

80.(0) Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ as centre lies an $2x - 2y + 9 = 0 \Rightarrow -2g + 2f + 9 = 0$

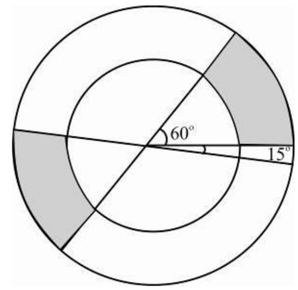
As it cuts $x^2 + y^2 - 4 = 0$ orthogonally $2g + 0 + 2f \times 0 = c - 4 = 0 \quad \Rightarrow c = 4$

So, the equation circle is $x^2 + y^2 + (2f + 9)x + 2fy + 4 = 0$

$x^2 + y^2 + 9x + 4 + 2f(x + y) = 0$ it passes through the through the intersection of

$x^2 + y^2 + 9x + 4 = 0$ and $x + y = 0 \Rightarrow$ Point as $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(-4, 4)$

81.(2) $4 \le a^2 + b^2 \le 9$ represents the region lying between the two concentric circles of radius of radius 2 and 3 having centre at $(0,0)$



For $b^2 - 4ab + a^2 \le 0$ Let $\frac{b}{a} = \tan\theta$

$$\left(\frac{b}{a}\right)^2 - \frac{4b}{a} + 1 \le 0$$

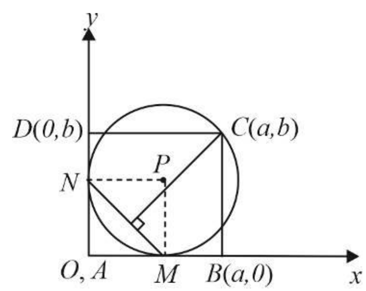
$\Rightarrow \quad \tan\theta \in \left[2-\sqrt{3}, 2+\sqrt{3}\right] \quad \Rightarrow \quad \theta \in 15°, 75°$

So the shaded region in the given figure is the region R

Hence $|a+b|$ is $|r \ \cos\theta + \sin\theta| = r\sqrt{2}|\sin \ \theta + 45°| \ge 2\sqrt{2}\sin \ 15° + 45°$ because $r \ge 2$

Minimum = $2\sqrt{2}\frac{\sqrt{3}}{2} = \sqrt{6}$

82.(25) Let r be the radius of circle



Equation of line MN is $x + y = r$

Now, $\frac{|a+b-r|}{\sqrt{2}} = 5 \Rightarrow a + b = 5\sqrt{2} + r$ ....(i)

Equation of the circle $x^2 + y^2 - 2rx - 2ry + r^2 = 0$; $\quad C(a,b)$ lies on the circle

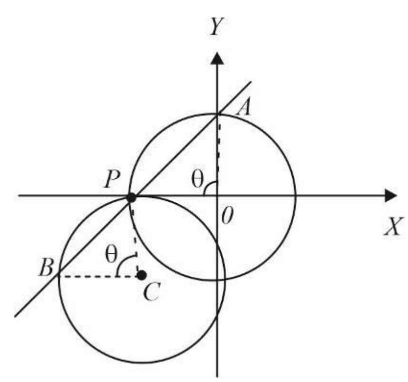
$a^2 + b^2 - 2ar - 2br + r^2 = 0$

By using (i)

$a^2 + b^2 = 2\left(5\sqrt{2} + r\right)r - r^2 = 10\sqrt{2}r + r^2$ ....(ii)

The required area of the rectangle $= ab = \frac{1}{2}\left[(a+b)^2 - \left(a^2+b^2\right)\right] = \frac{1}{2}\left[\left(5\sqrt{2}+r\right)^2 - \left(10\sqrt{2}r + r^2\right)\right] = 25$

83.(0.5) Solving $x^2 + y^2 = 1$



And $x^2 + y^2 + 2x + 4y + 1 = 0$

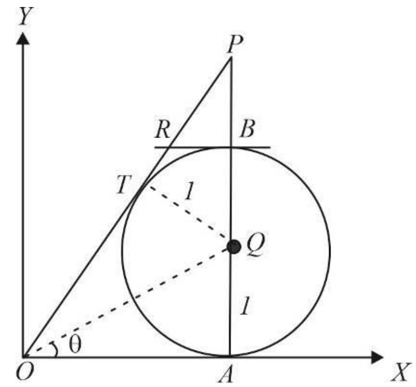
We get, $y = 0$ or $-\frac{4}{5}$; $\quad x = -1 or\, x = \frac{3}{5}$

The circles intersect at $(-1, 0)$ and $\left(\frac{3}{5}, -\frac{4}{5}\right)$

The coordinates of P are integral, so P is (–1, 0) $\Rightarrow \quad 0 = -2m + 1 \quad \Rightarrow \quad m = 0.5$

84.(1.66) Let $OA = h$ and $PQ = x$,

Triangles OAP and QTP are similar



$\Rightarrow \quad \frac{AP}{PT} = \frac{OA}{QT} = \frac{OP}{PQ} \quad \Rightarrow \quad \frac{x+1}{PT} = \frac{h}{1} = \frac{OP}{x}$ ....(i)

From equation (i), we get $OP = hx$

Perimeter of $\Delta OAP = 8$

$OA + AP + OP = 8$

$(h+1)(x+1) = 8$ ....(ii)

From equation (i) $PT = \frac{x+1}{h}$

Again, from equation (i), $OP = hx \Rightarrow OT + PT = hx$ or $PT = hx - h$

Thus, $\frac{x+1}{h} = hx - h$ or $h^2(x-1) = x+1$

$h^2\left[\frac{8}{h+1} - 2\right] = \frac{8}{h+1}$ ....from (ii)

$h^3 - 3h^2 + 4 = 0 \Rightarrow (h-2)^2(h+1) = 0$; $h = 2$

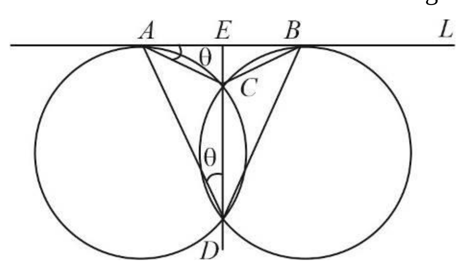
From the equation (ii) we have $(h+1)(x+1) = 8$; $x = \frac{5}{3} \Rightarrow PQ = \frac{5}{3}$

85.(1) Let R_1 be the radius of the circumcircle of ΔACB, then $R_1 = \frac{AB}{2\sin(\angle ACB)}$

Let R_2 be the radius of the circumcircle of ΔADB, then $R_2 = \frac{AB}{2\sin(\angle ADB)}$

Let the common chord CD meets the common tangent L at E. since the common chord bisects the common tangent,

So, $AE = BE$



Also $\angle EAC = \angle ADE = \theta$

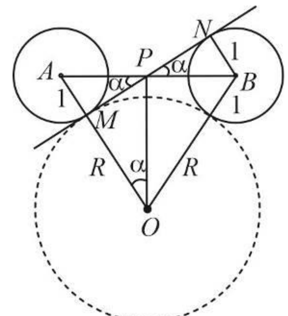
$\angle ACE = 90 - \theta$

$\angle ACB = 180 - 2\theta$

And $\angle ADB = 2\theta$

$R_1 = \frac{AB}{2\sin(180-2\theta)} = \frac{AB}{2\sin 2\theta} = R_2$; $\frac{R_1}{R_2} = 1$

86.(8) From the figure, we have $PB = 3, BN = 1$



$\tan\alpha = \frac{1}{2\sqrt{2}}$

And ΔABO is isosceles

OP is perpendicular to AB, so $\angle AOP = \alpha \Rightarrow \tan\alpha = \frac{2\sqrt{2}}{R}$

Hence, $\frac{1}{2\sqrt{2}} = \frac{2\sqrt{2}}{R} \Rightarrow R = 8$

87.(4) On adding and subtracting the curves we get $(x+3)^2 = 4(y-1)$, $y^2 - 20y + 59 = 0$ respectively

Let point of intersection be $(x_1, y_1)(x_2, y_1)(x_3, y_2)(x_4, y_2)$ $\Rightarrow$ $y_1 + y_2 = 20$ and $y_1, y_2 > 0$

The required sum of distances = $\sum\sqrt{(x_1+3)^2+(y_1-2)^2} = \sum\sqrt{4(y_1-1)+(y_1-2)^2} = \sum y_1 = 2(y_1+y_2) = 40$

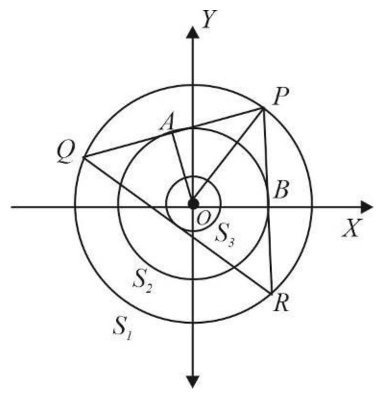
88.(40) Let us select the centre of the series of circles as origin O,

Let the tangents drawn from any point on the circle $S_1 : x^2 + y^2 = a^2$ to $S_2 : x^2 + y^2 = b^2$ be such that the chord of contact touches the circle $S_3 : x^2 + y^2 = c^2$.

Any point, P on the circle S_1 is $(a\cos\theta, a\sin\theta)$

The chord of contact of tangents drawn from P to the circle S_2 is $(a\cos\theta)x + (a\sin\theta)y = b^2$

Which touches S_3 if $b^2 = ac$



a, b, and c are in geometric progression.

Thus, the conditions of the problem hold if $r_1, r_2, r_3, ...$ are in G.P.

Again, the angle between the tangents PA and PB is $\frac{\pi}{3}$

$\angle OPQ = \frac{\pi}{6}$

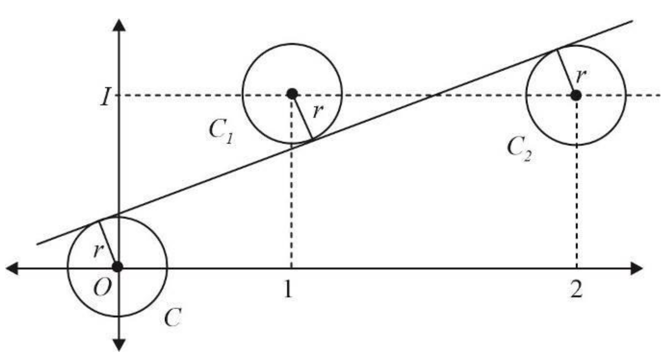
$\sin\frac{\pi}{6} = \frac{OA}{OP} = \frac{b}{a} \Rightarrow \frac{b}{a} = \frac{1}{2}$

Thus, the common ratio of G.P. is $\frac{1}{2}$

Hence, the radii $r_1, r_2, r_3, ...$ form a G.P. of common ratio $\frac{1}{2}$

Thus, $\lim\limits_{n\to\infty}\sum_{i=1}^{n} r_i = \lim\limits_{n\to\infty}(r_1 + r_2 + ... + r_n) = \lim\limits_{n\to\infty}\frac{r_1\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = \lim\limits_{n\to\infty} 2r_1\left[1-\left(\frac{1}{2}\right)^n\right] = 2r_1 = 2(20) = 40$

89. (5)



The equation of the line joining the origin and the centre of circle $C_2(2,1)$ is

$y = \frac{x}{2}$ or $x - 2y = 0$

Let the equation of common tangent be $x-2y+c=0$ …..(i)

Perpendicular distance from (0, 0) on this line $=$ Perpendicular distance from (1, 1)

Or $\left|\frac{c}{\sqrt{5}}\right|=\left|\frac{c-1}{\sqrt{5}}\right|$ or $c=1-c\,or\,c=\frac{1}{2}$

The equation of common tangent is $x-2y+\frac{1}{2}=0$ *or* $2x-4y+1=0$ ………..(ii)

The length of perpendicular from $(2,1)$ on the line (ii) is $r=\left|\frac{4-4+1}{\sqrt{20}}\right|=\frac{1}{2\sqrt{5}}=\frac{\sqrt{5}}{10}$

90.(1) $x^2+y^2+(3+\sin\beta)x+(2\cos\alpha)y=0$ ……..(i)

$x^2+y^2+(2\cos\alpha)x+2cy=0$ ………….(ii)

Since both the circles are passing through the origin $(0,0)$, the equation of tangent at $(0,0)$

Tangent at $(0,0)$ to circle (ii) is therefore, $(3+\sin\beta)x+(2\cos\alpha)y=0$

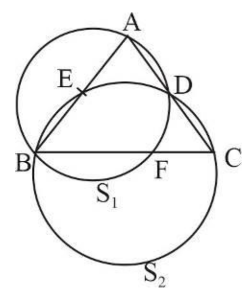
Tangent at $(0,0)$ to circle (ii) is $(2\cos\alpha)x+2cy=0$

Therefore, (i) and (ii) must be identical

Comparing (i) and (ii), we get $\frac{3+\sin\beta}{2\cos\alpha}=\frac{2\cos\alpha}{2c}$ or $c=\frac{2\cos^2\alpha}{3+\sin\beta}$ or $c_{max}=1\ when\sin\beta=-1 and\ \alpha=0$

91.(3) Let circumcircle of triangle BDC be S_1 and the circumcircle of triangle ABD be S_2

So, $\frac{AD}{CD}=\frac{AB}{BC}$ ……….(i)



Now, for circle S_1; $AE\times AB=AD\times AC$

$AE=\frac{AD\times AC}{AB}$ ………(ii)

For circle S_2; $CF\times CB=CD\times CA$

$CF=\frac{CD\times CA}{AB}$ ………(iii)

From equation (ii) and (iii), we get $\frac{AE}{CF}=\frac{AD\times CB}{AB\times CD}$ …….(iv)

From equation (i) and (iv), $AE=CF$ So, $CF=3$

92.(3) The circumcenter of the triangle formed by any three points lies at the origin. The orthocenter of the triangle formed by the point $A_1(x_1,y_1),A_2(x_2,y_2)\,and\,A_3(x_3,y_3)$ is given by $H(x_1+x_2+x_3,y_1+y_2+y_3)=(a,b)$

The centroid of the triangle formed by the other three points is given by

$G\left(\frac{x_4+x_5+x_6}{3},\frac{y_4+y_5+y_6}{3}\right)or\,G\left(\frac{8-a}{3},\frac{4-b}{3}\right)$

Equation of the line joining H and G, is $y-b=\frac{\frac{4-b}{3}-b}{\frac{8-a}{3}-a}(x-a)$ or $(2-a)(y-b)=(1-b)(x-a)$ ………(i)

The above equation holds for all values of 'a' and 'b' if $x=2$ and $y=1$

Thus, the line (i) always passes through the fixed point (2, 1).

93.(0) Let point on circle be $(\cos\theta, \sin\theta)$

If this point lies on the curve $\sin x = \cos y$, we have

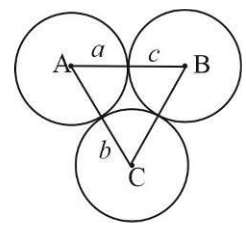
$$\sin(\cos\theta) = \cos(\sin\theta) \Rightarrow \frac{\pi}{2} - \cos\theta = 2n\pi \pm \sin\theta,\ n \in Z$$

$$\cos\theta \pm \sin\theta = \frac{\pi}{2} - 2n\pi$$

But $\left|\frac{\pi}{2} - 2n\pi\right| > \sqrt{2}$ for $\forall\ n \in Z$

$\Rightarrow$ No solution $\Rightarrow$ Number of points of intersection = 0.

94.(8) In radius of ΔABC is 4



We have to find $\frac{abc}{2(a+b+c)}$

$$r = \frac{\Delta}{S}$$

$$S = a+b+c,\ \Delta = \sqrt{abc(a+b+c)} \Rightarrow 4 = \sqrt{\frac{abc}{a+b+c}} \Rightarrow \frac{abc}{2(a+b+c)} = 8$$

95.(5) Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Since, it passes through (3, 4), $6g + 8f + c = -25$. As it cuts the circle $x^2 + y^2 = a^2$ orthogonally

$$2g \times 0 + 2f \times 0 = c - a^2 \Rightarrow c = a^2$$

$\Rightarrow$ $6g + 8f + a^2 + 25 = 0$

$\therefore$ Locus of the centre $(-g, -f)$ is $6x + 8y - (a^2 + 25) = 0$.

$\therefore$ Distance of the line from the origin is $817 = \frac{a^2+25}{\sqrt{36+64}} \Rightarrow a^2 + 25 = 8170 \Rightarrow a^2 = 8145$

96.(2) $S_1 : C_1(-5, 12)$

$S_2 : C_2(5, 12)$

$r_1 = 16,\ r_2 = 4$

$CC_2 = r + 4$ (Where C is the centre of circle touching C_2 externally and C_1 internally)

$CC_1 = 16 - r$ (Not $r - 16, \because S_2$ is contained by S_1)

$\Rightarrow$ $CC_1 + CC_2 = 20$

$\therefore$ Locus of C is the ellipse with foci at C_1 and C_2 and length of major axis = 20

$\therefore$ Locus of C is

$$\frac{x^2}{100} + \frac{(y-12)^2}{75} = 1 \qquad \text{........(i)}$$

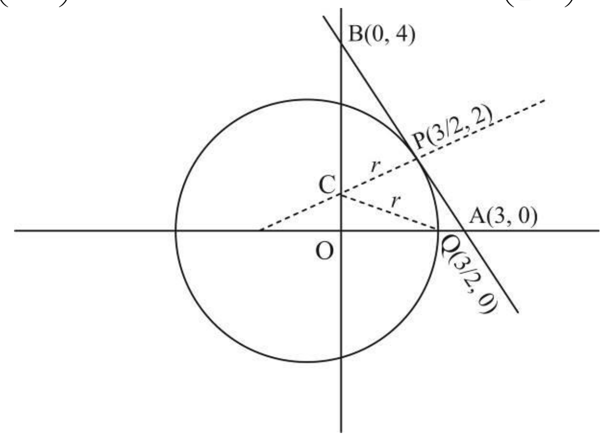
According to question $y = ax$ is tangent to this ellipse from (0, 0)

Equation of tangent to (i) is $y - 12 = mx + \sqrt{100m^2 + 75}$

But this is passing through (0, 0)

$\Rightarrow$ $-12 = \sqrt{100m^2 + 75} \Rightarrow m^2 = \frac{69}{100} \Rightarrow p + q = 169 \Rightarrow k = 2$.

97.(5) Let the centre be C(h, k). And P be the mid point of $AB \Rightarrow P = \left(\frac{3}{2}, 2\right)$ and also Q is mid point of $OQ \Rightarrow Q = \left(\frac{3}{2}, 0\right)$



$\because \quad CP \perp AB \Rightarrow \dfrac{2-k}{\frac{3}{2}-h} = \dfrac{3}{4}$

$\Rightarrow \quad 6h - 8k = -7 \qquad$ **. . . .(i)**

$\because \quad$ CP = CQ

$\Rightarrow \quad \left(h - \frac{3}{2}\right)^2 + (k-2)^2 = \left(h - \frac{3}{2}\right)^2 + k^2$

$\Rightarrow \quad k = 1$, putting in equation (i), we get :

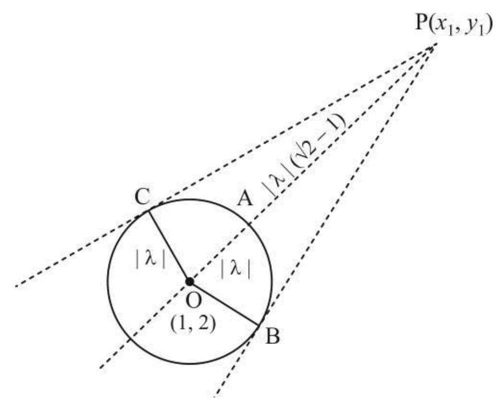
$$6h = 1 \Rightarrow h = \frac{1}{6}$$

$\therefore \quad$ Radius (r) = $CQ = \sqrt{\left(\frac{1}{6} - \frac{3}{2}\right)^2 + 1} = \sqrt{\left(\frac{1-9}{6}\right)^2 + 1} = \frac{5}{3} \qquad \therefore \qquad 3r = 5.$

98.(6) Since the line $y = x + c$ is normal to the given circle, $c = 1$.

So the equation of line is $y = x + 1 \qquad$ **. . . .(i)**

Also the radius of the circle is $|\lambda|$.



Given $AP = |\lambda|\left(\sqrt{2} - 1\right)$

$\Rightarrow \quad OP = \sqrt{2}|\lambda| \qquad \Rightarrow \qquad PC = |\lambda|$

$\Rightarrow \quad$ Area of quadrilateral OBPC

$$= 2 \times \frac{1}{2}|\lambda|^2 = 36 \Rightarrow \lambda = \pm 6$$

99.(7) Given circles are $x^2 + y^2 = 4 \qquad$ **. . . .(i)**

and $\quad x^2 + y^2 - 6x - 8y + m^2 = 0 \qquad$ **. . . .(ii)**

Centre of circle (i) is $A(0,0)$ and that of circle (ii) is $B(3,4)$

Radius of circle (i), $r_1 = 2$ and radius of circle (ii), $r_2 = \sqrt{25 - m^2}$

For $\sqrt{25 - m^2}$ to be defined $25 - m^2 \ge 0 \Rightarrow -5 \le m \le 5 \qquad$ **. . . .(A)**

For circles (i) and (ii) to have exactly two common tangents $|r_1 - r_2| < AB < r_1 + r_2$

(i) $\quad$ Now, $AB < r_1 + r_2 \Rightarrow 5 < 2 + \sqrt{25 - m^2}$

$\Rightarrow \quad 9 < 25 - m^2 \Rightarrow m^2 - 16 < 0 \Rightarrow -4 < m < 4 \qquad$ **. . . .(I)**

$\therefore \quad \sqrt{25 - m^2} > 3 \Rightarrow r_2 > r_1$

(ii) $\quad AB > |r_1 - r_2|$

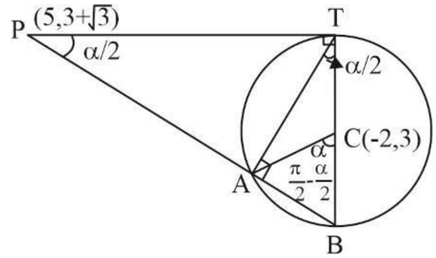
$5 > r_2 - r_1 \Rightarrow 5 > \sqrt{25 - m^2} - 2 \Rightarrow 49 > 25 - m^2 \Rightarrow m^2 + 24 > 0 \Rightarrow -\infty < m < \infty$ **. . . .(II)**

From (I) and (II), $-4 < m < 4$

Therefore, possible integral values of m are : $-3, -2, -1, 0, 1, 2, 3$

Hence number of possible integral values of m is 7

100.(4) $CT = CB = r = 4$



$$PT = \sqrt{5^2 + \left(3+\sqrt{3}\right)^2 + 4\times 5 - 6\left(3+\sqrt{3}\right) - 3}$$

$PT = 6$

Let $\angle ACB = \alpha$

$$Ar\left(\Delta CAB\right) = \frac{1}{2} r.r\, sin\,\alpha = 8\, sin\,\alpha$$

$$Ar\left(\Delta CAT\right) = \frac{1}{2} r.r\, sin\left(\pi - \alpha\right) = 8\, sin\,\alpha$$

In ΔPBT $tan\frac{\alpha}{2} = \frac{BT}{PT} = \frac{8}{6}$

$$sin\,\alpha = \frac{2\, tan\frac{\alpha}{2}}{1 + tan^2\frac{\alpha}{2}} = \frac{2\times\frac{4}{3}}{1+\frac{16}{9}} = \frac{24}{25}$$

$$Ar\left(\Delta CAB\right) + Ar\left(\Delta CAT\right) = 16\, sin\,\alpha = 16\times\frac{24}{25} = \frac{384}{25}$$

$\lambda = 384\, hence \left[\sqrt{384}\right] - 15 = 19 - 15 = 4$

101.(3) The equation of first circle $(x-1)^2 + (y-3)^2 = r^2$

whose centre is $C_1(1,\, 3)$ and radius $r_1 = r$ and equation of second circle $x^2 + y^2 - 8x + 2y + 8 = 0$

whose centre is $C_2(4,\, -1)$ and radius $r_2 = \sqrt{4^2 + 1^2 - 8} = \sqrt{17-8} = \sqrt{9} = 3$

Two circles intersect in two distinct points, then:

$$|r_1 - r_2| < C_1C_2 < r_1 + r_2 \quad \Rightarrow \quad |r-3| < \sqrt{(4-1)^2 + (-1-3)^2} < r+3$$

$$\Rightarrow \quad |r-3| < \sqrt{9+16} < r+3 \quad \Rightarrow \quad |r-3| < 5 < r+3$$

$$\Rightarrow \quad |r-3| < 5 \text{ and } 5 < r+3 \quad \Rightarrow \quad 2 < r < 8 \quad \Rightarrow \quad r = 3,\, 5,\, 7$$

CONIC SECTION

1.(A) Let the point be $P(at^2, 2at)$.

Then according to question, $SP = at^2 + a = k$ **. . . .(i)**

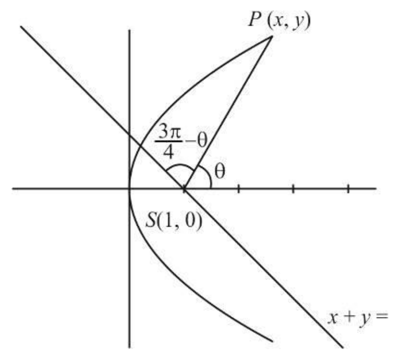
Let (α, β) is the moving point, then $\alpha = at^2, \beta = 2at$

$\Rightarrow \quad \frac{\alpha}{\beta} = \frac{t}{2}$ and $a = \frac{\beta^2}{4\alpha}$ $\quad$ ($\because$ point (α, β) lies on $y^2 = 4ax$)

On substituting these values in equation (i),

$$\frac{\beta^2}{4\alpha}\left(1 + \frac{4\alpha^2}{\beta^2}\right) = k \Rightarrow \beta^2 + 4\alpha^2 = 4k\alpha \Rightarrow 4x^2 + y^2 - 4kx = 0$$ is the required locus.

2.(A) Projection on $x + y = 1$ is $L = SP\cos\left(\frac{3\pi}{4} - \theta\right)$ where $\tan\theta$ is slope of SP



$$SP = x + 1$$

$$\frac{dL}{dt} = \frac{d}{dt}\left[(x+1)\cos\left(\frac{3\pi}{4} - \theta\right)\right]$$

$$\tan\theta = \frac{y}{x-1}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{(x-1)}\frac{dy}{dt} - \frac{y}{(x-1)^2}\frac{dx}{dt}$$

P lies on $y^2 = 4x$

Hence, $\quad 2y\frac{dy}{dt} = \frac{4dx}{dt}, \quad \frac{dy}{dt} = \frac{8}{y}$

So, at $x = y = 4$, we have $\sec^2\theta = 1 + \left(\frac{4}{3}\right)^2, \quad \frac{dy}{dt} = 2$

Hence, $\quad \frac{25}{9}\frac{d\theta}{dt} = \frac{2}{3} - \frac{16}{9}$

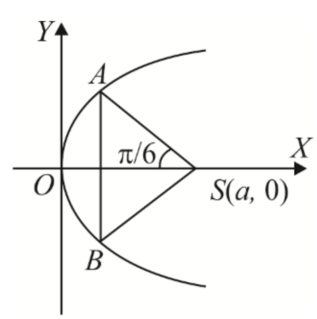
$$\frac{25}{9}\frac{d\theta}{dt} = -\frac{10}{9} \Rightarrow \frac{d\theta}{dt} = -\frac{2}{5}$$

Hence, $\quad \frac{dL}{dt} = \cos\left(\frac{3\pi}{4} - \theta\right) \times 4 + (x+1)\sin\left(\frac{3\pi}{4} - \theta\right) \times \frac{d\theta}{dt}$

Also, $\quad \sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5} \Rightarrow \frac{dL}{dt} = -\sqrt{2}$

3.(B) Let $A = (at_1^2, 2at_1), B \equiv (at_1^2, -2at_1)$.

We have $\quad m_{AS} = \tan\left(\frac{5\pi}{6}\right)$



$$\Rightarrow \frac{2at_1}{at_1^2 - a} = -\frac{1}{\sqrt{3}} \Rightarrow t_1^2 + 2\sqrt{3}t_1 - 1 = 0 \Rightarrow t_1 = -\sqrt{3} \pm 2$$

Clearly $t_1 = -\sqrt{3} - 2$ is rejected.

Thus, $\quad t_1 = (2 - \sqrt{3})$

Hence, $\quad AB = 4at_1 = 4a(2 - \sqrt{3})$

4.(B) Let $y_1 = 12 - \sqrt{1 - x_1^2}$ and $y_2 = \sqrt{4x_2}$ or $x_1^2 + (y_1 - 12)^2 = 1$ and $y_2^2 = 4x_2$

Thus (x_1, y_1) lies on the circle $x^2 + (y-12)^2 = 1$ and (x_2, y_2) lies on the parabola $y^2 = 4x.$

Thus, given expression is the shortest distance between the curves $x^2 + (y-12)^2 = 1$ and $y^2 = 4x.$

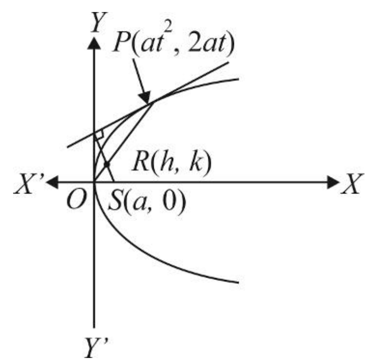
Now the shortest distance always occurs along common normal to the curves and normal to circle passes through the centre of the circle.

Normal to the parabola $y^2 = 4x$ is $y = mx - 2m - m^3$ passes through (0, 12) gives $m^3 + 2m + 12 = 0,$ which has only one real root $m = -2.$

Hence, corresponding point on the parabola is (4, 4).

Thus, required minimum distance $= \sqrt{4^2 + 8^2} - 1 = 4\sqrt{5} - 1.$

5.(B) Tangent at point P is $ty = x + at^2.$ . . . .(i)



Line perpendicular to equation (i) passes through (a, 0)

$\therefore \quad y - 0 = -t(x - a)$ or $tx + y = ta$ or $y = t(a - x)$ . . . .(ii)

Equation of OP

$$y - \frac{2}{t}x = 0 \text{ or } y = \frac{2}{t}x \quad \text{. . . .(iii)}$$

From equations (ii) and (iii), eliminating t, we get

$$y^2 = 2x(a - x) \quad \text{or} \quad 2x^2 + y^2 - 2ax = 0$$

6.(D) For the two ellipses to intersect at four distinct points, $a > 1$

$\Rightarrow b^2 - 5b + 7 > 1 \Rightarrow b^2 - 5b + 6 > 0 \Rightarrow b \in (-\infty, 2) \cup (3, \infty) \Rightarrow b$ does not lie in [2, 3]

7.(B) Since there are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$ whose distance from centre is same, the points would be either end points of the major axis or of the minor axis.

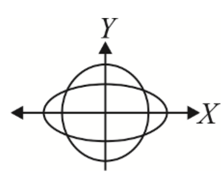
But $\sqrt{\frac{a^2 + 2b^2}{2}} > b,$ so the points are the vertices of major axis.

Hence, $a = \sqrt{\frac{a^2 + 2b^2}{2}} \quad \Rightarrow \quad a^2 = 2b^2 \quad \Rightarrow \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$

8.(B) Radius of the circle having SS' as diameter is $r = ae$

If it cuts an ellipse, then $r > b$

$\Rightarrow \quad ae > b \quad \Rightarrow \quad e^2 > \frac{b^2}{a^2}$

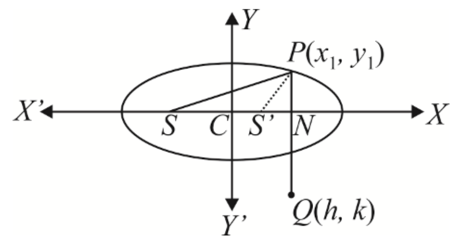


$\Rightarrow \quad e^2 > 1 - e^2 \quad \Rightarrow \quad e^2 > \frac{1}{2}$

$\Rightarrow \quad e > \frac{1}{\sqrt{2}} \quad \Rightarrow \quad e \in \left(\frac{1}{\sqrt{2}}, 1\right)$

9.(B) $a^2 = 25$ and $b^2 = 16$

$\Rightarrow \quad e = \sqrt{1 - \dfrac{16}{25}} = \dfrac{3}{5}$



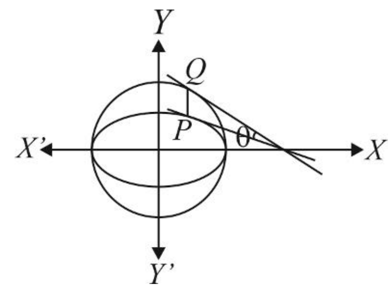
Let point Q be (h, k), where $k < 0$

Given that $|k| = SP = a + ex_1$, where $P(x_1, y_1)$ lies on the ellipse

$\Rightarrow \quad |k| = a + eh$ (as $x_1 = h$)

$\Rightarrow \quad -y = a + ex \Rightarrow 3x + 5y + 25 = 0$

10.(A) Tangent to the ellipse at $P(a\cos\alpha, b\sin\alpha)$ is $\dfrac{x}{a}\cos\alpha + \dfrac{y}{b}\sin\alpha = 1$



Tangent to the circle at $Q(a\cos\alpha, a\sin\alpha)$ is $\cos\alpha x + \sin\alpha y = a$

Now angle between tangents is θ,

Then $\quad \tan\theta = \left|\dfrac{-\dfrac{b}{a}\cot\alpha - (-\cot\alpha)}{1 + \left(-\dfrac{b}{a}\cot\alpha\right)(-\cot\alpha)}\right|$

$$= \left|\frac{\cot\alpha(1 - \frac{b}{a})}{1 + \frac{b}{a}\cot^2\alpha}\right| = \left|\frac{a-b}{a\tan\alpha + b\cot\alpha}\right| = \left|\frac{a-b}{(\sqrt{a\tan\alpha} - \sqrt{b\cot\alpha})^2 + 2\sqrt{ab}}\right|$$

Now the greatest value of the above expression is $\left|\dfrac{a-b}{2\sqrt{ab}}\right|$ when $\sqrt{a\tan\alpha} = \sqrt{b\cot\alpha}$

$\Rightarrow \quad \theta_{\text{maximum}} = \tan^{-1}\left(\dfrac{a-b}{2\sqrt{ab}}\right)$

11.(B) For hyperbola $\dfrac{x^2}{5} - \dfrac{y^2}{5\cos^2\alpha} = 1$

We have $e_1^2 = 1 + \dfrac{b^2}{a^2} = 1 + \dfrac{5\cos^2\alpha}{5} = 1 + \cos^2\alpha$

For ellipse $\dfrac{x^2}{25\cos^2\alpha} + \dfrac{y^2}{25} = 1$

We have $e_2^2 = 1 - \dfrac{25\cos^2\alpha}{25} = \sin^2\alpha$

Given that $e_1 = \sqrt{3}e_2$

$\Rightarrow \quad e_1^2 = 3e_2^2 \quad \Rightarrow \quad 1 + \cos^2\alpha = 3\sin^2\alpha$

$\Rightarrow \quad 2 = 4\sin^2\alpha \quad \Rightarrow \quad \sin\alpha = \dfrac{1}{\sqrt{2}}$

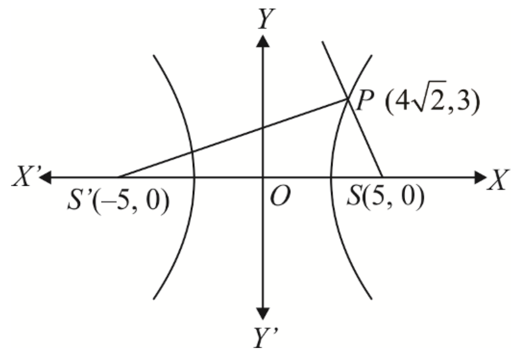
12.(D) The given hyperbola is $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1 \;\Rightarrow\; e=\dfrac{5}{4}$

Its foci are $(\pm 5, 0)$.

The ray is incident at $P(4\sqrt{2}, 3)$.

The incident ray passes through $(5, 0)$; so the reflected ray must pass through $(-5, 0)$.

Its equation is $\dfrac{y-0}{x+5}=\dfrac{3}{4\sqrt{2}+5}$ or $3x-y(4\sqrt{2}+5)+15=0$

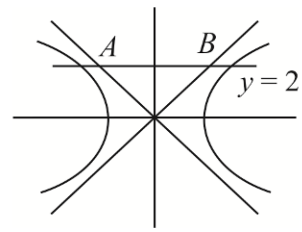


13.(A) For two distinct tangents on different branches the point should lie on the line $y=2$ and between A and B (where A and B the points on the asymptotes).

Equations of asymptotes are $4x=\pm 3y$

Solving with $y=2$, we have

$$x=\pm\frac{3}{2} \qquad \therefore \qquad -\frac{3}{2}<\alpha<\frac{3}{2}$$



14.(A) Any tangent to hyperbola forms a triangle with the asymptotes which has constant area ab.

Given $ab=a^2\tan\lambda$

$$\Rightarrow \quad \frac{b}{a}=\tan\lambda \;\Rightarrow\; e^2-1=\tan^2\lambda \;\Rightarrow\; e^2=1+\tan^2\lambda=\sec^2\lambda \;\Rightarrow\; e=\sec\lambda$$

15.(A) If (x_t, y_t) is the point of intersection of given curves, then $\dfrac{\sum_{i=1}^{4}x_i}{4}=\dfrac{1+1}{2}$ and $\dfrac{\sum_{i=1}^{4}y_i}{4}=0$

Now $\dfrac{\sum_{i=1}^{3}x_i}{3}=\dfrac{4-x_4}{3}$ and $\dfrac{\sum_{i=1}^{3}y_i}{3}=-\dfrac{y_4}{3}$

Centroid $\left(\dfrac{\sum_{i=1}^{3}x_i}{3}, \dfrac{\sum_{i=1}^{3}y_i}{3}\right)$ lies on the line $y=3x-4$. Hence, $\dfrac{-y_4}{3}=\dfrac{3(4-x_4)}{3}-4 \;\Rightarrow\; y_4=3x_4$

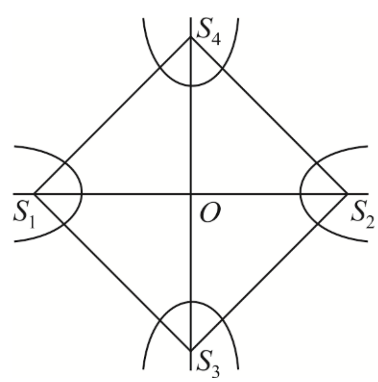
16.(B) Required area $=4$ area $\Delta S_2OS_4=4\times\dfrac{1}{2}ae\times be_1$

$$=4\times\frac{1}{2}\times 2\times 3\times ee_1 \qquad \text{....(i)}$$

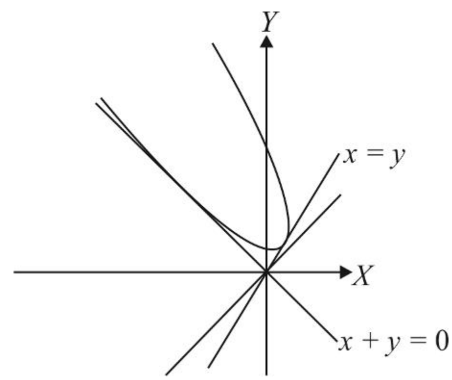
$$b^2=a^2(e^2-1)\Rightarrow e^2=\frac{9}{4}+1=\frac{13}{4}$$

Also $\dfrac{1}{e_1^2}=1-\dfrac{1}{e^2}=1-\dfrac{4}{13}=\dfrac{9}{13}$

$e_1^2=\dfrac{13}{9}$ Required area $=12\times\dfrac{\sqrt{13}}{2}\times\dfrac{\sqrt{13}}{3}=2\times 13=26$



17-19. **17.(A)** **18.(B)** **19.(C)**



We know that foot of perpendicular from focus upon any tangent lies on the tangent at vertex of the parabola.

Now, if foot of perpendicular of (2, 3) on the line $x-y=0$ is (x_1, y_1),

then $\dfrac{x_1-2}{1}=\dfrac{y_1-3}{-1}=-\dfrac{2-3}{2} \Rightarrow x_1=\dfrac{5}{2}$ and $y_1=\dfrac{5}{2}$

If foot of perpendicular of $(2,3)$ on the line $x+y=0$ is (x_2, y_2),

then $\dfrac{x_2-2}{1}=\dfrac{y_2-3}{1}=-\dfrac{2+3}{2} \Rightarrow x_2=-\dfrac{1}{2}$ and $y_2=\dfrac{1}{2}$

Now tangent at vertex passes through the points $\left(\dfrac{5}{2},\dfrac{5}{2}\right)$ and $\left(-\dfrac{1}{2},\dfrac{1}{2}\right)$.

Then, its equation is $y-\dfrac{1}{2}=\dfrac{2}{3}\left(x+\dfrac{1}{2}\right)$ or $4x-6y+5=0$

Latus rectum of the parabola

$$=4\times\text{(Distance of focus from tangent at vertex)} =4\times\left|\frac{8-18+5}{\sqrt{52}}\right|=\frac{10}{\sqrt{13}}$$

We know that $\dfrac{1}{SP}+\dfrac{1}{SQ}=\dfrac{1}{a}$, where a is $\left(\dfrac{1}{4}\right)$th of latus rectum.

$$\Rightarrow \quad \frac{1}{SP}+\frac{1}{SQ}=\frac{2\sqrt{13}}{5}$$

20-22. **20.(B)** **21.(A)** **22.(C)**

$$21x^2-6xy+29y^2+6x-58y-151=0$$

$$3(x-3y+3)^2+2(3x+y-1)^2=180$$

$$\Rightarrow \quad \frac{(x-3y+3)^2}{60}+\frac{(3x+y-1)^2}{90}=1 \Rightarrow \left(\frac{x-3y+3}{\sqrt{1+3^2}\sqrt{6}}\right)^2+\left(\frac{3x+y-1}{3\sqrt{1+3^2}}\right)^2=1$$

Thus C is an ellipse whose lengths of axes are $6, 2\sqrt{6}$.

Hence e = $1/\sqrt{3}$

The major and the minor axes are $x-3y+3=0$ and $3x+y-1=0$, respectively.

Their point of intersection gives the centre of the conic. $\therefore$ Centre $\equiv (0,1)$

23-25.

23.(C) $2a=3$

Distance between the foci (1, 2) and (5, 5) is 5.

$\therefore \quad 2ae=5 \qquad \therefore \quad e=\dfrac{5}{3}$

Now if e' is eccentricity of the corresponding conjugate hyperbola, then

$$\frac{1}{e^2}+\frac{1}{e'^2}=1 \quad \Rightarrow \quad e'=\frac{5}{4}$$

24.(D) Director circle is given by $(x-h)^2+(y-k)^2=a^2-b^2$

Where (h,k) is centre, i.e. the midpoint of foci $\left(\frac{1+5}{2},\frac{2+5}{2}\right)\equiv\left(3,\frac{7}{2}\right)$.

$$b^2=a^2(e^2-1)=\left(\frac{3}{2}\right)^2\left(\left(\frac{5}{3}\right)^2-1\right)=4$$

Therefore, the director circle is $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{9}{4}-4 \quad\Rightarrow\quad (x-3)^2+\left(y-\frac{7}{2}\right)^2=-\frac{7}{4}$

This does not represent any real point.

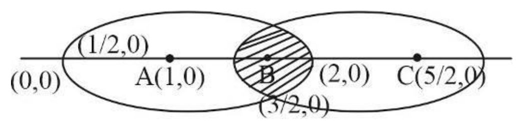
25.(B) Slope of transverse axis is $\frac{3}{4}$.

Therefore, the angle of rotation is $\theta=\tan^{-1}\frac{3}{4}$.

26-28 **26.(A)** **27.(D)** **28.(B)**

$PA+PB<2 \text{ and } PB+PC<2$

Region inside ellipse with foci A and B $2a=2, a=1$



$2ae=\frac{3}{2}-\frac{1}{2}=1$

$b^2=a^2-a^2e^2=1-\frac{1}{4}=\frac{3}{4}$

Equation of ellipse is $\frac{(x-1)^2}{1}+\frac{y^2}{3/4}=1$

$PB+PC<2\Rightarrow P$ lies inside ellipse with foci B and C

Whose equation is $\frac{(x-2)^2}{1}+\frac{y^2}{3/4}=1$

Locus of P is shown by shaded region which is symmetric about x-axis

Area of region $=4\int_1^{3/2}\frac{\sqrt{3}}{2}\sqrt{1-(x-2)^2}\,dx=\sqrt{3}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$

29.(B) Any normal to $y^2=8x \text{ is } mx-4m-2m^3$ and this passes through the point P if

$2m^3+(4-h)m=0$ ……….(i)

Let m_1,m_2,m_3 be roots, then $m_1+m_2+m_3=0$

$m_1m_3+m_3m_1+m_1m_2=\frac{4-h}{2} \text{ and } m_1m_2m_3=-\frac{k}{2}$

$m_1m_2=-1 \text{ and } m_3=\frac{k}{2}$ ……..(ii)

m_3 is the root of equation (i) $\frac{k^3}{4}+(4-h)\frac{k}{2}+k=0$ i.e. $k^2=2(h-6)$

$y^2=2(x-6)$

30.(A) Latus rectum of $y^2 = 8x \, is \, 8$

latus rectum of $y^2 = 2(x-6) \, is \, 2 \; ratio = 4:1$

31.(B) From (ii), $m_3 = \frac{k}{2}$ it passes through (x_1, y_1)

$m_3 = \frac{y_1}{2}$

32.(A) Equation of tangent in parametric form $\frac{x-1}{-1/\sqrt{2}} = \frac{y-1}{1/\sqrt{2}} = \pm 3\sqrt{2}$

$A \equiv (4,-2), B \equiv (-2,4)$

Equation of asymptotes (OA and OB)

$y+2 = \frac{-2}{4}(x-4) \Rightarrow 2y+4+x-4=0$

$2y+x=0 \; and \; y-4 = \frac{4}{-2}(x+2) \Rightarrow y-4=-2x-4$

$2y+x=0, (2x+y)(x+2y)=0 \Rightarrow 2x^2+2y^2+5xy=0$

33.(C) $m_{OQ} = -\frac{1}{2}, m_{OB} = -2$

$tan\,\theta = \left|\frac{-1/2+2}{1+1}\right| = \frac{3}{4}; \; sin\,\theta = \frac{3}{5}, \theta = sin^{-1}\frac{3}{5}$

34.(B) Equation of the hyperbola is $2x^2+2y^2+5xy+\lambda=0$. It passes through $(1,1)$

So, $2+2+5+\lambda=0 \Rightarrow \lambda=-9$

Hyperbola $2x^2+2y^2+5xy=9$

Tangent at $\left(-1,\frac{7}{2}\right)$

$$2x(-1)+2y\left(\frac{7}{2}\right)+5\frac{x\left(\frac{7}{2}\right)+(-1)y}{2}=9$$

$3x+2y=4$

35.(AC) The line $y=2x+c$ is a tangent to $x^2+y^2=5.$

If $\quad c^2=25$

$\Rightarrow \quad c=\pm 5$

Let the equation of parabola be $y^2=4ax.$ Then

$$\frac{a}{2}=\pm 5 \qquad \Rightarrow \qquad a=\pm 10$$

$\Rightarrow$ Equation of the parabola is $y^2=\pm 40x$

$\Rightarrow$ Equation of the directrix are $x=\pm 10.$

36.(BCD) Let $(x_1, y_1) \equiv (at^2, 2at)$

Tangent at this point is $ty = x + at^2$.

Any point on this tangent is $\left(h, \left(\frac{h + at^2}{t}\right)\right)$.

Chord of contact of this point with respect to the circle $x^2 + y^2 = a^2$ is

$$hx + \left(\frac{h + at^2}{t}\right)y = a^2 \quad \text{or} \quad (aty - a^2) + h\left(x + \frac{y}{t}\right) = 0$$

Which is a family of straight lines passing through point of intersection of

$$ty - a = 0 \text{ and } x + \frac{y}{t} = 0$$

So, the fixed point is $\left(-\frac{a}{t^2}, \frac{a}{t}\right)$ $\therefore$ $x_2 = -\frac{a}{t^2}, y_2 = \frac{a}{t}$

Clearly, $x_1 x_2 = -a^2, y_1 y_2 = 2a^2$

Also, $\frac{x_1}{x_2} = -t^4$

$$\frac{y_1}{y_2} = 2t^2 \qquad \Rightarrow \qquad 4\frac{x_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0$$

37.(BD) Given parabola is $x^2 - ky + 3 = 0$ or $x^2 = k\left(y - \frac{3}{k}\right)$

Let $x = X, y - \frac{3}{k} = Y$

Then the parabola is $X^2 = kY$

Whose focus is $\left(0, \frac{k}{4}\right)$.

Therefore, the focus of $x^2 = k\left(y - \frac{3}{k}\right)$ is $\left(0, \frac{3}{k} + \frac{k}{4}\right) \equiv (0, 2)$

$$\therefore \quad \frac{3}{k} + \frac{k}{4} = 2 \quad \Rightarrow \quad 12 + k^2 = 8k$$

$$\Rightarrow \quad k^2 - 8k + 12 = 0 \quad \Rightarrow \quad (k-6)(k-2) = 0 \quad \Rightarrow \quad k = 2, 6$$

38.(AD) Here $x^2 = -\lambda\left(y + \frac{\mu}{\lambda}\right)$

Therefore, vertex $= \left(0, -\frac{\mu}{\lambda}\right)$ and the directrix is $\left(y + \frac{\mu}{\lambda}\right) - \frac{\lambda}{4} = 0$.

Comparing with the given data, $-\frac{\mu}{\lambda} = 1$ and $\frac{\mu}{\lambda} - \frac{\lambda}{4} = -2$

$$\therefore \quad -1 - \frac{\lambda}{4} = -2 \quad \text{or} \quad \lambda = 4 \Rightarrow \mu = -4.$$

39.(AC) Given that the extremities of the latus rectum are (1, 1) and (1, –1)

$\Rightarrow$ $4a = 2 \Rightarrow a = \frac{1}{2}$ $\Rightarrow$ The focus of the parabola is (1, 0).

$\Rightarrow$ The vertex can be $\left(\frac{1}{2}, 0\right)$ and $\left(\frac{3}{2}, 0\right)$.

$\Rightarrow$ The equations of the parabola can be $y^2 = 2\left(x - \frac{1}{2}\right)$

or $y^2 = -2\left(x - \frac{3}{2}\right)$ $\Rightarrow$ $y^2 = 2x - 1$ or $y^2 = -2x + 3$

40.(AC) Let the possible point be $(t^2, 2t)$. Equation of tangent at this point is $yt = x + t^2$.

It must pass through $(6, 5)$. (Since normal to circle always passes through its centre)

$\Rightarrow$ $t^2 - 5t + 6 = 0$ $\Rightarrow$ $t = 2, 3$ $\Rightarrow$ Possible points are (4, 4), (9, 6).

41.(CD) $t_2 = -t_1 - \frac{2}{t_1}$

Also, $\frac{2at_1}{at_1^2} \times \frac{2at_2}{at_2^2} = -1$ $\Rightarrow$ $t_1 t_2 = -4$

$\therefore$ $\frac{-4}{t_1} = -t_1 - \frac{2}{t_1}$ $\Rightarrow$ $t_1^2 + 2 = 4$ and $t_1 = \pm\sqrt{2}$ So, point can be $(2a, \pm 2\sqrt{2}a)$.

42.(ABCD) Any point on the parabola is $P(at^2, 2at)$.

Therefore, midpoint of $S(a, 0)$ and $P(at^2, 2at)$ is $R\left(\frac{a + at^2}{2}, at\right) \equiv (h, k)$.

$\therefore$ $h = \frac{a + at^2}{2}, k = at$

Eliminate 't'

i.e., $2x = a\left(1 + \frac{y^2}{a^2}\right) = a + \frac{y^2}{a}$ i.e., $2ax = a^2 + y^2$ i.e., $y^2 = 2a\left(x - \frac{a}{2}\right)$

It's a parabola with vertex at $\left(\frac{a}{2}, 0\right)$, latus rectum $= 2a$

Directrix is $x - \frac{a}{2} = -\frac{a}{2}$, i.e., $x = 0$

Focus is $x - \frac{a}{2} = \frac{a}{2}$, i.e., $x = a$ i.e., $(a, 0)$

43.(AC) $r^2 - r - 6 > 0$ and $r^2 - 6r + 5 > 0$

$\Rightarrow$ $(r-3)(r+2) > 0$ and $(r-1)(r-5) > 0$

$\Rightarrow$ $(r < -2 \text{ or } r > 3)$ and $(r < 1$ or $r > 5)$

$\Rightarrow$ $r < -2$ or $r > 5$

44.(AC) Tangent and normal are bisectors of $\angle SPS'$, P(0,0)

Now equation of SP is $y = 3x/2$ and that of $S'P$ is $y = 5x$

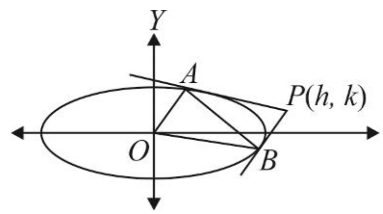
Then equations of their bisectors are $\dfrac{3x-2y}{\sqrt{13}} = \pm\dfrac{5x-y}{\sqrt{26}}$

or $\quad 3x - 2y = \pm\dfrac{5x-y}{\sqrt{2}} \Rightarrow$ Lines are $(3\sqrt{2}-5)x + (1-2\sqrt{2})y = 0$ and $(3\sqrt{2}+5)x - (2\sqrt{2}+1)y = 0$

Now (2, 3) and (1, 5) lie on the same side of $(3\sqrt{2}-5)x + (1-2\sqrt{2})y = 0,$ which is equation of tangent.

Points (2, 3) and (1, 5) lie on the different sides of $(3\sqrt{2}+5)x - (2\sqrt{2}+1)y = 0,$ which is equation of normal.

45.(AC) Let the point of intersection of tangents at A and B be $P(h,k)$, then equation of AB is



$$\frac{xh}{4} + \frac{yk}{1} = 1 \qquad \textbf{....(i)}$$

Homogenizing the equation of ellipse using Equation (i), we get :

$$\frac{x^2}{4} + \frac{y^2}{1} = \left(\frac{xh}{4} + \frac{yk}{1}\right)^2$$

$$\Rightarrow \quad x^2\left(\frac{h^2-4}{16}\right) + y^2(k^2-1) + \frac{2hk}{4}xy = 0 \qquad \textbf{....(ii)}$$

Given equation of OA and OB is $x^2 + 4y^2 + \alpha xy = 0 \qquad$ **....(iii)**

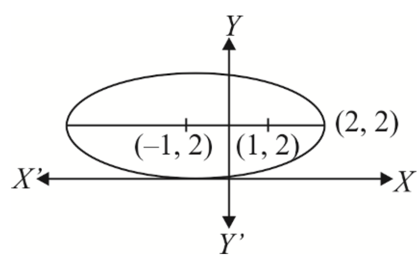
$\because$ Equation (ii) and (iii) represent same lines,

Hence, $\dfrac{h^2-4}{16} = \dfrac{k^2-1}{4} = \dfrac{hk}{2\alpha} \Rightarrow h^2 - 4 = 4(k^2-1) \Rightarrow h^2 - 4k^2 = 0 \Rightarrow$ Locus $(x-2y)(x+2y) = 0$

46.(ABC)

(A) The given equation is $\left(x - \dfrac{1}{13}\right)^2 + \left(y - \dfrac{2}{13}\right)^2 = \dfrac{1}{a^2}\left(\dfrac{5x+12y-1}{13}\right)^2$

It represents ellipse, if $\dfrac{1}{a^2} < 1 \Rightarrow a^2 > 1 \Rightarrow a > 1$



(B) $4x^2 + 8x + 9y^2 - 36y = -4$

$\Rightarrow \quad 4(x^2 + 2x + 1) + 9(y^2 - 4y + 4) = 36$

$\Rightarrow \quad \dfrac{(x+1)^2}{9} + \dfrac{(y-2)^2}{4} = 1$

Hence, (–1, 2) is centre and (1, 2) lies on the major axis.

Then required minimum distance is 1.

(C) Equation of normal at $P(\theta)$ is $5\sec\theta\, x - 4\operatorname{cosec}\theta\, y = 25 - 16,$ and it passes through $P(0,\alpha)$

$\therefore \quad \alpha = \dfrac{-9}{4\operatorname{cosec}\theta} \Rightarrow \alpha = \dfrac{-9}{4}\sin\theta \Rightarrow |\alpha| < \dfrac{9}{4}$

(D) $\dfrac{2b^2}{a} = \dfrac{2a}{3} \Rightarrow 3b^2 = a^2 \Rightarrow$ From $b^2 = a^2(1-e^2), 1 = 3(1-e^2) \Rightarrow e = \sqrt{2/3}$

47.(AC) Let $P(h,k)$ be the point of intersection of E_1 and E_2

$$\Rightarrow \quad \frac{h^2}{a^2}+k^2=1 \quad \Rightarrow \quad h^2=a^2(1-k^2) \quad \text{and} \quad \frac{h^2}{1}+\frac{k^2}{a^2}=1 \quad \Rightarrow \quad k^2=a^2(1-h^2)$$

Eliminating a from equations, we get :

$$\frac{h^2}{1-k^2}=\frac{k^2}{1-h^2} \quad \Rightarrow \quad h^2(1-h^2)=k^2(1-k^2) \quad \Rightarrow \quad (h-k)(h+k)(h^2+k^2-1)=0$$

Hence, the locus is a set of curves consisting of the straight lines $y=x, y=-x$ and circle $x^2+y^2=1$.

48.(AC) $f(x)$ is a decreasing function and for major axis to be *x*-axis

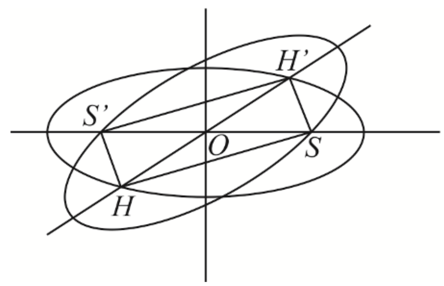
$$f(k^2+2k+5)>f(k+11) \quad \Rightarrow \quad k^2+2k+5<k+11 \quad \Rightarrow \quad k\in(-3,2)$$

Then for the remaining values of k, i.e., $k\in(-\infty,-3)\cup(2,\infty)$, major axis is *y*-axis.

49.(ABC) Clearly *O* is the midpoint of *SS'* and *HH'*

$\Rightarrow$ Diagonals of quadrilateral $HSH'S'$ bisect each other, so it is a parallelogram.

Let $H'OH=2r \Rightarrow OH=r=ae'$



H lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (suppose)

$$\therefore \quad \frac{r^2\cos^2\theta}{a^2}+\frac{r^2\sin^2\theta}{b^2}=1$$

$$e'^2\cos^2\theta+\frac{e'^2\sin^2\theta}{1-e^2}=1 \qquad [\because b^2=a^2(1-e^2)]$$

$$\Rightarrow \quad e'^2\cos^2\theta-\frac{e'^2\cos^2\theta}{1-e^2}=1-\frac{e'^2}{1-e^2} \quad \Rightarrow \quad \cos^2\theta=\frac{1}{e^2}+\frac{1}{e'^2}-\frac{1}{e^2e'^2} \quad \Rightarrow \quad \theta=\cos^{-1}\sqrt{\frac{1}{e^2}+\frac{1}{e'^2}-\frac{1}{e^2e'^2}}$$

$$\text{For } \theta=90^\circ, \frac{e^2+e'^2}{e^2e'^2}=\frac{1}{e^2e'^2} \quad \Rightarrow \quad e^2+e'^2=1$$

50.(AD) The equation of the tangent at $(t^2,2t)$ to the parabola $y^2=4x$ is

$$2ty=2(x+t^2) \quad \Rightarrow \quad ty=x+t^2$$

$$\Rightarrow \quad x-ty+t^2=0 \qquad \text{....(i)}$$

The equation of the normal at point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2+5y^2=20$ is

$$\Rightarrow \quad (\sqrt{5}\sec\theta)x-(2\operatorname{cosec}\theta)y=5-4$$

$$\Rightarrow \quad (\sqrt{5}\sec\theta)x-(2\operatorname{cosec}\theta)y=1 \qquad \text{....(ii)}$$

Given that equations (i) and (ii) represent the same line.

$$\Rightarrow \quad \frac{\sqrt{5}\sec\theta}{1}=\frac{-2\operatorname{cosec}\theta}{-t}=\frac{-1}{t^2} \quad \Rightarrow \quad t=\frac{2}{\sqrt{5}}\cot\theta \text{ and } t=-\frac{1}{2}\sin\theta$$

$$\Rightarrow \quad \frac{2}{\sqrt{5}}\cot\theta=-\frac{1}{2}\sin\theta \quad \Rightarrow \quad 4\cos\theta=-\sqrt{5}\sin^2\theta$$

$$\Rightarrow \quad 4\cos\theta=-\sqrt{5}(1-\cos^2\theta) \quad \Rightarrow \quad \sqrt{5}\cos^2\theta-4\cos\theta-\sqrt{5}=0$$

$\Rightarrow \quad (\cos\theta-\sqrt{5})(\sqrt{5}\cos\theta+1)=0 \quad \Rightarrow \quad \cos\theta=-\dfrac{1}{\sqrt{5}} \quad [\because \cos\theta\neq\sqrt{5}]$

$\Rightarrow \quad \theta=\cos^{-1}\left(-\dfrac{1}{\sqrt{5}}\right)$

Putting $\cos\theta=-\dfrac{1}{\sqrt{5}}$ in $t=-\dfrac{1}{2}\sin\theta,$ we get : $\quad t=-\dfrac{1}{2}\sqrt{1-\dfrac{1}{5}}=-\dfrac{1}{\sqrt{5}}$

Hence, $\theta=\cos^{-1}\left(-\dfrac{1}{\sqrt{5}}\right)$ and $t=-\dfrac{1}{\sqrt{5}}.$

51.(A) We have, $\left|\sqrt{x^2+(y-1)^2}-\sqrt{x^2+(y+1)^2}\right|=K$

Which is equivalent to $|S_1P-S_2P|=$ constant, where $S_1\equiv(0,1), S_2\equiv(0,-1)$ and $P\equiv(x,y).$

The above equation represents a hyperbola. So, we have $2a=K$

And $\quad 2ae=S_1S_2=2$

Where $2a$ is the transverse axis and e is the eccentricity.

Dividing, we have $e=\dfrac{2}{K}$

Since, $e>1$ for a hyperbola, therefore $K<2.$

Also, K must be a positive quantity. $\quad$ So, we have $K\in(0,2).$

52.(ABC) $3x^4-2(19y+8)x^2+[(19y^2)+(10)^2+(10)^2+y^4+y^4+8^2]=2(19\times10y+10y^2-8y^2)$

$\Rightarrow \quad 3x^4-2(19y+8)x^2+(19y-10)^2+(10-y^2)^2+(y^2+8)^2=0$

$\Rightarrow \quad 3x^4-2(19y-10+10-y^2+y^2+8)x^2+(19y-10)^2+(10-y^2)^2+(y^2+8)^2=0$

$\Rightarrow \quad [x^2-(19y-10)]^2+[x^2-(10-y^2)]^2+[x^2-(y^2+8)]^2=0$

$\Rightarrow \quad x^2-19y+10=0, x^2-10+y^2=0$ and $x^2-y^2-8=0$

The three curves represented by the given equation are $x^2=19y-10$ (parabola), $x^2+y^2=10$ (circle) and $x^2-y^2=8$ (hyperbola).

53.(B) Locus of point of intersection of perpendicular tangents is director circle of $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1.$ Equation of director circle is $x^2+y^2=a^2-b^2$ which is real if $a>b.$

54.(ABCD) Given hyperbola can be written as $\dfrac{(x-1)^2}{16}-\dfrac{(y-1)^2}{9}=1$

$\Rightarrow \quad \dfrac{X^2}{16}-\dfrac{Y^2}{9}=1 \quad$ (where $X=x-1, Y=y-1$)

$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{9}{16}}=\frac{5}{4}$$

Directrices are $X = \pm\frac{a}{e} \quad \Rightarrow \quad x-1 = \pm\frac{16}{5} \Rightarrow x = \frac{21}{5}$ and $x = -\frac{11}{5}$

Length of latus rectum $= \frac{2b^2}{a} = \frac{9}{2}$

The foci are given by $X = \pm ae, Y = 0 \Rightarrow \quad (6,1), (-4,1)$

55.(AD) Circle with points $P(2t_1, 2/t_1)$ and $Q(2t_2, 2/t_2)$ as diameter is given by

$$(x-2t_1)(x-2t_2) + \left(y - \frac{2}{t_1}\right)\left(y - \frac{2}{t_2}\right) = 0 \qquad \textbf{....(i)}$$

Also, slope of PQ is given by $-\frac{1}{t_1 t_2} = 1 \Rightarrow t_1 t_2 = -1$

Hence, from (1), circle is $(x^2 + y^2 - 8) - 2(t_1 + t_2)(x - y) = 0$

Which is of the form $S + \lambda L = 0.$

Hence, circles pass through the points of intersection of the circle $x^2 + y^2 - 8 = 0$ and the line $x = y.$

The points of intersection are (2, 2) and (–2, –2).

56.(BCD) $(x-\alpha)^2 + (\gamma - \beta)^2 = k(lx + my + n)^2$

$$\Rightarrow \qquad \sqrt{(x-\alpha)^2 + (y-\beta)^2} = \sqrt{k}\sqrt{l^2+m^2}\,\frac{|lx+my+n|}{\sqrt{l^2+m^2}} \qquad \Rightarrow \qquad \frac{PS}{PM} = \sqrt{k}\sqrt{l^2+m^2}$$

where point $P(x, y)$ is any point on the curve.

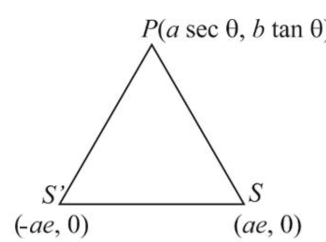
Fixed point $S(\alpha,\beta)$ is focus and fixed line $lx + my + n = 0$ is directrix.

If $k(l^2+m^2) = 1, P$ lies on parabola. If $k(l^2+m^2) < 1, P$ lies on ellipse.

If $k(l^2+m^2) > 1, P$ lies on hyperbola. If $k = 0, P$ lies on a point circle.

57.(AB) Let (h,k) be excentre opposite $\angle S'$, then

$$h = \frac{+ae(ae\sec\theta + a) + ae(-a + ae\sec\theta) + 2ae(a\sec\theta)}{2a(e-1)}$$



$h = ae\sec\theta\left(\frac{e+1}{e-1}\right), k = \frac{be\tan\theta}{e-1}$ (for $a\sec\theta > 0$), locus is a hyperbola

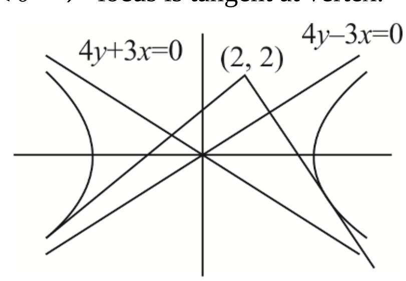
Again let (h,k) be excentre opposite $\angle P$

$$\therefore \qquad h = \frac{-2a^2e\sec\theta + a^2e^2\sec\theta + a^2e - a^2e^2\sec\theta + a^2e}{2ae\sec\theta - 2ae}$$

$\Rightarrow \qquad h = -a, \text{for}\, a\sec\theta > 0$ or $h = a, \text{for}\, a\sec\theta < 0 \;\Rightarrow\;$ locus is tangent at vertex.

58.(CD) Equations of asymptotes are

$$4y - 3x = 0 \text{ and } 4y + 3x = 0$$



As point (2, 2) lies above the asymptotes.

Hence, points of contact of the tangents are in III and IV quadrants.

59.(ABC) Locus of point of intersection of perpendicular tangents is director circle $x^2+y^2=a^2-b^2$.

Now, $e^2=1+\frac{b^2}{a^2}$

If $a^2>b^2$, then there are infinite (or more than 1) points on the circle $\Rightarrow e^2<2 \Rightarrow e<\sqrt{2}$.

If $a^2<b^2$, then there does not exist any point on the plane $\Rightarrow e^2>2 \Rightarrow e>\sqrt{2}$.

If $a^2=b^2$, there is exactly one point (centre of the hyperbola) $e=\sqrt{2}$.

60.(BC) Equation of the chords of contact of the given points $P(x_1,y_1)$ *and* $Q(x_2,y_2)$ are respectively

$xy_1+x_1y=2c^2$ ……….(i) $\qquad$ $xy_2+x_2y=2c^2$ …….(ii)

Equation of any conic section passing through the points of intersection of the curve $xy=c^2$ and the lines (i) and (ii) is given by $\left(xy_1+x_1y-2c^2\right)\left(xy_2+x_2y-2c_2\right)+\lambda\left(xy-c^2\right)=0$ ……..(iii)

It represents a circle if $y_1y_2=x_1x_2$ *and* $x_1y_2+x_2y_1+\lambda=0$ …….(iv)

Also, the circle (iii) passes through the points P and Q, so $\left(2x_1y_1-2c^2\right)\left(x_1y_2+x_2y_1-2c^2\right)+\lambda\left(x_1y_1-c^2\right)=0$

$2\left(x_1y_2+x_2y_1-2c^2\right)+\lambda=0$ ……….(v)

From equation (iv) and (v) we have $2\left(-\lambda-2c^2\right)+\lambda=0$

$\lambda=-4c^2$; $\quad x_1y_2+x_2y_1=4c^2$

61.(CD) $\tan(\alpha-\beta)=\tan\{(\theta+\alpha)-(\theta+\beta)\}=\dfrac{\frac{x}{a}-\frac{y}{b}}{1+\frac{x}{a}\frac{y}{b}}$

Or $xy-(bx-ay)\cot(\alpha-\beta)+ab=0$, which represents a rectangular hyperbola

We may rewrite the equation as $\left[x+a\cot(\alpha-\beta)\right]\left[y-b\cot(\alpha-\beta)\right]+ab\operatorname{cosec}^2(\alpha-\beta)=0$

Thus, the asymptotes are $x+a\cot(\alpha-\beta)=0$ and $y-b\cot(\alpha-\beta)=0$

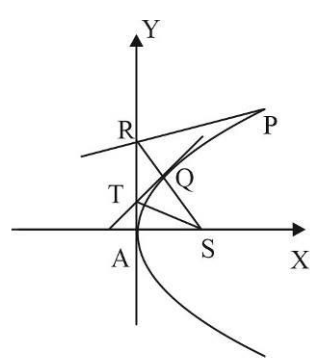
62.(ABCD) $R\equiv(1,1), T\equiv(2,2)$

Equation of RS is $4x-3y-1=0$

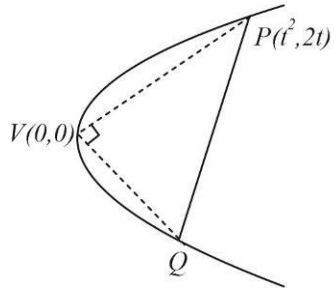
Equation of TS is $3x-2y-2=0$

Length of latus rectum $=4\times 1/\sqrt{2}=2\sqrt{2}$

Axis is $x+y-9=0 \Rightarrow vertex \equiv\left(\frac{9}{2},\frac{9}{2}\right)$



63.(AB) m of $PV = \frac{2t-0}{t^2-0} = \frac{2}{t}$



the equation of QV is $y = -\frac{t}{2}x$

solving it with $y^2 = 4x, Q = \left(\frac{16}{t^2}, \frac{-8}{t}\right)$

Now, $ar(\Delta PVQ) = \frac{1}{2} \times PV.VQ = 20(given)$

$PV^2.VQ^2 = 40^2$ Or $\left(\left(t^2\right)^2 + (2t)^2\right)\left\{\left(\frac{16}{t^2}\right)^2 + \left(\frac{-8}{t}\right)^2\right\} = 40^2$

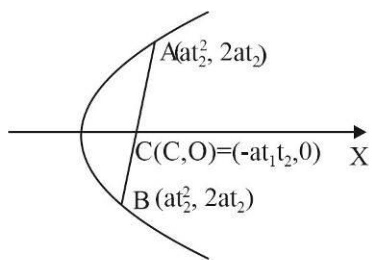
$\left(t^2\right)\left(4+t^2\right)\frac{1}{t^2}\left(\frac{256}{t^2}+64\right) = 40^2$

$\left(\frac{256\times 4}{t^2} + 256 + 256 + 64t^2\right) = 40^2$

$\left(t^2-16\right)\left(t^2-1\right) = 0 \therefore t = \pm 4, \pm 1$

64.(BD) Let $C(c, 0)$

$t_1 t_2 = \frac{-c}{a} \Rightarrow c = -at_1 t_2$



$\frac{AC^2}{CB^2} = \frac{\left(at_1^2 + at_1t_2\right)^2 + 4a^2t_1^2}{\left(at_2^2 + at_1t_2\right)^2 + 4a^2t_2^2} = \frac{t_1^2}{t_2^2}$

$\frac{CB}{AC} = \pm\frac{t_2}{t_1} \Rightarrow \frac{CB}{AC} + 1 = \pm\frac{t_2}{t_1} + 1$

$\frac{AB}{AC} = \frac{t_1+t_2}{t_1} = 3$

$\frac{AB}{AC} = \frac{t_1-t_2}{t_1} = 3$

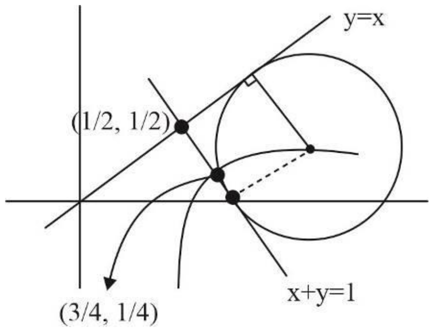
$t_2 + t_1 = 3t_1 \; or \; t_1 - t_2 = 3t_1$

$2t_1 - t_2 = 0 \; or \; 2t_1 + t_2 = 0$

$6t_1^2 - 2t_2^2 - t_1t_2 = 0 = 6t_1^2 + 3t_1t_2 - 4t_1t_2 - 2t_2^2$

$\left(3t_1 - 2t_2\right)\left(2t_1 + t_2\right) = 0$

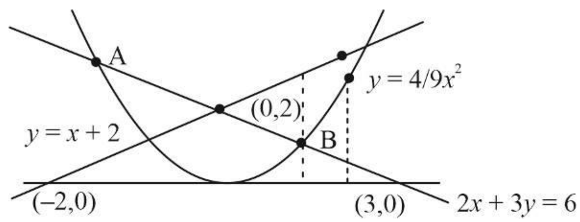
65.(BCD)



$$4a = 2\left(\frac{|0-1|}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

66.(AC) For point *A*, *B*

$$2x + 3\left(\frac{4}{9}x^2\right) = 6$$



$$2x^2 + 3x - 9 = 0$$

$$2x^2 + 6x - 3x - 9 = 0 = (2x-3)(x+3) = 0$$

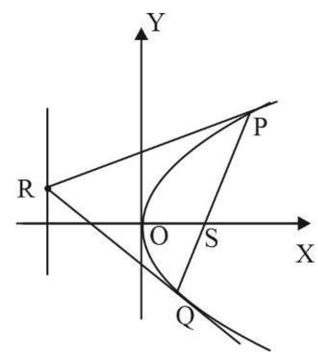
$$A \equiv (-3,4)\, and\, B \equiv \left(\frac{3}{2},1\right)$$

$$\frac{3}{2}\alpha = -3 \Rightarrow \alpha = -2$$

$$\frac{3}{2}\alpha = \frac{3}{2} \Rightarrow \alpha = 1$$

$$\alpha \in (-\infty,-2) \cup (0,1)$$

67.(ABD) The tangents at the ends of a focal chord of a parabola intersect at the directrix perpendicularly. So, the orthocenter being the vertex of the triangle at which right angle occurs therefore lies on the directrix. Also, the circumcircle of ΔPQR will have PQ as diameter.



Area of circumcircle $= \frac{\pi}{4}(PQ)^2$

Further the minimum length of a focal chord equals to the latus rectum, so the minimum area of the circumcircle.

$$= \frac{\pi}{4}(4a)^2 = 4\pi a^2$$

Again the tangents at P and Q are say

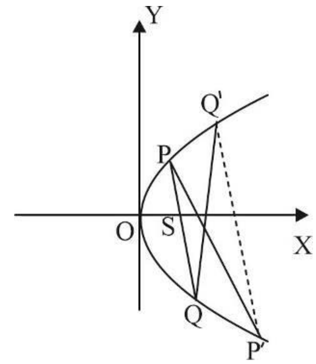
$t_1 y = x + at_1^2\, and\, t_2 y = x + at_2^2$ respectively and $t_1 t_2 = -1$ their distance from the origin are

$$d_1 = \frac{at_1^2}{\sqrt{1+t_1^2}}\, and\, d_2 = \frac{at_2^2}{\sqrt{1+t_2^2}} = \frac{a}{\sqrt{t_1^2 + t_1^4}}$$

Clearly, $d_1\, and\, d_2$ cannot be equal for all values of t_1 . So, the incentre cannot lie at the vertex.

68.(BC) Let the coordinates of P and Q be $\left(at_1^2, 2at_1\right)$ *and* $\left(at_2^2, 2at_2\right)$ then $t_1t_2 = -1$ *and* $PQ = a\left(t_1 - t_2\right)^2$

If the normal at P meets the parabola again at $P'\left(t_1'\right)$, then $t_1' = -t_1 - \dfrac{2}{t_1}$ ……...(i)



And if the normal at Q meets the parabola again at $Q'\left(t_2'\right)$ then $t_2' = -t_2 - \dfrac{2}{t_2}$ …….(ii)

From the equation (i) and (ii) we have,

$$t_1' + t_2' = -\left(t_1 + t_2\right) - \left(\frac{2}{t_1} + \frac{2}{t_2}\right) = -\left(t_1 + t_2\right) - \frac{2}{t_1t_2}\left(t_1 + t_2\right) = t_1 + t_2$$

$$t_1' - t_2' = t_2 - t_1 + \frac{2}{t_2} - \frac{2}{t_1} = \left(t_2 - t_1\right) + \frac{2}{t_1t_2}\left(t_1 - t_2\right) = 3\left(t_2 - t_1\right) \text{ …….(iv)}$$

Now the slope of the line PQ is $m_1 = \dfrac{2}{t_1 + t_2}$ and the slope of the line $P'Q'$ is $m_2 = \dfrac{2}{t_1' + t_2'} = \dfrac{2}{t_1 + t_2}$

Thus, the lines PQ and $P'Q'$ are parallel,

Further $P'Q' = a\left|t_1' - t_2'\right|\sqrt{\left(t_1' + t_2'\right)^2 + 4} = 3a\left(t_1 - t_2\right)^2 = 3PQ$

69.(AD) Equation of a normal to $y^2 = 4ax$ is $y = mx - 2am - am^3$ ……….(i)

Equation of the normal to $y^2 = 4c\left(x - d\right)$ with the same slope 'm' is $y = m\left(x - d\right) - 2cm - cm^3$ ……..(ii)

The equation (i) and (ii) represent the same straight lines if $-2am - am^3 = -dm - 2cm - cm^3$

$$\left(c - a\right)m^3 = \left(2a - d - 2c\right)m$$

Either $m = 0$, which implies that one normal is always X-axis, Or $m^2 = \dfrac{2a - d - 2c}{c - a}$

For non-zero roots $m^2 > 0$

$$\frac{2a - d - 2c}{c - a} > 0$$

$c - a > 0$ and $2a - d - 2c > 0$; $c - a < 0$ and $2a - d - 2c < 0$

70.(AC) Let $y^2 = 4ax$ be the parabola and Q be the point $\left(h, 0\right)$. If P and R are $\left(at_1^2, 2at_1\right)$ and $\left(at_2^2, 2at_2\right)$, then the equation of PR is $\left(t_1 + t_2\right)y = 2x + 2at_1t_2$

If this line passes through the point $\left(h, 0\right)$, then we have $0 = 2h + 2at_1t_2 \Rightarrow t_1t_2 = -\dfrac{h}{a}$

Now $OM.ON = at_1^2.at_2^2 = a^2\left(t_1t_2\right)^2 = a^2\left(-\dfrac{h}{a}\right)^2 = h^2 = OQ^2$

And $PM.RN = \left|2at_1\right|.\left|2at_2\right| = 4a^2\left|t_1t_2\right| = 4a^2\left|\left(-\dfrac{h}{a}\right)\right| = 4ah = 4aOQ$

71.(ABD) Equation of the parabola is $x^2 - 8x - 16y = 0 \Rightarrow x^2 - 8x + 16 = 16 + 16y$; $(x-4)^2 = 16(y+1)$

Shifting the origin to $(4,-1)$, above equation becomes $X^2 = 16Y$ ……….(i)

Equation of a normal to (i) is $X = mY - 8m - 4m^3$ and foot of the normal is $\left(-8m, 4m^2\right)$.

It passes through $(7,14)$, which transforms to $(3,15)$ in the new system. So, we have

$3 = 15m - 8m - 4m^3 \Rightarrow 4m^3 - 7m + 3 = 0$

$(m-1)(2m-1)(2m+3) = 0 \Rightarrow m = 1, \frac{1}{2}, -\frac{3}{2}$

The coordinates of the feet in the shifted coordinates system are $(-8,4),(-4,1)$ and $(12,9)$

The required coordinates are therefore (changing to original coordinate system) $(-4,3),(0,0)\, and\, (16,8)$

72. (ABCD) Point of contact will lie an $y = x$ (let say (α,α)) and slope of tangent can be ± 1 and $\alpha = a\,\alpha^2 + a\alpha + \frac{1}{24}$

eliminating α we get $\left(\frac{\pm 1 - a}{2a}\right)^2 = a\left(\frac{\pm 1 - a}{2a}\right)^2 + a + \frac{1}{24}$

On solving we get $a = \frac{2}{3}, \frac{3}{2}, \frac{13 \pm \sqrt{601}}{12}$

73.(ABC) Facts

74.(BC) By properties $PA = PB = \sqrt{10}$

$\frac{x-1}{-1/\sqrt{10}} = \frac{y-5}{3/\sqrt{10}} = \pm\sqrt{10}$

$A = (0,8);\ B = (2,2)$

Slopes of asymptotes are -7 and 1

If angles between asymptotes be θ then $tan\,\theta = \left|\frac{-7-1}{1-7}\right| = \frac{4}{3}$

Now, $2\,tan^{-1}\frac{b}{a} = tan^{-1}\frac{4}{3} \Rightarrow \frac{\frac{2b}{a}}{1-\frac{b^2}{a^2}} = \frac{4}{3}$

$\frac{b}{a} = \frac{1}{2}$ ………..(i)

$\frac{b^2}{a^2} = e^2 - 1 \Rightarrow e = \frac{\sqrt{5}}{2}$

Also, area of $\Delta CAB = ab$

$ab = \frac{1}{2}\left|1(8-2) + 0(2-1) + 2(1-8)\right| = 4$ ………(ii)

$b^2 = 2, a = 2b = 2\sqrt{2}$, length of latus rectum $= \frac{2b^2}{a} = \frac{4}{2\sqrt{2}} = \sqrt{2}$

75.(BD) As tangents are perpendicular to each other and its points of intersections $P\left(\frac{7}{5},\frac{-4}{5}\right)$ lies on directrix. AB is focal chord

Equation of AB is $x-y=4$, as focus is (α,β) $so\ \alpha-\beta=4 \Rightarrow \beta=\alpha-4$

Now $\angle ASP=90°$

$$\frac{-2-\alpha+4}{2-\alpha}\times\frac{-\frac{4}{5}-\alpha+4}{\frac{7}{5}-\alpha}=-1 \quad ; \quad \frac{2-\alpha}{2-\alpha}\times\frac{\frac{16}{5}-\alpha}{\frac{7}{5}-\alpha}=-1$$

$$\frac{16}{5}-\alpha=\alpha-\frac{7}{5};\ \alpha=\frac{23}{10}$$

Focus is $\left(\frac{23}{10},-\frac{17}{10}\right)$

Also, $l_1=AS=\frac{3\sqrt{2}}{10}, l_2=BS=\frac{27\sqrt{2}}{10}$

$$\frac{1}{a}=\frac{1}{l_1}+\frac{1}{l_2}\Rightarrow a=\frac{27\sqrt{2}}{100}\Rightarrow 4a=\frac{27\sqrt{2}}{25}$$

76.(BC) Ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1\Rightarrow$ eccentricity $e=\frac{\sqrt{3}}{2}$; Extremities of latus rectums are $\left(\pm\sqrt{3},\pm\frac{1}{2}\right)$

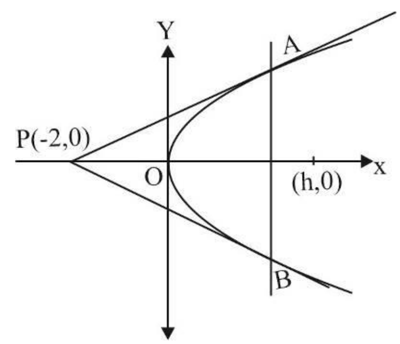
Given $y_1<0, y_2<0$

$P\left(\sqrt{3},-\frac{1}{2}\right)$ and $Q\left(-\sqrt{3},-\frac{1}{2}\right)$

$PQ=2\sqrt{3}=$ length of latus rectum of parabola

Two parabolas with vertices $\left(0,-\frac{\sqrt{3}}{2}-\frac{1}{2}\right)$ and $\left(0,\frac{\sqrt{3}}{2}-\frac{1}{2}\right)$ are possible

77.(AB) Point 'P' clearly lies on the directrix of $y^2=8x$



Thus slope of PA and PB are 1 and -1 respectively

Equation of PA $: y=x+2$, equation of $PB: y=-x-2$, equation of $AB: x=2$

Let the centre of the circle be $(h,0)$ and radius be 'r'

$$\Rightarrow\left|\frac{h+2}{\sqrt{2}}\right|=\frac{|h-2|}{1}=r\Rightarrow h^2+4+4h=2\left(h^2+4-4h\right)\Rightarrow h^2-12h+4=0$$

$$h=\frac{12\pm8\sqrt{2}}{2}=6\pm4\sqrt{2}\Rightarrow|h-2|=4\left(\sqrt{2}-1\right),4\left(\sqrt{2}+1\right)$$

78.(AB) Any point on the line $x+y=1$ can be taken $(t,1-t)$

Equation of the chord, with this as mid-point is

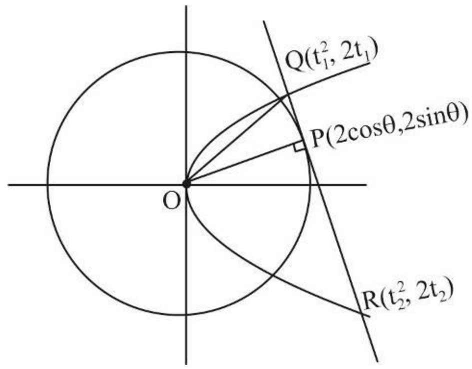
$y(1-t)-2a(x+t)=(1-t)^2-4at$, it passes through $(a,2a)$

So $t^2-2t+2a^2-2a+1=0$, this should have 2 distinct real roots so $a^2-a<0, 0<a<1$,

so length of latus-rectum <4

79.(AB) It is a well-known property of a parabola that a tangent and normal to it from focus intersect at tangent at vertex.

80.(D) $OQ^2 = OP^2 + PQ^2$



$t_1^4 + 4t_1^2 = 5$

$t_1 = 1$

$Q \equiv (1,2)$

$PQ = 1 \Rightarrow \cos\theta + 2\sin\theta = 2$

$\cos\theta = \dfrac{4}{5}, \sin\theta = \dfrac{3}{5}$

$P = \left(\dfrac{8}{5}, \dfrac{6}{5}\right)$

Now R lies on tangent $4x + 3y = 10$

$t_2 = -\dfrac{5}{2}$; $\quad R = \left(\dfrac{25}{4}, -5\right)$ *and* $S = \left(-\dfrac{5}{2}, -\dfrac{3}{2}\right)$

81. **[A-r] [B-s] [C-p] [D-q]**

Locus of point of intersection of perpendicular tangent is directrix which is $12x - 5y + 3 = 0$.

Parabola is symmetrical about its axis, which is a line passing through the focus (1, 2) and perpendicular to the directrix, which has equation $5x + 12y - 29 = 0$.

Minimum length of focal chord occurs along the latus rectum line, which is a line passing through the focus and parallel to directrix, i.e,. $12x - 5y - 2 = 0$.

Locus of foot of perpendicular from focus upon any tangent is tangent at the vertex, which is parallel to directrix and equidistant from directrix and latus rectum line, i.e., $12x - 5y + \lambda = 0$

where $\dfrac{|\lambda - 3|}{\sqrt{12^2 + 5^2}} = \dfrac{|\lambda + 2|}{\sqrt{12^2 + 5^2}} \Rightarrow \lambda = \dfrac{1}{2}$. Hence, equation of tangent at vertex is $24x - 10y + 1 = 0$.

82. **[A-p, r] [B-p, q, r] [C-q] [D-q, s]**

Points through which perpendicular tangent can be drawn to the parabola $y^2 = 4x$ lie on the directix. Points (–1, 2) and (–1, –5) lie on the directrix. Also from these points only one normal can be drawn.

83. **[A-p, q, r, s] [B-p, q, r, s] [C-q, r, s] [D-p]**

(A) Equation of tangent at $\left(\dfrac{\cos\theta}{2}, \dfrac{\sin\theta}{3}\right)$ is $2x\cos\theta + 3y\sin\theta = 1$ which is parallel to the given line $8x = 9y$

$\therefore \quad \cos\theta = \pm\dfrac{4}{5}, \sin\theta = \mp\dfrac{3}{5}$

Hence, points are $\left(\dfrac{2}{5}, -\dfrac{1}{5}\right)$ and $\left(-\dfrac{2}{5}, \dfrac{1}{5}\right)$.

Distance between the points is $\sqrt{\dfrac{16}{25} + \dfrac{4}{25}} = \dfrac{2}{\sqrt{5}}$ which is less than 1.

(B) The given equation is $\frac{(x+1)^2}{9}+\frac{(y+2)^2}{25}=1 \Rightarrow e^2=1-\frac{9}{25}=\frac{16}{25}\Rightarrow e=\frac{4}{5}$

Hence, the foci are $S, S' \equiv (-1, -2\pm 4) \equiv S(-1, 2)$ and $S'(-1, -6)$

The required sum of distances $=2+6=8.$

(C) Equation of normal at $(3\cos\theta, 2\sin\theta)$ is $3x\sec\theta-2y\operatorname{cosec}\theta=5$

which is parallel to the given line $2x+y=1.$ Therefore, $\cos\theta=\mp\frac{3}{5}, \sin\theta=\pm\frac{4}{5}$

Hence, points are $\left(\frac{-9}{5}, \frac{8}{5}\right)$ and $\left(\frac{9}{5}, -\frac{8}{5}\right)$. The required sum of distances $=\frac{16}{5}.$

(D) Consider any point $(t, t+2), t\in R,$ on the line $x-y+2=0.$

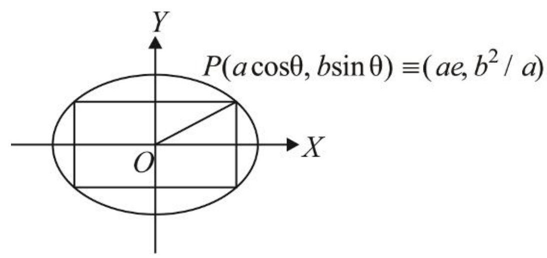
The chord of contact of this point w.r.t. ellipse is $xt+2y(t+2)=2 \Rightarrow (4y-2)+t(x+2y)=0$

These lines passes through point of intersection of lines $4y-2=0$ and $x+2y=0$

$\therefore$ $x=-1$, y = 1/2 Hence, the point is $(-1, 1/2),$ whose distance from (2, 1/2) is 3.

84. **[A-q]** **[B-q]** **[C-s]** **[D-p]**

(A) Let one of the vertices of the rectangle be $P(a\cos\theta, b\sin\theta).$



Then its area $A=2a\cos\theta\times 2b\sin\theta=2ab\sin 2\theta.$

Hence, $A_{max}=2ab$

Now area of rectangle formed by extremities of $LR=(2ae)(2b^2/a)=4eb^2.$

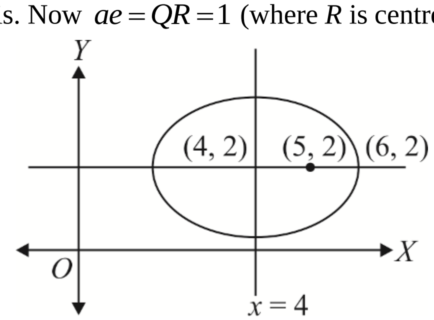
Given that $2ab=4eb^2 \Rightarrow \frac{2b}{a}e=1 \Rightarrow \frac{4b^2}{a^2}e^2=1$

$\Rightarrow 4(1-e^2)e^2=1 \Rightarrow 4e^4-4e^2+1=0 \Rightarrow (2e^2-1)^2=0 \Rightarrow e=\frac{1}{\sqrt{2}}$

(B) For ellipse, distance between the foci, $2ae=8$ and length of semi-minor axis, is $b=4.$

Now, $b^2=a^2-a^2e^2 \Rightarrow 16=a^2-16 \Rightarrow a^2=32 \Rightarrow e=\sqrt{1-\frac{16}{32}}=\frac{1}{\sqrt{2}}$

(C) Normal at point $P(6, 2)$ to the ellipse passes through its focus $Q(5, 2).$ Then P must be extremity of the major axis. Now $ae=QR=1$ (where R is centre) and $a-ae=1.$



$\therefore$ $a=2$

$b^2=a^2-a^2e^2=4-1=3 \Rightarrow e=\sqrt{1-\frac{3}{4}}=\frac{1}{2}$

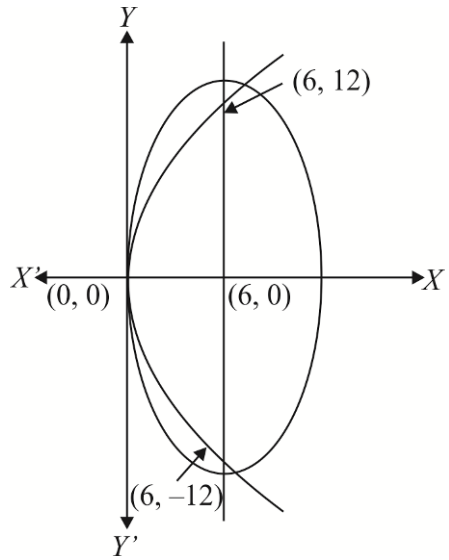
(D) Extremities of LR of parabola $y^2 = 24x$ are $(6, \pm 12)$.

For ellipse, $2be = 24$ and extremity of minor axis is (0, 0). Hence, $a = 6$.

Now, $a^2 = b^2 - b^2e^2$

$\Rightarrow \quad b^2 = 36 + 144 = 180$

$\Rightarrow \quad e = \sqrt{1 - \frac{36}{180}} = \sqrt{1 - \frac{1}{5}} = \frac{2}{\sqrt{5}}$



85. **[A-p, s]** **[B-q]** **[C-r]** **[D-p, q, s]**

(A) We must have

$e_1 < 1 < e_2 \Rightarrow f(1) < 0 \Rightarrow 1 - a + 2 < 0 \Rightarrow a > 3$

(B) We must have

I. $D \geq 0$ or $a^2 - 8 \geq 0$ or $a \in (-\infty, -2\sqrt{2}] \cup [2\sqrt{2}, \infty)$

II. $1 \cdot f(1) > 0$ or $1 - a + 2 > 0$ or $a < 3$ **III.** $\frac{a}{2} > 1 \Rightarrow a > 2$

From equation (i), (ii) and (iii) we have $a \in (2\sqrt{2}, 3)$.

(C) We must have $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$

$\Rightarrow \quad \frac{(e_1 + e_2)^2 - 2e_1e_2}{e_1^2e_2^2} = 1 \quad \Rightarrow \quad \frac{a^2 - 4}{4} = 1 \quad \Rightarrow \quad a = \pm 2\sqrt{2}$

(D) We must have $\sqrt{2} < e_1$ since product of roots is 2, the other root must lie in $(0, \sqrt{2})$

$\Rightarrow \quad f(\sqrt{2}) < 0 \quad \Rightarrow \quad 2 - a\sqrt{2} + 2 < 0 \quad \Rightarrow \quad a > 2\sqrt{2}$

86.(A) (P) Let an end of latus rectum be $\left(ae, b\sqrt{1-e^2}\right)$, then the equation of normal at this end is $\frac{x - ae}{ae/e^2} = \frac{y - b\sqrt{1-e^2}}{\frac{b\sqrt{1-e^2}}{b^2}}$

It will pass through the end $(0, -b)$ if $-a^2 = \frac{-b^2\left(1 + \sqrt{1-e^2}\right)}{\sqrt{1-e^2}}$ or $\frac{b^2}{a^2} = \frac{\sqrt{1-e^2}}{1 + \sqrt{1-e^2}}$

or $1 + \sqrt{1-e^2} = \frac{\sqrt{1-e^2}}{1-e^2} \Rightarrow \sqrt{1-e^2} + \left(1 - e^2\right) = 1 \Rightarrow e^4 + e^2 = 1$

(Q) Let PQ be the double ordinate of the parabola $y^2 = 4ax$. R and S be the points of trisection of $PQ \Rightarrow PR = RS = SQ$

If $R(h, k) \Rightarrow S(h, -k)$

$RS = 2k, PR = 2k, PL = 3k \Rightarrow P(h,k)$

But P lies on $y^2 = 4ax \Rightarrow (3k)^2 = 4ah \Rightarrow 9k^2 = 4ah$

Locus of $R(h,k) is\, 9y^2 = 4ax$

Length of the latus rectum $= \dfrac{4a}{9} \Rightarrow k = \dfrac{1}{9}$

(R) let $P\left(a_1^2, 2at_1\right) Q\left(a_2^2, 2at_2\right)$ be the extremes $\Rightarrow t_1t_2 = -1$

$e = SP = \sqrt{\left(at_1^2 - a\right)^2 + 4a^2t_1^2} = a\left(1+t_1^2\right)$ *and* $e' = SQ = a\left(1+t_2^2\right)$

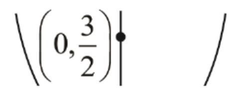
$(e-a)(e'-a) = a^2t_1^2t_2^2 = a^2 \Rightarrow ee' - a(e+e') = 0 \Rightarrow \dfrac{1}{e} + \dfrac{1}{e'} = \dfrac{1}{a}$

(S) Let $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ be the given hyperbola $\Rightarrow b^2 = a^2\left(e^2 - 1\right)$ and conjugate is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1 \Rightarrow a^2 = b^2\left(e'^2 - 1\right)$

On eliminating, $\dfrac{a^2}{b^2}$, we get $e'^2 - 1 = \dfrac{1}{e^2 - 1} \Rightarrow \dfrac{1}{e^2} + \dfrac{1}{e'^2} = 1$

87.(1) $y = x^2$

Normal at (t, t^2) is given by



$x + 2ty = t + 2t^3$ if it passes through $\left(0, \dfrac{3}{2}\right)$

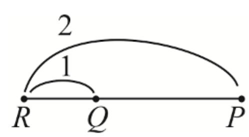
$\Rightarrow \quad t^3 = t \Rightarrow t = 0, 1, -1 \quad \Rightarrow \quad$ points are $P(0,0), Q(1,1), R(-1,1)$

$\angle P = 90° \quad \Rightarrow R = \dfrac{QR}{2} = 1$

88.(4) $(y-1)^2 = 4x - 4 = 4(x-1)$

Let $P(t^2+1, 2t+1)$, tangent is given by $x - ty + t^2 + t - 1 = 0$. It intersects directrix $x = 0$

at $\quad Q\left(0, \dfrac{t^2+t-1}{t}\right)$



$\Rightarrow \quad R$ is given by $\left(-t^2 - 1, \dfrac{t-2}{t}\right)$

$x = -t^2 - 1, y = 1 - \dfrac{2}{t} \Rightarrow t = \dfrac{2}{1-y} \quad \Rightarrow \quad x + 1 = -\dfrac{4}{(y-1)^2} \Rightarrow (y-1)^2(x-1) + 4 = 0 \quad \Rightarrow \quad \lambda = 4$

89.(1) $A(at_1^2, 2at_1), B(at_2^2, 2at_2), C(at_3^2, 2at_3), D(at_4^2, 2at_4)$

$\angle ABC = 90° \Rightarrow \dfrac{2}{t_1+t_2} \times \dfrac{2}{t_2+t_3} = -1 \Rightarrow (t_1+t_2)(t_2+t_3) = -4$

$\angle ADC = 90° \Rightarrow \dfrac{2}{t_1+t_4} \times \dfrac{2}{t_3+t_4} = -1 \Rightarrow (t_1+t_4)(t_3+t_4) = -4$

t_2, t_4 are roots of $(t+t_1)(t+t_3) + 4 = 0 \Rightarrow t^2 + (t_1+t_3)t + t_1t_3 + 4 = 0 \Rightarrow t_2 + t_4 = -(t_1+t_3)$

90.(2) $\frac{tx}{4}-\frac{y}{3}+t=0$ … (1) and $\frac{x}{4}+t\frac{y}{3}-t=0$ … (2)

Locus of point of intersection of (1) and (2) is given by $t=\frac{\frac{y}{3}}{\frac{x}{4}+1}=\frac{\frac{x}{4}}{1-\frac{y}{3}}$

$$\frac{y}{3}-\frac{y^2}{9}=\frac{x^2}{16}+\frac{x}{4} \quad \ldots (3)$$

Equation of pair of straight lines joining origin to the points of intersection of $ax+by+c=0$ & (3) is given by

$\frac{x^2}{16}+\frac{y^2}{9}+\left(\frac{x}{4}-\frac{y}{3}\right)\left(\frac{ax+by}{-c}\right)=0$. If they are at right angles, then

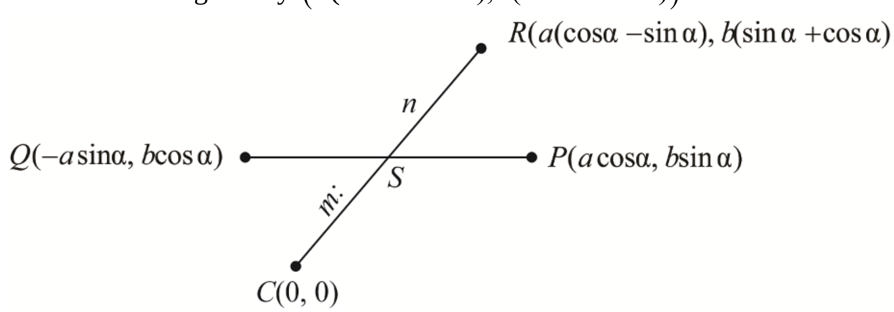
$$\left(\frac{1}{16}-\frac{a}{4c}\right)+\left(\frac{1}{9}+\frac{b}{3c}\right)=0 \quad \Rightarrow \quad \frac{c-4a}{16c}+\frac{c+3b}{9c}=0$$

$$9c-36a+16c+48b=0 \quad \Rightarrow \quad 36a-48b-25c=0 \quad \text{Hence } \left[\frac{3a-4b}{c}\right]=\left[\frac{25}{12}\right]=2$$

91.(1) Let $P(a\cos\alpha, b\sin\alpha)$ & $Q(-a\sin\alpha, b\cos\alpha)$

Tangents at $P\,\&\,Q$ are given by $\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1 \quad \Rightarrow \quad \frac{-x\sin\alpha}{a}+\frac{y\cos\alpha}{b}=1$

Point of intersection R is given by $\left(a(\cos\alpha-\sin\alpha), b(\sin\alpha+\cos\alpha)\right)$



$$S\left(\frac{ma(\cos\alpha-\sin\alpha)}{m+n}, \frac{mb(\sin\alpha+\cos\alpha)}{m+n}\right)$$

As P, Q, S are collinear then $\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ -a\sin\alpha & b\cos\alpha & 1 \\ \frac{ma(\cos\alpha-\sin\alpha)}{m+n} & \frac{mb(\sin\alpha+\cos\alpha)}{m+n} & 1 \end{vmatrix}=0$

$$mR_1+mR_2-(m+n)R_3 \Rightarrow \begin{vmatrix} 0 & 0 & m-n \\ -a\sin\alpha & b\cos\alpha & 1 \\ \frac{ma(\cos\alpha-\sin\alpha)}{m+n} & \frac{mb(\sin\alpha+\cos\alpha)}{m+n} & 1 \end{vmatrix}=0$$

$m=n \Rightarrow m:n:1:1$

92.(6) $\sqrt{(x-2)^2+(y+3)^2} = \frac{1}{2}\left|\frac{3x-4y+7}{5}\right|$

Focus at $(2,-3)$, directrix at $3x-4y+7=0$ and e = ½

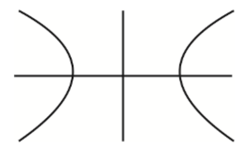
Distance of focus from directrix is 5 which is given by $\frac{a}{e}-ae$ i.e. $\frac{3a}{2}$ Hence $a=\frac{10}{3}$, $A=\frac{20}{3}$

93.(3) Normal is given by $13\sec\theta x - 5\operatorname{cosec}\theta y = 144$

If it passes through (0, 6), then $\sin\theta = \frac{-5}{24}$

Corresponding to it $\cos\theta$ has 2 values, hence these two normals and one normal is y axis

94.(4) $\frac{1}{e^2}+\frac{1}{e'^2}=1 \quad \Rightarrow \quad e' = f(e) = \frac{e}{\sqrt{e^2-1}} \quad \Rightarrow \quad e'' = f(e') = e$



$f(f(f(......f(e)))) = \begin{cases} \frac{e}{\sqrt{e^2-1}} & n \text{ odd} \\ e & n \text{ even} \end{cases}$; $\int_1^3 e\,de = 4$

95.(8) Normal to the hyperbola is given by $2\cos\theta x + \cot\theta y = 5$, if its intercepts are equal then

$2\cos\theta = \cot\theta$ i.e. $\sin\theta = \frac{1}{2}$, hence normal is given by $x+y=\frac{5}{\sqrt{3}}$

If it touches given ellipse then $c^2 = a^2m^2+b^2$ i.e. $a^2+b^2=\frac{25}{3}$

96.(4) $\frac{(x+1)^2}{9}-\frac{(y-2)^2}{16}=1$; $k=\frac{2\times 16}{3}=\frac{32}{3}$

97.(2) Let the point is $P(h,k)$, then $k = mh-2am-am^3 \Rightarrow am^3+(2a-h)m+k=0$ ……….(i)

The slopes m_1, m_2, m_3 of the normal drawn from P are the roots of the equation (i), hence,

And $m_1m_2m_3 = -\frac{k}{a}$

Also, according to the question $m_1m_2 = p$

From the last of the above three relations, we get $m_3 = -\frac{k}{ap}$

m_3 is a root of the equation (i), so we get $a\left(-\frac{k}{ap}\right)^3+(2a-h)\left(-\frac{k}{ap}\right)+k=0$

$-\frac{k}{a^2p^3}\left[k^2+(2a-h)ap^2-a^2p^3\right]=0$

$k^2+(2a-h)ap^2-a^2p^3=0$

The locus of $P(h,k)$ is $y^2+(2a-x)ap^2-a^2p^3=0$

$y^2 = ap^2x+(p-2)a^2p^2$ ……….(ii)

Equation (ii) will represent the given parabola $y^2=4ax$ if $p^2=4$ *and* $p-2=0 \Rightarrow p=2$

98.(0) Tangents at (x_1, y_1) on the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$ ….(i)

Any point on this line (i) may be selected as $P\left(t, \dfrac{2a(t+x_1)}{y_1}\right)$

Equation of chord of contact of the tangents drawn from P to the circle $x^2 + y^2 = a^2$ is $tx + \dfrac{2a(t+x_1)}{y_1}y = a^2$

$$txy_1 + 2a(t + x_1)y - a^2 y_1 = 0$$

$$\left(2ax_1 y - a^2 y_1\right) + t(xy_1 + 2ay) = 0 \text{ …….(ii)}$$

The line (ii), for all values of t passes through the point of intersection of the lines

$$2ax_1 y - a^2 y_1 = 0 \text{ and } xy_1 + 2ay = 0$$

$$y = \frac{ay_1}{2x_1} \text{ and } x = -\frac{a^2}{x_1}$$

According to question the line (ii) passes through (x_2, y_2), so, $y_2 = \dfrac{ay_1}{2x_1}$ and $x_2 = -\dfrac{a^2}{x_1}$

Now, $4\left(\dfrac{x_1}{x_2}\right) + \left(\dfrac{y_1}{y_2}\right)^2 = 4\left(x_1 / \dfrac{-a^2}{x_1}\right) + \left(y_1 / \dfrac{ay_1}{2x_1}\right)^2 = -\dfrac{4x_1^2}{a^2} + \dfrac{4x_1^2}{a^2} = 0$

99.(0) Let $y = mx + c$

Equation of pair of asymptotes is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$

For intersection points of chord and pair of asymptotes $\dfrac{x^2}{a^2} - \dfrac{(mx+c)^2}{b^2} = 0 \begin{smallmatrix}\nearrow x_1 \\ \searrow x_2\end{smallmatrix}$ ……….(i)

For intersection points of chord and hyperbola, $\dfrac{x^2}{a^2} - \dfrac{(mx+c)^2}{b^2} = 1 \begin{smallmatrix}\nearrow x_3 \\ \searrow x_4\end{smallmatrix}$ ………..(ii)

$x_1 + x_2 = x_3 + x_4$; $y_1 + y_2 = y_3 + y_4$

Mid points of PP' and QQ' are same

$RQ = RQ'$ ……….(i)

$RP = RP'$ ……..(ii)

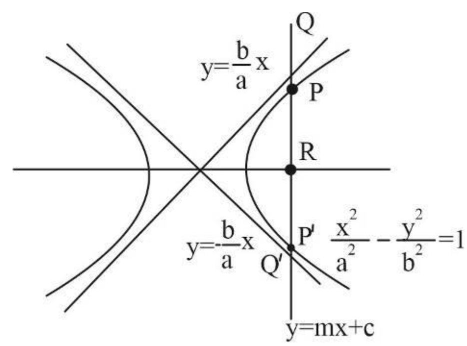
(i) – (ii)

$PQ = P'Q'$

$PQ + PP' = P'Q' + PP'$

$P'Q = PQ'$

$(PQ - P'Q') + (PQ' - P'Q) = 0$



100. (3) Equation of normal at $(2\sec\theta, \tan\theta)$ is $\dfrac{2x}{\sec\theta} + \dfrac{y}{\tan\theta} = 5$

$Slope = -1 \Rightarrow -\dfrac{2\tan\theta}{\sec\theta} = -1$; $\theta = \dfrac{\pi}{6}$

Normal becomes $y = -x + \dfrac{5}{\sqrt{3}}$

It is tangent to ellipse $\Rightarrow a^2 + b^2 = \dfrac{25}{3}$; $\dfrac{9}{25}\left(a^2 + b^2\right) = 3$

101.(6.33) Let $X = x - \sqrt{2}, Y = y - \frac{1}{\sqrt{2}}$

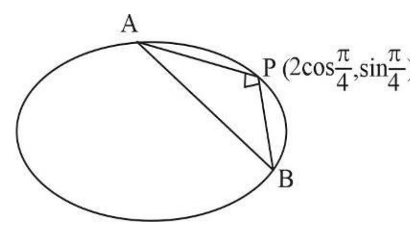
Equation of ellipse $\frac{x^2}{4} + y^2 = 1$ becomes $\frac{\left(X+\sqrt{2}\right)^2}{4} + \left(Y + \frac{1}{\sqrt{2}}\right)^2 = 1$

$$\frac{(X)^2}{4} + (Y)^2 + \frac{1}{\sqrt{2}}X + \sqrt{2}Y = 0$$

Let equation of AB be $aX + bY = 1$ Homogenising, we get $\frac{(X)^2}{4} + (Y)^2 + \left(\frac{1}{\sqrt{2}}X + \sqrt{2}Y\right)(aX + bY) = 0$

$PA \perp PB \Rightarrow coeff.of\ X^2 + coeff.of\ Y^2 = 0$

$$\frac{a}{\sqrt{2}} + \sqrt{2}b + \frac{5}{4} = 0 \ ; \ -\frac{4}{5\sqrt{2}}a - \frac{4\sqrt{2}}{5}b = 1$$



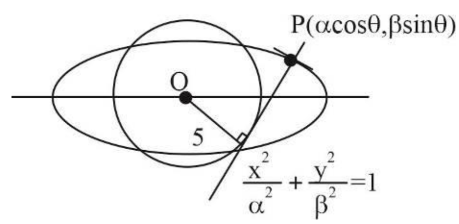
AB passes through fixed point $\left(-\frac{4}{5\sqrt{2}}, \frac{-4\sqrt{2}}{5}\right) = \left(-\frac{4}{5\sqrt{2}} + \sqrt{2}, \frac{-4\sqrt{2}}{5} + \frac{1}{\sqrt{2}}\right)$

$$\left(\frac{6}{5\sqrt{2}}, \frac{-3}{5\sqrt{2}}\right)$$

$$a = \frac{6}{5\sqrt{2}}, b = \frac{-3}{5\sqrt{2}} \ ; \ a^2 + b^2 = \frac{36+9}{50} = \frac{9}{10} = \frac{m}{n}$$

102.(2) Equation of normal at P is $\alpha x \sec\theta - \beta y \operatorname{cosec}\theta = \alpha^2 - \beta^2$

$$r = \frac{\left|\alpha^2 - \beta^2\right|}{\sqrt{\alpha^2 \sec^2\theta + \beta^2 \operatorname{cosec}^2\theta}} \ ; \ r = \frac{\left|\alpha^2 - \beta^2\right|}{\sqrt{\alpha^2 + \beta^2 + \alpha^2 \tan^2\theta + \beta^2 \cot^2\theta}}$$

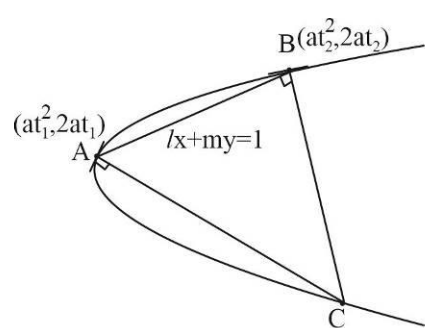


$$\le \frac{\left|\alpha^2 - \beta^2\right|}{\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta}}$$

$$r_{max} = |\alpha - \beta| = a^2 + 2a + 2 - \left(a^2 + 1\right) = 2a + 1 \ ; \quad 5 = 2a + 1 \Rightarrow a = 2$$

103.(4) Put $\left(at^2, 2at\right)$ to equation of AB i.e., $lx + my = 1$

$$lat^2 + 2mat - 1 = 0 \begin{array}{l} \nearrow t_1 \\ \searrow t_2 \end{array}$$

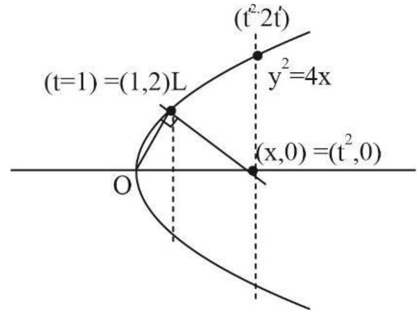


$$t_1 + t_2 = -\frac{2m}{l}$$

$$t_1 + t_2 + t_3 = 0$$

$$t_3 = \frac{2m}{l}$$

$$C = \left(at_3^2, 2at_3\right) = \left(\frac{4m^2 a}{l^2}, \frac{4am}{l}\right)$$



104.(80) $\dfrac{0-2}{x-1}=-\dfrac{1}{2}\Rightarrow x-1=4\Rightarrow x=5$

$t'^2=5\Rightarrow t'=\sqrt{5}$

Length of double ordinate $4t'=4\sqrt{5}=\sqrt{80}$

$N=80$

105.(2) If P and Q are α and β, then $|\alpha-\beta|=\dfrac{\pi}{2}$

Equation of PQ is $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2}+\dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2}=\cos\dfrac{\alpha-\beta}{2}=\dfrac{1}{\sqrt{2}}$

This line and the given line must be identical.

106.(4) The point P lies on the director circle $x^2+y^2=3$

Let P be $\left(\sqrt{3}\cos\theta,\sqrt{3}\sin\theta\right)$. Chord of contact is $x\sqrt{3}\,cos\,\theta+2y\sqrt{3}\,sin\,\theta=2$. If it is a tangent to the circle $x^2+y^2=a^2$.

Then, $a=\dfrac{2}{\sqrt{3cos^2\theta+12\,sin^2\theta}}=\dfrac{2}{\sqrt{\dfrac{15}{2}-\dfrac{9}{2}cos\,2\theta}}$

Now, $-\dfrac{9}{2}\le-\dfrac{9}{2}cos\,2\theta\le\dfrac{9}{2}\Rightarrow\dfrac{2}{\sqrt{12}}\le a\le\dfrac{2}{\sqrt{3}}$

Thus, $\dfrac{\max imum\,area}{\min imum\,area}=\dfrac{\left(\dfrac{2}{\sqrt{3}}\right)^2}{\left(\dfrac{2}{\sqrt{12}}\right)^2}=4$

107.(2) Tangent at $P\left(t,\dfrac{1}{t}\right)$ is $x+t^2y=2t\Rightarrow T(2t,0)$ and $T'\left(0,\dfrac{2}{t}\right)$

Normal at $P\left(t,\dfrac{1}{t}\right) is\, t^3x-ty=t^4-1\Rightarrow N\left(t-\dfrac{1}{t^3},0\right)$ and $P\left(t,\dfrac{1}{t}\right) is\, t^3x-ty=t^4-1\Rightarrow N'\left(0,\dfrac{1}{t}-t^3\right)$

Now the area of $\Delta PTN\, is\, \Delta=\dfrac{1}{2}\left|\dfrac{1}{t}\left(t+\dfrac{1}{t^3}\right)\right|=\dfrac{\left(1+t^4\right)}{2t^4}$

And the area of $\Delta PT'N'\, is\, \Delta=\dfrac{1}{2}\left|t\left(\dfrac{1}{t}+t^3\right)\right|=\dfrac{\left(1+t^4\right)}{2}$

Hence, $\dfrac{1}{\Delta}+\dfrac{1}{\Delta'}=2\left(\dfrac{t^4+1}{1+t^4}\right)=2$

$\dfrac{1}{\Delta}+\dfrac{1}{\Delta'}$ is constant for all positions of P.

108.(8) Let the coordinates of P are $\left(t, \frac{1}{t}\right)$.

Equation of the tangent at P is $x + t^2 y = 2t$

It meets $y = x \text{ and } y = -x$ at the points $Q\left(\frac{2t}{1+t^2}, \frac{2t}{1+t^2}\right)$ and $R\left(\frac{2t}{1-t^2}, -\frac{2t}{1-t^2}\right)$ respectively. Hence,

$$\Delta_1 = \frac{1}{2}\left[\left(\frac{2t}{1+t^2}\right)\left(\frac{2t}{1-t^2}\right) + \left(\frac{2t}{1-t^2}\right)\left(\frac{2t}{1+t^2}\right)\right] = \left|\frac{4t^2}{1-t^4}\right|$$

Again, the normal at P is $t^3 x - ty = t^4 - 1$

It meets the X-axis and the Y axis at $M\left(\frac{t^4-1}{t^3}, 0\right)$ and $N\left(0, \frac{1-t^4}{t}\right)$ respectively, hence.

$$\Delta_2 = \frac{1}{2}\left|\left(\frac{t^4-1}{t^3}\right)\left(\frac{1-t^4}{t}\right)\right| = \left|\frac{\left(t^4-1\right)^2}{2t^4}\right| \; ; \; \Delta_1^2 \Delta_2 = 8$$

109.(3) $ae = 5\sqrt{14} \text{ and } e = \sqrt{2} \Rightarrow a^2 = 175$

So, the equation of the hyperbola is $x^2 - y^2 = 175$ Or $(x-y)(x+y) = 175$

$175 = 5^2 .7 \text{ and } x > y$ so, we can have $x - y = 1, x + y = 175 \Rightarrow x = 88 \text{ and } y = 87$

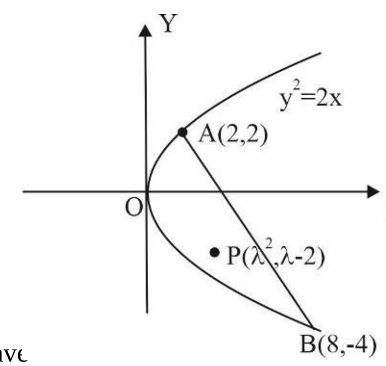
$x - y = 5, x + y = 35 \Rightarrow x = 20 \text{ and } y = 15;$ $\quad x - y = 7, x + y = 25 \Rightarrow x = 16 \text{ and } y = 9$

110.(3) If coordinates of A, B, C and D are $(x_i, y_i); i = 1,2,3,4,$ then we have $\sum_{i=1}^{4} x_i = h, \sum_{i=1}^{4} y_i = k, \sum_{i=1}^{4} x_i^2 = h^2 \text{ and } \sum_{i=1}^{4} y_i^2 = k^2,$

Now $PA^2 + PB^2 + PC^2 + PD^2 = \sum_{i=1}^{4}\left[(h - x_i)^2 + (k - y_i)^2\right]$

$$= 4\left(h^2 + k^2\right) - 2h\sum_{i=1}^{4} x_i - 2k\sum_{i=1}^{4} y_i + \sum_{i=1}^{4} x_i^2 + \sum_{i=1}^{4} y_i^2 \quad = 4\left(h^2 + k^2\right) - 2h^2 - 2k^2 + h^2 + k^2 = 3\left(h^2 + k^2\right)$$

111.(1) The points $P\left(\lambda^2, \lambda - 2\right)$ lies inside the parabola $y^2 = 2x$, so



$(\lambda - 2)^2 - 2\lambda^2 < 0$; $\lambda^2 + 4\lambda - 4 > 0$

The roots of the equation $\lambda^2 + 4\lambda - 4 = 0$ are $\lambda = -2 - 2\sqrt{2}$ and $-2 + 2\sqrt{2}$

So the solution of the above inequality is

$\lambda < -2 - 2\sqrt{2}$ or $\lambda > -2 + 2\sqrt{2}$ ……….(i)

Further, the equation of the chord is $y - 2 = \frac{-4-2}{8-2}(x - 2) \Rightarrow x + y - 4 = 0$

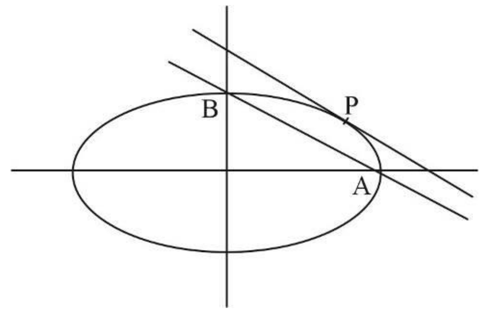
The origin O and the point $P\left(\lambda^2, \lambda - 2\right)$ lies on the same side of the line, so we have

$\lambda^2 + \lambda - 2 - 4 < 0 \Rightarrow \lambda^2 + \lambda - 6 < 0$

$(\lambda + 3)(\lambda - 2) < 0 \Rightarrow -3 < \lambda < 2$ ………(ii)

The common values from (i) and (ii) are $-2 + 2\sqrt{2} < \lambda < 2 \Rightarrow \lambda \in \left(-2 + 2\sqrt{2}, 2\right)$

112.(3) Let perpendicular distance of P from the line be h



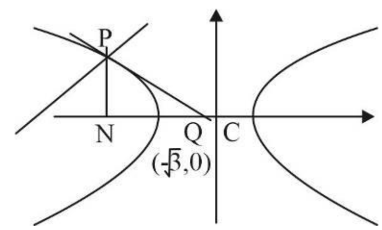
$\frac{1}{2}\times h\times 5=6(\sqrt{2}-1)$ (as $\Delta PAB=6(\sqrt{2}-1)$)

$h=\frac{12\left(\sqrt{2}-1\right)}{5}$

Now distance of tangent parallel to AB i.e. $4y+3x=12\sqrt{2}$,

from line AB is $\frac{12\left(\sqrt{2}-1\right)}{5}$. There are just three such points.

113.(3) The corresponding normals are $y=x\pm\frac{5\sqrt{3}}{3}$.



Considering $y=x+\frac{5\sqrt{3}}{3}$, corresponding tangent is $y=-x-\sqrt{3}$

Point of contact is $\left(\frac{-4}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

Length of sub tangent $=NQ=NC-QC=\frac{4}{\sqrt{3}}-\sqrt{3}=\frac{1}{\sqrt{3}}\Rightarrow k=3$

114(1). equation of normal is $y-t^2=-\frac{1}{2t}(x-t)$ it passes through $\left(0,\frac{3}{2}\right)$

$t=0$ or $t=1,-1$

hence P,Q,R are (0, 0) (1, 1) and (–1, 1), PQR is a right triangle, radius of circum circle is 1.

115.(4) Let $A_r=\left(2t,t^2\right)$

Slope of BA, $BA_r=\frac{t^2-1}{2t}=tan\left(\frac{\pi}{2}+\theta_r\right)$

$tan\left(\theta_r\right)=\frac{2t}{1-t^2}=tan\,2\phi$

$\phi=\frac{\theta_r}{2}=\frac{r\pi}{4n}$ where $tan\,\phi=t$ and $BA_r=\sqrt{\left(t^2-1\right)^2+4t^2}=t^2+1=sec^2\phi=sec^2\left(\frac{r\pi}{4n}\right)$

$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}BA_r=\int_0^1 sec^2\left(\frac{\pi x}{4}\right)dx=\frac{4}{\pi}$

116.(1.32) Image of $\left(h,k\right)$ on ellipse about $x-y-2=0$ is say $\left(h',k'\right)$

$\frac{h'-h}{1}=\frac{k'-k}{-1}=-2\frac{h-k-2}{1+1}=-h+k+2$

$h'=k+2,k'=h-2$; $\left(h,k\right)=\left(k'+2,h'-2\right)$

$\frac{\left(h-4\right)^2}{16}+\frac{\left(k-3\right)^2}{9}=1$; $\frac{\left(k'+2-4\right)^2}{16}+\frac{\left(h'-2-3\right)^2}{9}=1$; $16h'^2+9k'^2-36k'-160h'+292=0$

Equation of reflection of ellipse is $16x^2+9y^2-160x-36y+292=0$

$\frac{k_1+k_2}{100}=\frac{292-160}{100}=1.32$

FUNCTIONS

1.(B) Check by drawing a rough graph.

2.(D) Domain of $f(x)$ is $[-1,1]$ and $f(x)$ is continuous and increasing. So, $\tan^{-1} x \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$ and $\frac{1}{2}\sin^{-1} x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

3.(C) As $-(x-1) = -x+1,$ the two graphs are symmetrical about the line $x-1=0$

4.(C) $f(x) = x$ on [2, 3], period = 2

$f(x) = x+4$ on $[-2,-1]$ and $f(x) = x+2$ on [0, 1]

So, $f(x) = 2-x$ on $[-1, 0]$ as it is an even function

Hence, on $[-2, 0]$, $f(x) = 3 - |x+1|$

5.(A) Notice that $f(x)$ is odd and increasing So, if $a+b \ge 0,\ a \ge -b$ then $f(a) \ge f(-b)$ or $f(a) \ge -f(b),$ $f(a) + f(b) \ge 0$ and these steps can be traced back.

6.(A) Simplify $f(x)$ to get $f(x) = 1 - \frac{1}{2}\sin 2x - \frac{1}{2}\sin^2 2x = 1 - \frac{1}{2}t - \frac{t^2}{2}, t = \sin 2x, t \in [-1,1]$

$$= \frac{9}{8} - \frac{1}{2}\left(t + \frac{1}{2}\right)^2$$

7.(B) Notice that $f(x) + f(\pi - x) = 2$. So, $\frac{1}{2} f(x) + \frac{1}{2} f(\pi - x) = 1.$ Hence, $a = b = \frac{1}{2}$ and $c = \pi$

8.(A) Note that $f(0) = 1$ and putting x = 0 in the given functional equation, we have $f(y) = e^{f(y)-1}$. Now consider a function $g(t) = t - e^{t-1}$. Here $g'(t) < 0$ if $t > 1$ and $g'(t) > 0$ if $t < 1$ and $g(1) = 0.$ So, $g(t)$ has only one root $t = 1.$

9.(A) $3^x = x(x+1)(x+2)+1$ LHS = multiple of 3 and RHS = not divisible by 3.

10.(B) $Q(x) = (x-1)^2 + 21,$ $Q(x)$ takes all values in interval $[21, \infty)$. Since $P(Q(x))$ has no real roots, all three roots of $P(x)$ must be less than 21.

P(22) = 21 gives

$(22-x_1)(22-x_2)(22-x_3) = 21$ = 1 × 3 × 7 where x_1, x_2, x_3 are roots of P(x).

So, $\{(22-x_1),(22-x_2),(22-x_3)\} = \{1,3,7\}$

So, $x_1 = 15,\ \ x_2 = 19,\ \ x_3 = 21$

Hence, $a = -(x_1 + x_2 + x_3) = -55$

11.(A) The given graph can be rewritten as $(y-3)(x+4) = \frac{-29}{3}$. [A rectangular hyperbola with centre at (– 4, 3)]

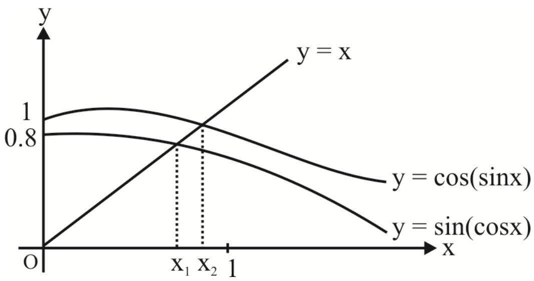
12.(A) $f(x) = \sqrt[3]{x^2 + 100}$ = increasing function

The given equation can be written as $f(f(x)) = x$ so, $f(x) = x \implies (f(x))^3 = x^3,$

$$x^2 + 100 = x^3 \quad ; \quad x^3 - x^2 - 100 = 0 \quad ; \quad (x-5)(x^2 + 4x + 20) = 0$$

Hence $x = 5$

13.(B)



14.(B) Given expression $= \dfrac{1}{-f(x)}$; where $f(x) = x^2 + b^2 + sin^2 x - 2bx = (x-b)^2 + sin^2 x$

Now, $x \in [-1, 0]$ and $b \in [2, 3]$ $\Rightarrow$ $|x-b| \in [2, 4] \Rightarrow b \in [4, 16]$ ad $sin^2 x \in \left[0, sin^2 1\right]$.

As x increases from -1 to 0 ; $sin^2 x$ and $(x-b)^2$ both decreases from $sin^2 1$ to 0 and $(1+b)^2$ to b^2 respectively and $b \in [2, 3]$.

$$\Rightarrow \quad f(x) \in \left[4, 16 + sin^2 1\right] \Rightarrow \frac{1}{f(x)} \in \left[\frac{1}{16 + sin^2 1}, \frac{1}{4}\right] \Rightarrow \frac{1}{-f(x)} \in \left[-\frac{1}{4}, -\frac{1}{16 + sin^2 1}\right]$$

So, least value of expression is $\dfrac{-1}{4}$

15.(C) $\because$ $f(x)$ is an even function

$$\Rightarrow \quad f(x) = f(-x) \ \forall \ x \in R - \left\{(2n+1)\frac{\pi}{2};\ n \in Z\right\}$$

$$\Rightarrow \quad \left([a]^2 - 5[a] + 4\right)x^3 - \left(6[a]^2 - 5\{a\} + 1\right)x - (tan\, x)\, sgn(x)$$

$$= -\left([a]^2 - 5[a] + 4\right)x^3 + \left(6[a]^2 - 5\{a\} + 1\right)x + tan\, x\, sgn(-x)$$

$$\Rightarrow \quad 2\left([a]^2 - 5[a] + 4\right)x^3 - 2\left(6\{a\}^2 - 5\{a\} + 1\right)x + tan\, x\, sgn(-x)$$

$$\Rightarrow \quad 2\left([a]^2 - 5[a] + 4\right)x^3 - 2\left(6\{a\}^2 - 5\{a\} + 1\right)x - tan\, x$$

$$\left(sgn(x) + sgn(-x)\right) = 0 \ \forall \ x \in R - \left\{(2n+1)\frac{\pi}{2};\ n \in Z\right\}$$

$$\Rightarrow \quad \left([a]^2 - 5[a] + 4\right)x^3 - \left(6\{a\}^2 - 5\{a\} + 1\right)x = 0 \ \forall \ x \in R - \left\{(2n+1)\frac{\pi}{2};\ n \in Z\right\}$$

But $sgn\, x + sgn(-x) = 0 \ \forall \ x \in R$

$$\Rightarrow \quad \left([a]^2 - 5[a] + 4\right)x^3 - \left(6\{a\}^2 - 5\{a\} + 1\right)x = 0 \ \forall \ x \in R - \left\{(2n+1)\frac{\pi}{2};\ n \in Z\right\}$$

$$\Rightarrow \quad [a]^2 - 5[a] + 4 = 0 \text{ and } 6\{a\}^2 - 5\{a\} + 1 = 0$$

$$\Rightarrow \quad ([a]-1)([a]-4) = 0 \text{ and } (3\{a\}-1)(2\{a\}-1) = 0$$

$\Rightarrow \quad [a]=1$ or $[a]=4$ and $\{a\}=\frac{1}{3}$ or $\{a\}=\frac{1}{2}$

$\therefore \quad [a]=1, \{a\}=\frac{1}{3} \Rightarrow \quad a=\frac{4}{3}$

$[a]=1, \{a\}=\frac{1}{2} \Rightarrow \quad a=\frac{3}{2}$

$[a]=4, \{a\}=\frac{1}{3} \Rightarrow \quad a=\frac{13}{3}$

$[a]=4, \{a\}=\frac{1}{2} \Rightarrow \quad a=\frac{9}{2}$

$\therefore$ Sum of all possible values of $a=\frac{4}{3}+\frac{3}{2}+\frac{13}{3}+\frac{9}{2}=\frac{8+9+26+27}{6}=\frac{35}{3}$

16.(ACD) $f(x)=5x,\ x\geq\frac{-1}{2}\ =\ x-2,\ x<\frac{-1}{2}$

$g(x)=\frac{x}{5},\ \ x\geq\frac{-5}{2}=\ x+2,\ \ x<\frac{-5}{2}$

For $x<\frac{-1}{2},\ g(f(x))=g(x-2)=(x-2)+2=x$ For $x\geq\frac{-1}{2},\ g(f(x))=g(5x)=\frac{1}{5}(5x)=x$

For $x<\frac{-5}{2},\ f(g(x))=f(x+2)=(x+2)-2=x$ For $x\geq\frac{-5}{2},\ f(g(x))=f\left(\frac{x}{5}\right)=5\left(\frac{x}{5}\right)=x$

17.(AC) From the given condition, $f(10+(10-x))=f(10-(10-x))$ i.e., $f(20-x)=f(x)$

Similarly, $-f(20+x)=f(x)$

So, $f(x)=-f(20+x)=f(20+(20+x))$

$f(x)=f(x+40)$. Periodic and $f(-x)=f(20+x)=-f(x)$ odd function.

18.(ABC) Adding the given equations, $[3x]+[-y]+\{y\}-\{x\}=2$

So, $\{y\}-\{x\}$ is an integer and since $-1<\{y\}-\{x\}<1$, so, $\{y\}-\{x\}=0$

19.(ABC) For horizontal tangent $(f'(x)=0)\Rightarrow x=e$

$f'(x)\gtrless 0$for $x\in(e,\infty)$ and $x\in(0,e)$ respectively. So, many one function.

For vertical tangent, $f'(x)=\infty\ \Rightarrow x=0$ which is not in the domain of $f(x)$

20.(ABCD) **(A)** Sgn $(e^{-x})=1$ **(B)** LCM of 2π and π

(C) $\min.(|x|, \sin x)=\sin x$ **(D)** Simplify to get $-\left\{x+\frac{1}{2}\right\}-\left\{x-\frac{1}{2}\right\}-2\{-x\}$

21.(ABCD) $f(x)=\frac{1-x}{x^2}\quad x<1,\ x\neq 0$ $\quad f'(x)=\frac{x-2}{x^3},\ x<1,\ x\neq 0$ and

$=\frac{x-1}{x^2}\quad x>1$ $\quad =\frac{2-x}{x^3},\ x>1$

22.(AD) Domain of $f(x)$ is $(-1,1)$. So, $[x^2]=0$

23.(AC) $f(x)$ = odd degree polynomial + bounded function $\cot^{-1}x\in(0,\pi)$ and $f'(x)>0$

24.(BD) $f(x)$ is odd function

25.(AB) $f(x) = x^2,\ x \in I = x^2 + x,\ x \notin I$

26.(AD) $g(x) = \frac{\sin \pi x}{x} + \frac{\sin \pi x}{1-x} = \frac{\sin \pi x}{x(1-x)} = \frac{2\sin\frac{\pi x}{2}\cos\frac{\pi x}{2}}{x(1-x)}$ and $g\left(\frac{1}{2}+x\right) = g\left(\frac{1}{2}-x\right)$

27.(ABCD) Simplify $f(-x)$ to get equal to $f(x)$. So, even function.

$f(x)$ is even so it is not one-one

$\lim\limits_{x\to 0} \frac{x}{e^x - 1} + \frac{x}{2} + 1 = 1 + 1 = 2.$

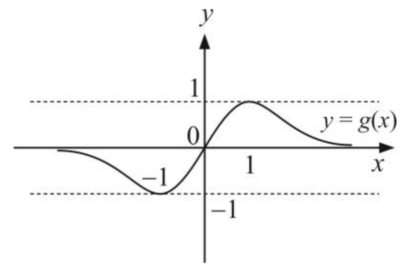
As $\lim\limits_{x\to\infty} f(x) \to \infty$ and $f(x)$ is even. So, $f(x) > 0$ and hence not onto.

28.(BCD) Domain of $f(x) = [0, \infty)$

Domain of $g(x) = (-\infty, \infty)$

$\Rightarrow$ Domain of $(g+2)(x) = (-\infty, \infty)$

$\therefore$ Domain of $(f+g+2)(x) = [0, \infty) \cap (-\infty, \infty) = [0, \infty)$



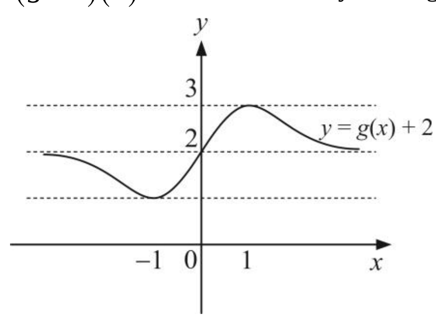
Further range of $f(x) = [2, \infty)$ as $f'(x) = \frac{1}{2\sqrt{x}} > 0\ \forall\ x \in (0, \infty)$

$\Rightarrow$ $f(x)$ is increasing and continuous function on $[0, \infty)$

Now $f(x) = \frac{2x}{1+x^2}$. Graph of $g(x)$ is as shown below

Clearly $g(x)$ has its range $[-1, 1]$

$\Rightarrow$ $(g+2)(x)$ can be obtained by shifting the graph of $g(x)$ by 2 units above x-axis as shown below



$\therefore$ Range of $(g+2)(x)$ is [1, 3]

$\therefore$ Range of $f(x) \cap$ range of $(g+2)(x) = [1, 3] \cap [2, \infty) = [2, 3]$

and range of $f(x) \cup$ range of $(g+2)(x) = [1, 3] \cap [2, \infty) = [1, \infty)$

29.(CD) If $f(x) = 2$

$\Rightarrow$ $\sin x = 1, \sin x\sqrt{3} = 1 \Rightarrow x = (4n+1)\frac{\pi}{2}$ and $x\sqrt{3} = (4m+1)\frac{\pi}{2}$

$\Rightarrow$ $\sqrt{3} = \frac{4m+1}{4n+1}$ which is a contradiction as LHS is irrational whereas RHS is rational

$\therefore$ $f(x)$ cannot attain value 2.

$\therefore$ 2 cannot be the maximum value of $f(x)$.

$\Rightarrow$ $sin\,x = sin\,x\sqrt{3} = -1$

Which is again impossible by same reason So, (B) is true.

Now period of $f(x) = LCM\left(2\pi, \dfrac{2\pi}{\sqrt{3}}\right)$ which does not exist as multiples of 2π are $\pm 2\pi, \pm 4\pi, \ldots$

whereas multiplies of $\dfrac{2\pi}{\sqrt{3}}$ are $\pm\dfrac{2\pi}{\sqrt{3}}, \pm\dfrac{4\pi}{\sqrt{3}}, \pm\dfrac{6\pi}{\sqrt{3}}$ Therefore, (C) is false.

Now, $f(0) = 0$ $\therefore$ $f(x) > 0\ \forall\ x \in R$ is false, i.e., (D) is false.

30. **[A-s] [B-r] [C-q] [D-p]**

Consider the ratio $f(x) = \dfrac{x^n}{e^x}$, $f(0) = 0$

$f'(x) = \dfrac{x^{n-1}(n-x)}{e^x}$ $\lim\limits_{x\to\infty} f(x) = 0$

f is increasing for $0 < x < a$ and decreasing for $x > a$

So, $f(x) = 1$ has two solutions if $f(a) > 1$, one if $f(a) = 1$ and none if $f(a) < 1$

31. **[A-r] [B-s] [C-p] [D-p]**

(A) $y = \dfrac{x^2+9+6x}{x^2+1} \Rightarrow (y-1)x^2 - 6x + (y-9) = 0$ $D \ge 0 \Rightarrow 0 \le y \le 10$

(B) Range of $A = [-5, 5]$ and of B is $R^+ \cup \{0\}$. Hence, range of $f(x)$ is $[0, 5]$

(C) $y = \dfrac{9}{2-\cos 3x} \Rightarrow \cos 3x = \dfrac{2y-9}{y} \in [-1, 1]$ $3 \le y \le 9$

(D) Domain $\left[\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right] \Rightarrow \sqrt{\dfrac{\pi^2}{16} - x^2} \in \left[0, \dfrac{\pi}{4}\right]$. So, $0 \le f(x) \le 3$

32. **[(i)-(c)] [(ii)-(d)] [(iii)-(b)] [(iv)-(c)]**

(i) When $\{x\} = 0$, then $sgn\{x\} = 0$ and when $0 < \{x\} < 1$, then $sgn\{x\} = \{0, 1\}$

(ii) For domain $-1 \le x \le 1$ and $-1 \le 1 - x \le 1$

$\Rightarrow$ $-1 \le x \le 1$ and $0 \le x \le 2 \Rightarrow 0 \le x \le 1 \Rightarrow$ Domain of function = [0, 1]

(iii) $-\dfrac{\pi}{2} < tan^{-1}x < \dfrac{\pi}{2}\ \forall\ x \in R$

$\Rightarrow$ $-1 < \dfrac{2}{\pi}tan^{-1}x < 1$; for $f(x) = \sqrt{\dfrac{2\,sin^{-1}x}{\pi}}$; $\dfrac{2\,tan^{-1}x}{\pi} \ge 0$

$\Rightarrow$ $0 \le \dfrac{2\,tan^{-1}x}{\pi} < 1 \Rightarrow 0 \le \sqrt{\dfrac{2}{\pi}tan^{-1}x} < 1 \Rightarrow$ Range of $f(x) = [0, 1)$

(iv) $\dfrac{3}{4} \le x^2 + x + 1 < \infty$, but for $sin^{-1}\left[x^2+x+1\right]$

$-1 \le \left[x^2+x+1\right] \le 1 \Rightarrow \dfrac{3}{4} \le x^2 + x + 1 < 2$

Case 1: When $\frac{3}{4} \le x^2 + x + 1 < 1$

$\Rightarrow \quad \left[x^2 + x + 1\right] = 0 \Rightarrow sin^{-1}\left[x^2 + x + 1\right] = 0 \Rightarrow \quad f(x) = \frac{2}{\pi} sin^{-1}\left[x^2 + x + 1\right] = 0$

Case 2: When $1 \le x^2 + x + 1 < 2$

$\Rightarrow \quad \left[x^2 + x + 1\right] = 1 \Rightarrow sin^{-1}\left[x^2 + x + 1\right] = \frac{\pi}{2} \Rightarrow f(x) = \frac{2}{\pi} . \frac{\pi}{2} = 1$

$\therefore$ Range of given function is {0, 1}

33.(2) Use the concept of inflexion point of a cubic i.e., a cubic graph is symmetrical about its inflexion point.

34.(3) Let $[x] = m$ then $\frac{x}{x+4} = \frac{5m-7}{7m-5}$,

Rearranging, $x = \frac{10m-14}{m+1}$

Now, we have $m \le x < m+1 \Rightarrow m \le \frac{10m-14}{m+1} < m+1$

Simplify to get, possible integral values of m to be 2, 6 and 7.

35.(22) Consider the function $f(t) = 144^t + 144^{1-t}$, $t = \sin^2 x$, $0 \le t \le 1$

$f'(t) = \log 144 \cdot \left(144^t - 144^{1-t}\right)$

So, f is decreasing on $\left(0, \frac{1}{2}\right)$ and increasing on $\left(\frac{1}{2}, 1\right)$.

So, $f_{\min} = f\left(\frac{1}{2}\right) = 24$ and $f_{\max} = f(0) = f(1) = 45$

36.(5) Use the concept, $[t] \le t < [t] + 1$, $t \in R$

So, $3x - 7 \le x^2 - 3x + 2 < (3x-7) + 1$ and $3x \in I$

Solving, $x \in \left\{\frac{7}{3}, \frac{8}{3}, \frac{9}{3}, \frac{10}{3}, \frac{11}{3}\right\}$

37.(8) Let $t = 2105$ then given expression is $\frac{(t+1)^3}{(t-1)t} - \frac{(t-1)^3}{t(t+1)} = 8 + \frac{16t}{t^3 - t} = 8 + \frac{16}{t^2 - 1}$

So, integral part = 8, $\frac{16}{t^2-1} < 1$

38.(4) Given function is $y = \frac{x+m}{x^2+1}$

$\Rightarrow \quad yx^2 - x + (y - m) = 0$

We must have $D \ge 0$ for $y \in [0, 1]$

$\Rightarrow \quad 1 - 4y(y-m) \ge 0$ for $y \in [0, 1] \Rightarrow g(y) = 4y^2 - 4my - 1 \le 0$ for $y \in [0, 1]$

$\Rightarrow \quad g(y) \le 0$ for $y \in [0, 1] \Rightarrow g(0) \le 0, g(1) \le 0$

$\Rightarrow \quad -1 \le 0$ and $3 - 4m \le 0 \Rightarrow m \ge \frac{3}{4} \Rightarrow k = 4$

39.(2049) Let $f(x)=(x^2+5x+4)(x^2+5x+6)+5$

$$=\left[(x^2+5x+5)-1\right]\left[(x^2+5x+5)+1\right]+5=(x^2+5x+5)^2-1+5$$

$\Rightarrow \quad f(x)=(x^2+5x+5)^2+4$

Hence, $f(x)$ has minimum value 4 when $x^2+5x+5=0$, i.e., $x=\dfrac{-5\pm\sqrt{5}}{2}\in[-6, 6]$

Also maximum occurs at $x = 6$.

$\Rightarrow \quad f(x)_{max}=(36+30+5)^2+4=(71)^2+4=5041+4=5045$

Range is [4, 5045] $\equiv$ $[a, b]$ (Given)

$\Rightarrow \quad a=4; b=5045 \;\Rightarrow\; a+b=5049$

40.(2) Given $f(x)=f\left(\dfrac{x+4}{x-2}\right)$

One of the possibilities is $x=\dfrac{x+4}{x-2} \;\Rightarrow\; x^2-2x-x-4=0 \Rightarrow x^2-3x-4=0$

Disc. = $9+16=25>0$

$\therefore$ Minimum number of roots = 2

41.(37) Given $f(2x+1)=4x^2+14x$ **. . . .(i)**

We are to find the sum of squares of roots of equation $f(x) = 0$. Putting $2x+1=y$

$\Rightarrow \quad x=\dfrac{y-1}{2}$ in (i), we have $f(y)=4\left(\dfrac{y-1}{2}\right)^2+14\left(\dfrac{y-1}{2}\right)$

$\Rightarrow \quad f(y)=y^2+1-2y+7y-7 \;\Rightarrow\; f(y)=y^2+5y-6$

$\therefore \quad f(x)=0$

$\Rightarrow \quad x^2+5x-6=0 \;\Rightarrow\; (x+6)(x-1)=0 \;\Rightarrow\; x=-6$ or $x=1 \quad \therefore \quad \alpha^2+\beta^2=36+1=37$

42.(7) Clearly α is a non-real complex root of equation $x^{13}=1$

But α is non-real complex, α is a root of $1+x+x^2+...+x^{12}=0$

Now, $\displaystyle\sum_{r=0}^{12} r(\alpha^{??}x)=f(x)+f(\alpha x)+f(\alpha^2 x)+...+f(\alpha^{12}x)$

$$=\left(7+\sum_{k=1}^{50}A_k x^k\right)+\left(7+\sum_{k=1}^{50}A_k\alpha^k x^k\right)+...+\left(7+\sum_{k=1}^{50}A_k\alpha^{124}x^k\right)$$

$$=91+\sum_{k=1}^{50}A_k x^k\left[1+\alpha^k+...+\alpha^{12k}\right]=91+\sum_{k=1}^{50}A_k x^k\,\frac{1-(\alpha^k)^{12}}{1-\alpha^k}=91+\sum_{k=1}^{50}A_k x^k\,\frac{(1-\alpha^{13k})}{(1-\alpha^k)}$$

$=91+0$ ($\because \alpha$ is a root of $x^{13}=1 \Rightarrow \alpha^{13}=1 \Rightarrow \alpha^{13}-1=0$)

$\therefore \quad \dfrac{1}{13}\displaystyle\sum_{r=0}^{13} f(\alpha^r x)=7$

43.(12) Given $g(x) = f\left(2^x, 0\right) = f\left(2, 2+2.0, 2.0-2.2^x\right)$

$$= f\left(2^{x+1}, -2^{x+1}\right) = f\left(2.2^{x+1} + 2\left(-2^{x+1}\right), 2\left(-2^{x+1}\right) - 2.2^{x+1}\right)$$

$$= f\left(0, -2^{x+3}\right) = f\left(-2^{x+4}, -2^{x+4}\right) = f\left(-2^{x+6}, 0\right)$$

$$= f\left(-2^{x+7}, 2^{x+7}\right) = f\left(0, 2^{x+9}\right) = f\left(2^{x+10}, 2^{x+10}\right)$$

$= f\left(2^{x+12}, 0\right) = g(x+12) \Rightarrow$ Period of g is 12.

44.(1) Let $y = g(x) = \dfrac{e^x - e^{-x}}{2} \quad \Rightarrow \quad e^{2x} - 1 = 2ye^x$

Substituting $e^x = t$, we have $t^2 - 2yt - 1 = 0$

$$\Rightarrow \qquad t = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$$

$$\Rightarrow \qquad t = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$$

$$\Rightarrow \qquad e^x = y + \sqrt{y^2+1} \qquad \left(\because e^x > 0\right)$$

$$\Rightarrow \qquad x = ln\left(y + \sqrt{y^2+1}\right) \Rightarrow g^{-1}(y) = ln\left(y + \sqrt{y^2+1}\right)$$

$$\therefore \qquad g^{-1}(x) = f(x) = ln\left(x + \sqrt{x^2+1}\right) \qquad \left(\because g\left(f(x)\right) = x \Rightarrow g^{-1}(x) = f(x)\right)$$

$$\therefore \qquad f\left(\frac{e^{22}-1}{2e^{11}}\right) = ln\left(\frac{e^{22}-1}{2e^{11}} + \sqrt{\left(\frac{e^{22}-1}{2e^{11}}\right)^2 + 1}\right) = ln\left(\frac{e^{22}-1}{2e^{11}} + \sqrt{\left(\frac{e^{22}+1}{2e^{11}}\right)^2}\right) = ln\left[e^{11}\right] = 11$$

DIFFERENTIAL CALCULUS-1

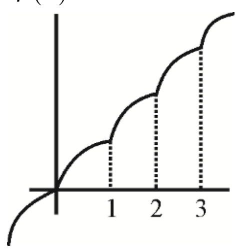
1.(D) $\lim_{x\to 0}\frac{((a-n)nx-\tan x)\sin nx}{x^2}=0$

$\lim_{x\to 0}\left(\left((a-n)n-\frac{\tan x}{x}\right)\frac{\sin nx}{nx}\right)n=0 \Rightarrow (a-n)n=1$

$a=\frac{1}{n}+n$

2.(D) $k=\lim_{x\to 0^+}\frac{(2e)^x-x-x\ln 2-1}{x^2\frac{\tan x}{x}}=\lim_{x\to 0^+}\frac{1+x\ln(2e)+\frac{x^2}{2!}(\ln(2e))^2+....-x-x\ln 2-1}{x^2\frac{\tan x}{x}}=\frac{1}{2}(\ln 2+1)^2=\frac{1}{2}(\ln 2)^2+\ln 2+\frac{1}{2}$.

3.(C) Graph of $f(x)$



4.(A) Let $\sin x=t$ and evaluate $\lim_{t\to 1}\frac{t^2}{1-t^2}\left[\sqrt{2t^2+3t+4}-\sqrt{t^2+6t+2}\right]$ by rationalization

5.(B) $\underset{x\to\infty}{Lim}\,[(x+a)(x+b)(x+c)]^{1/3}-x$ (Using $a-b=\frac{a^3-b^3}{a^2+ab+b^2}$)

$=\frac{[x^3+(a+b+c)x^2+(ab+bc+ca)x+abc]-x^3}{[(x+a)(x+b)(x+c)]^{2/3}+x^2+x[(x+a)(x+b)(x+c)]^{1/3}}$

$\underset{x\to\infty}{Limit}\frac{x^2\left[(a+b+c)+\frac{ab+bc+ca}{x}+\frac{abc}{x^3}\right]}{x^2\left[\left(1+\frac{a+b+c}{x}+\frac{ab+bc+ca}{x^2}+\frac{abc}{x^2}\right)+1+\left(1+\frac{a+b+c}{x}+\frac{ab+bc+ca}{x^2}+\frac{abc}{x^3}\right)^{1/3}\right]^{1/3}}=\frac{a+b+c}{3}$

6.(D) $\underset{h\to 0}{Limit}$ g(n + h) = f [{n + h}] = f (h) = $\underset{h\to 0}{Limit}\frac{e^h-\cos 2h-h}{h^2}=\frac{(e^h-1)+(1-\cos 2h)-h}{h^2}$

$=\underset{h\to 0}{Limit}\frac{e^h-h-1}{h^2}+\underset{h\to 0}{Limit}\frac{(1-\cos 2h)}{4h^2}.4=\frac{1}{2}+2=\frac{5}{2}$

$\underset{h\to 0}{Limit}$ g(n – h) = $\underset{h\to 0}{Limit}$ f (1 – {n–h}) = $\underset{h\to 0}{Limit}$ f (h)

$=\frac{e^h-\cos 2h-h}{h^2}\ (\{n-h\}=\{-h\}=1-h)=g(n)=\frac{5}{2}$.

Hence $g(x)$ is continuous at $\forall x\in I$. Hence $g(x)$ is continuous $\forall x\in R$

7.(C) 1^∞ form: $l = e^{\lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^\alpha + \sin\frac{1}{n} - 1\right)} = e^{\lim_{n\to\infty} n\sin\frac{1}{n} + \lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^\alpha - 1\right)} = e \cdot e^l$

Consider $l = \lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^\alpha - 1\right) = \lim_{n\to\infty} n\left(\left(\frac{1}{1+1/n}\right)^\alpha - 1\right)$; put n = $\frac{1}{y}$

$= \lim_{y\to 0}\frac{1}{y}\left(\left(\frac{1}{1+y}\right)^\alpha - 1\right) = \lim_{y\to 0}\frac{1-(1+y)^\alpha}{y} = -\alpha$ (using binomial);

8.(C) Let $f(n) = \frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + ... + \frac{n}{n^2+n+n}$

Consider g (n) = $\frac{1}{n^2+n+n} + \frac{2}{n^2+n+n} + ... \frac{n}{n^2+n+n} = \frac{1+2+3+....+n}{n^2+2n} = \frac{n(n+1)}{2(n^2+2n)}$

$g(n) < f(n)$ ....(1)

ly $h(n) = \frac{1}{n^2+n+1} + \frac{2}{n^2+n+1} + ... + \frac{n}{n^2+n+1} = \frac{n(n+1)}{2(n^2+n+1)}$

$\therefore$ $f(n) < h(n)$ ....(2)

From (1) and (2) : $g(n) < f(n) < h(n)$

But $\lim_{n\to\infty} g(n) = \lim_{n\to\infty} h(n) = \frac{1}{2}$; Hence using Sandwich theorem $\therefore$ $\lim_{n\to\infty} f(n) = \frac{1}{2}$ Ans.

9.(B) n < $\sqrt{n^2+n+1}$ < n + 1 Hence $\left[\sqrt{n^2+n+1}\right]$ = n

After rationalization $\lim_{h\to\infty}\left[\frac{h+1}{\sqrt{n^2+n+1}+n}\right] = \frac{1}{2}$

10.(B) $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(h)+|x|h+xh^2}{h}$ where $x = h$ and $y = x$

$\because$ $f(0) = 0$; hence $f'(x) = \lim_{h\to 0}\left(\frac{f(h)-f(0)}{h} + |x| + xh\right)$

$f'(x) = f'(0) + |x| = |x|$

11.(A) $\lim_{x\to\infty}\frac{1^2 n + 2^2(n-1) + 3^2(n-2) + ... + n^2(n-(n-1))}{\sum r^3}$

N^r. = $n(1^2+2^2+.....n^2) - (1.2^2 + 2.3^2 + 3.4^2 + ... + (n-1)n^2)$

$= n\sum r^2 - \sum_{r=2}^{n}(r-1).r^2 = n\sum r^2 - \sum_{r=1}^{n}(r^3 - r^2) = n\sum r^2 - \left[\sum r^3 - \sum r^2\right]$

$= (n+1)\sum r^2 - \sum r^3 \quad \therefore \lim_{x\to\infty}\frac{(n+1)\sum r^2 - \sum r^3}{\sum r^3}$

$\lim_{x\to\infty}\frac{(n+1)n(n+1)(2n+1)}{6n(n+1)n(n+1)} - 1; \quad \frac{4}{3} - 1 = \frac{1}{3} \Rightarrow A$

12.(A) $\lim_{x\to\infty} \dfrac{\cot^{-1}\left(\dfrac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\dfrac{a^x}{\log_a x}\right)}$; $\underset{x\to\infty}{as}\left(\dfrac{\log_a x}{x^a}\right)\to 0$ and $\left(\dfrac{a^x}{\log_a x}\right)\to\infty$ (using L'Hospital rule) $\therefore \lim_{x\to\infty} = \dfrac{\pi/2}{\pi/2} = 1$

13.(B) L = $\underset{n\to\infty}{Lim}\left(\dfrac{p^{1/n}+q^{1/n}}{2}\right)^n = e^{\underset{n\to\infty}{Lim}\, n\left(\frac{p^{1/n}+q^{1/n}}{2}-1\right)} = e^{\underset{n\to\infty}{Lim}\, n\left(\frac{(p^{1/n}-1)+(q^{1/n}-1)}{2}\right)} = e^{\underset{n\to\infty}{Lim}\,\frac{1}{2}\left(\frac{(p^{1/n}-1)}{1/n}+\frac{(q^{1/n}-1)}{1/n}\right)}$

$= e^{\left(\frac{\ln p+\ln q}{2}\right)} = e^{\left(\ln\sqrt{pq}\right)} = \sqrt{pq}$

14.(D) $a = [(x+1)^2+2]_{\min} \Rightarrow a = 2$ $\qquad b = \underset{x\to 0}{Lim}\dfrac{\sin 2x}{\dfrac{2(e^{2x}-1).2x}{2x}} = \dfrac{1}{2}$

Now $\displaystyle\sum_{r=0}^{n} a^r b^{n-r} = \sum 2^r\left(\frac{1}{2}\right)^{n-r} = \frac{1}{2^n}\sum_{r=0}^{n}2^{2r} = \frac{1}{2^n}\sum_{r=0}^{n}4^r = \frac{1}{2^n}$ $[1 + 4 + 4^2 + ... + 4^n]$ $= \dfrac{1}{2^n}\left[\dfrac{4^{n+1}-1}{3}\right] = \dfrac{4^{n+1}-1}{3.2^n}$

15.(A) $\underset{x\to\infty}{Lim}\sqrt{x+1}-\sqrt{x} = 0 \qquad \Rightarrow \cot^{-1}(0) = \pi/2$

$\underset{x\to\infty}{Lim}\left(\dfrac{2x+1}{x-1}\right)^x \to \infty \qquad \Rightarrow sec^{-1}(\infty) = \pi/2 \qquad \therefore \qquad \lim_{x\to\infty}\dfrac{\pi/2}{\pi/2} = 1$

16.(B) $\underset{x\to 0}{Lim}\dfrac{e^{2x}-(1+4x)^{1/2}}{\ln(1-x^2)}$; $\quad \underset{x\to 0}{Lim}\dfrac{e^{2x}-(1+4x)^{1/2}}{\dfrac{\ln(1-x^2)}{-x^2}\cdot(-x^2)}$; $\quad \underset{x\to 0}{Lim}\dfrac{(1+4x)^{1/2}-e^{2x}}{x^2}$

$\underset{x\to 0}{Lim}\dfrac{\left(\cancel{1}+\frac{1}{2}4x+\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\frac{1}{2!}16x^2+......\right)-\left(\cancel{1}+\frac{2x}{1!}+\frac{4x^2}{2!}+.......\right)}{x^2} = -2-2 = -4$

17.(C) $x \to 0$, $\qquad x^3 - x^2 = x^2(x-1) \to 0^-$

$x \to 0$, $\qquad 2x^4 - x^5 = x^4(2-x) \to 0^+$

$2\,(3) = \lambda\,(2) \Rightarrow \lambda = 3$

18.(B) $\underset{x\to 0^+}{Lim}\dfrac{f(-x)\,x^2}{\left(\dfrac{1-\cos x}{[f(x)]}\right)-\left[\dfrac{1-\cos x}{[f(x)]}\right]} = \dfrac{3x^2}{\dfrac{1-\cos x}{2}-0} = 6\times 2 = 12$

19.(B) For $x\to 0^-$ $\left[f\left(\dfrac{x^3-\sin^3 x}{x^4}\right)\right] = \left[f\left(\dfrac{x-\sin x}{x^3}\right)\left(\dfrac{x^2+\sin^2 x+x\sin x}{x}\right)\right]$

$= \left[f(0^-)\right] = \left[3^+\right] = 3$ for $x\in(-\infty, 0)$ decreasing function

For $x\to 0^-$ $\qquad f\left[\dfrac{\sin x^3}{x}\right] = f\left[\dfrac{\sin x^3}{x^3}x^2\right] = f[(0^+)] = f(0) = 4$

$\lim_{x\to 0^-}(9-4) = 5$

20.(AB) For L.H.L. at $x = o^-$

$\{-h\} = 1-h$

L.H.L. $= \lim\limits_{x\to 0^-} \dfrac{\cos^{-1}(1-\{-h\}^2).\sin^{-1}(1-\{-h\})}{\{-h\}(1-\{-h\}^2)} = \lim\limits_{h\to 0} \dfrac{\cos^{-1}(1-(1-h)^2)\cdot\sin^{-1}(1-(1-h))}{(1-h)(1-(1-h)^2)}$

$= \lim\limits_{h\to 0} \dfrac{\cos^{-1}(h(2-h))\cdot\sin^{-1}h}{(1-h)\cdot(h(2-h))} = \lim\limits_{h\to 0} \dfrac{\cos^{-1}(h(2-h))}{(1-h)(2-h)}\cdot\left(\dfrac{\sin^{-1}h}{h}\right) = \dfrac{\pi}{4}$

R.H.L. $= \lim\limits_{x\to 0^+} \dfrac{\cos^{-1}(1-h^2)\cdot\sin^{-1}(1-h)}{h(1-h^2)}$

$= \lim\limits_{h\to 0} \dfrac{\cos^{-1}(1-h^2)}{h(1-h^2)}\cdot\sin^{-1}(1)$ .... $(\text{let } \cos^{-1}(1-\text{h}^2) = \text{t})$

$= \dfrac{\pi}{2}\lim\limits_{t\to 0} \dfrac{t}{\sqrt{2}\sin t/2} = \dfrac{\pi}{2}\lim\limits_{t\to 0}\left(\dfrac{t/2}{\sin t/2}\right) = \dfrac{\pi}{2}$

Let $\cos^{-1}(1-h^2) = t$

$h^2 = 1-\cos t \quad ; \quad h^2 = 2\sin^2\dfrac{t}{2}$

21.(AC) Let $2^{32} = n$

$k = \lim\limits_{x\to 1} \dfrac{x^n - nx + n - 1}{(x-1)^2} = \lim\limits_{x\to 1} \dfrac{x^n - 1 - n(x-1)}{(x-1)^2}$

$= \lim\limits_{x\to 1} \dfrac{(x^{n-1} + x^{n-2} + \text{..........} + 1) - n}{x-1} = (n-1) + (n-2) + \text{.........}1 = \dfrac{2^{32}(2^{32}-1)}{2} = 2^{63} - 2^{31}$

22.(AC) (A) is true

$\lim\limits_{h\to 0} \dfrac{f(c+h) - f(c) + f(c) - f(c-h)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(c+h) - f(c)}{h} + \lim\limits_{h\to 0} \dfrac{f(c-h) - f(c)}{-h} = f'(c) + f'(c) = 2f'(c)$ (f is differentiable)

(B) is false. Existence of limit is no guarantee for differentiability

(C) is true

(D) is false

23.(AB) Use expansion of 2^x.

24.(AB) $g(x) = \begin{cases} \dfrac{f(x)}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$h(x) = g'(x) = \begin{cases} \dfrac{xf'(x) - f(x)}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

For $h(x) = g'(x)$

L.H.S. $= \lim\limits_{x\to 0} \dfrac{-h.f'(-h) - f'(-h)}{h^2} = \lim\limits_{x\to 0} \dfrac{-h.f'(-h) - f'(-h) + f'(-h)}{2h} = \dfrac{f''(0)}{2} = 0$

R.H.L $= \lim\limits_{h\to 0} \dfrac{h.f'(h) - f(h)}{h^2} = \lim\limits_{h\to 0} \dfrac{k.f''(h) + f'(h) - f'(h)}{2h} = \dfrac{f''(0)}{2} = 0$

25.(AC) We get $(a, b) = (-1,1)$, $\left(1,1+\sqrt{2}\right),\left(1,1-\sqrt{2}\right)$

26.(AB) Using first principle method $f'(0) = \lim_{x\to 0} \dfrac{x + 2x^2 \sin\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0}\left(1 + 2x\sin\dfrac{1}{x}\right) = 1$

$$f'(x) = 1 - 2\cos\frac{1}{x} + 4x\sin\frac{1}{x}$$

Function f is differentiable for all x, but derivative of $f(x)$ is not continuous at $x = 0$

$$f'\left(\frac{1}{2k\pi}\right) = -1 < 0$$

There is not interval around 0 where f is increasing. So it should not make sense to say that f is increasing at $x = 0$.

27.(AC) $f(x) = \begin{cases} x^3 & if\ \ x \le 1 \\ x^2 & if\ \ x > 1 \end{cases}$

$g(x) = [x]^2 + \sqrt{\{x\}^2}$

$f(x)$ is continuous every where

Continuity of $g(x)$ need to be checked only at integral points $(n \in I)$

$g(n) = n^2$

$\lim_{h\to 0} g(n^+) = \lim_{h\to 0}\left(n^2 + \sqrt{h^2}\right) = n^2$

$\lim_{h\to 0} g(n^-) = \lim_{h\to 0}\left((n-1)^2 + \sqrt{\{-h\}^2}\right) = \lim_{h\to 0}\left((n-1)^2 + \sqrt{(1-h)^2}\right) = (n-1)^2 + 1$

For continuity

L.H.L. = R.H.L = $g(n)$

$n^2 = (n-1)^2 + 1$; $n^2 = n^2 + 1 - 2n + 1$; $n = 1$

So $g(x)$ is continuous at only one integral value $x = 1$

28.(ABD) $f(x) = \underset{n\to\infty}{Lim} \dfrac{2x^{2n}\sin\frac{1}{x} + x}{1 + x^{2n}} = \begin{cases} 2\sin\dfrac{1}{x}, & x > 1\ or\ x < -1 \\ \dfrac{2(\sin 1) - 1}{2}, & x = 1 \\ \dfrac{-2(\sin 1) - 1}{2}, & x = -1 \\ x, & -1 \le x < 1 \end{cases}$

$\underset{x\to\infty}{Lim}\ x f(x) = \underset{x\to\infty}{Lim}\ 2x\sin\dfrac{1}{x} = 2$

$\underset{x\to 1^+}{Lim}\ 2\sin\dfrac{1}{1+h} = 2\sin 1,\ \lim_{x\to 1^-} x = 1$

$\underset{x\to 1}{Lim}\ f(x)$ does not exist.

$\underset{x\to 0}{Lim}\ f(x)$ exist

$\underset{x\to -\infty}{Lim}\ f(x) = \underset{x\to -\infty}{Lim}\ 2\sin\dfrac{1}{x} = 0$

29.(AD) $\lim\limits_{\theta\to-1}\dfrac{\theta^2+\theta-2}{\theta+3}=\lim\limits_{\theta\to-1}\dfrac{f(\theta)}{\theta^2}\le\lim\limits_{\theta\to-1}\dfrac{\theta^2 2\theta-1}{\theta+3}$

$-1\le\lim\limits_{\theta\to-1}\dfrac{f(\theta)}{\theta^2}\le-1$

Using squeeze theorem or sand with theorem $\lim\limits_{\theta\to-1}\dfrac{f(\theta)}{\theta^2}=-1$

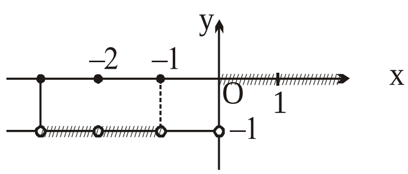
$\lim\limits_{\theta\to-1}f(\theta)=-1$

30.(AC) $\lim\limits_{x\to0^+}\dfrac{\tan^2\{x\}}{x^2-[x]^2}\quad x>0\;=\lim\limits_{x\to0}\dfrac{\tan^2 h}{h^2}=1$

$\lim\limits_{x\to0^-}\sqrt{\{x\}\cot\{x\}}\;=\lim\limits_{x\to\infty}\sqrt{\dfrac{1-h}{\tan(1-h)}}=\sqrt{\cot 1}$

31.(ABCD) $[|x|]-|[x]|=\begin{cases}0 & x=-1\\ -1 & -1<x<0\\ 0 & 0\le x\le 1\\ 0 & 1<x\le 2\end{cases}\Rightarrow$ range is $\{0,-1\}$

The graph is



32.(ABD) $f(x)=\dfrac{|x|}{\sqrt{1+\sqrt{1-x^2}}}$

$f'(0^+)=\lim\limits_{h\to0}\dfrac{\dfrac{h}{\sqrt{1+\sqrt{1-h^2}}}-0}{h}=\dfrac{1}{\sqrt{2}}$

$f'(0^-)=\lim\limits_{h\to0}\dfrac{\dfrac{h}{\sqrt{1+\sqrt{1-x^2}}}-0}{-h}=-\dfrac{1}{\sqrt{2}}$

33.(AC) If $f(b)=g(b)$ then

L.H.L. = $h(b^-)=f(b^-)=f(b)$

R.H.L. $=h(b^+)=g(b^+)=g(b)$

For $h(x)$ to have nonremovable discontinuity

L.H.L. = R.H.L.

$f(b)=g(b)$ (given)

$h(x)$ has a removable discontinuity at $x=b$

$g(b^-)$ and $f(b^+)$ have no existence.

34.(ABCD) **(A)** f (x) = x – cos x ; f (0) < 0, f (π/2) > 0

(B) f (x) = x + sin x – 1

f (0) = – 1 < 0; f (π/6) = $\frac{\pi}{6}+\frac{1}{2}-1>0$

(C) f (x) = a(x – 3) + b(x – 1) in [1, 3]

f (1) = – 2a < 0; f (3) = 2b > 0 ⇒ f (x) = 0 in (1, 3)

(D) h (x) = f (x) – g (x)

h (a) = f (a) – g (a) > 0

h (b) = f (b) – g (b) < 0

Hence using IVT all the four have at least one root in indicated interval.

35.(AC) $f'(0^+) = \lim\limits_{h\to 0}\frac{h\ln(\cos h)}{h\left(\frac{\ln(1+h^2)}{h^2}\right)\times h^2} = \lim\limits_{h\to 0}\frac{\ln(\cosh)}{h^2} = \lim\limits_{h\to 0}\frac{-\sinh}{(\cosh)\times 2h} = -\frac{1}{2}$

$f'(0^-) = \lim\limits_{h\to 0}\frac{-h\ln(\cosh)}{-h\left(\frac{\ln(1+h^2)}{h^2}\right)\times h^2} = -\frac{1}{2}$

36.(BC) (A) $f(3^+) = 3-5 = -2,\ f(3^-) = 2-4 = -2$ **(B)** $f(1^+) = 1-1 = 0,\ f(1^-) = -1$

(C) $f(0^+) = -1,\ f(0^-) = 1$ **(D)** $f(0^+) = \tan 1,\ f(0^-) = \tan 1$

37.(ABC) $\underset{x\to 0}{Lim}\frac{(1-\sin x)\sin x}{\cos x}\cdot\frac{1+\sin x}{1+\sin x}$

$\underset{x\to\pi/2}{Lim}\frac{\sin x\cos x}{1+\sin x} = 0$

38.(AC) We have, $f(x) = |x-1|([x]-[-x])$

$\therefore\quad f(x) = \begin{cases}(x-1)(1+2), \text{when } 1<x<2\\ 0, \quad when\ x=1\\ (1-x), \quad when\ 0<x<1\end{cases}$

$\Rightarrow\quad f'(1^+) = \lim\limits_{h\to 0}\frac{3(h)-0}{h} = 3$ and $f'(1^-) = \lim\limits_{h\to 0}\frac{(1-(1-h))-0}{-h} = -1$

$\Rightarrow\quad f(x)$ is continuous at $x=1$ and non-derivable at $x=1$

39.(AD) Here, $x = 2t-|t-1|$ and $y = 2t^2+t|t|$

Now, when $t<0$,

$x = 2t-\{-(t-1)\} = 3t-1$ and $y = 2t^2-t^2 = t^2 \Rightarrow y = \frac{1}{9}(x+1)^2$

When $0\le t<1$,

$x = 2t-(-(t-1)) = 3t-1$ and $y = 2t^2+t^2 = 3t^2 \Rightarrow y = \frac{1}{3}(x+1)^2$

When $t\ge 1$ $x = 2t-(t-1) = t+1$ and $y = 2t^2+t^2 = 3t^2 \Rightarrow y = 3(x-1)^2$

Thus, $y = f(x) = \begin{cases} \frac{1}{9}(x+1)^2, & x < -1 \\ \frac{1}{3}(x+1)^2, & -1 \le x < 2 \\ 3(x-1)^2, & x \ge 2 \end{cases}$

$f(x)$ is continuous at $x = 2$.

Now, to check differentiability at $x = -1$ and 2.

Differentiability at $x = -1$,

LHD = $Lf'(-1) = \lim_{h\to 0} \frac{f(-1-h) - f(-1)}{-h} = \lim_{h\to 0} \frac{\frac{1}{9}(-1-h+1)^2 - 0}{-h} = 0$

RHD = $Rf'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\frac{1}{3}(-1+h+1)^2 - 0}{h} = 0$

Hence, $f(x)$ is differentiable at $x = -1$.

Differentiability at x = 2.

LHD $= Lf'(2) = \lim_{h\to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h\to 0} \frac{\frac{1}{3}(2-h+1)^2 - 3}{-h} = 2$

RHD $= Rf'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{3(2+h-1)^2 - 3}{h} = 6$

Hence, $f(x)$ is not differentiable at $x = 2$

40.(ABD) Graph is symmetrical about (4, 0)

$\Rightarrow \quad f(4+x) = -f(4-x) \quad \Rightarrow \quad f(x) = -f(8-x)$

Since *f* is invertible, replace *x* by $f^{-1}(x)$

$x = -f(8 - f^{-1}(x))$ or $f^{-1}(-x) + f^{-1}(x) = 8$

$\Rightarrow \quad f^{-1}(2010) + f^{-1}(-2010) = 8 \quad \Rightarrow$ Option (a) is true.

and $\int_{-2010}^{2018} f(x)dx = \int_{-2010}^{2018} f(8-x)dx = -\int_{2010}^{2018} f(x)dx = 0$

$\Rightarrow$ Option (b) is true.

Also, $D = (f'(10))^2 + 4f'(10) > 0$

As, $f'(-100) > 0 \Rightarrow f'(10) \ge 0$

$f'(-100) > 0 \Rightarrow f'(10) \ge 0 \qquad cx^2 - f'(10)x - f'(10) = 0$ has real roots.

$\Rightarrow$ Option (c) is false.

As, $f'(4+x) = f'(4-x)$

$f'(x)$ is symmetric about $x = 4 \quad \Rightarrow \quad f'(10) = f'(-2) = 20 \quad \Rightarrow$ Option (d) is true.

41.(B, C) Here, $f'(x)=\frac{1}{x}+\sqrt{1+\sin x}, x>0$ but $f'(x)$ is not differentiable in $(0,\infty)$ as $\sin x$ may be -1 and then

$f''(x)=-\frac{1}{x^2}+\frac{1}{2}\frac{\cos x}{\sqrt{1+\sin x}}$ will not exist.

$\Rightarrow$ $f'(x)$ is continuous for all $x\in(0,\infty)$ but $f'(x)$ is not differentiable on $(0,\infty)$.

$\therefore$ Option (b) is true.

Also, $f'(x)\le 3,$ if $x>1$ and $f(x)>3,$ if $x>e^3$

$\therefore$ Let $\alpha=e^3$ $\Rightarrow$ Option (c) is true.

(d) is not possible, as $f(x)\to\infty$ when $x\to\infty$.

42.(ABC) RHL $=\lim\limits_{x\to 0^+}\left(3-\left[\cot^{-1}\left(\frac{2x^3-3}{x^2}\right)\right]\right)=3-\left[\cot^{-1}(-\infty)\right]=3-3=0$

LHL $=\lim\limits_{x\to 0}\{x^2\}\cos\left(e^{-\frac{1}{x}}\right)=\lim\limits_{h\to 0}h^2\cos\left(e^{\frac{1}{h}}\right)=0$ and $f(0)=0$

Hence, $f(x)$ is continuous at $x=0$.

43.(AC) $\lim\limits_{x\to -1^+}f(x)=\lim\limits_{h\to 0}\left(b\left([-1+h]^2+[-1+h]\right)+1\right)=\lim\limits_{h\to 0}\left(b\left((-1)^2-1\right)+1\right)=1$

$\Rightarrow$ $b\in R$

$\lim\limits_{x\to -1^-}f(x)=\lim\limits_{x\to -1^-}\sin(\pi(x+a))=\lim\limits_{h\to 0}\sin\left(\pi((-1-h)+a)\right)$

$\therefore$ $\sin\pi a=-1$

$\pi a=2n\pi+\frac{3\pi}{2}\Rightarrow a=2n+\frac{3}{2}$

44.(BCD)
$$f(x)=\begin{cases}\dfrac{\sin\frac{\pi}{4}}{1}=\dfrac{1}{\sqrt{2}}, & 1\le x<2\\[2ex] \dfrac{\sin\frac{\pi}{2}}{1}=\dfrac{1}{2}, & 2\le x<3\end{cases}$$

Hence, $f(x)$ is continuous at $f(x)\,\frac{3}{2},$ differentiable at $\frac{4}{3}$ and discontinuous at 2.

45.(AD)
$$H(x)=\begin{cases}\cos x, & 0\le x<\frac{\pi}{2}\\[1ex] \frac{\pi}{2}-x, & \frac{\pi}{2}<x\le 3\end{cases}$$

$H'\left(\frac{\pi-}{2}\right)=-\sin x=-1$

$H'\left(\frac{\pi^+}{2}\right)=-1$

Hence, $H(x)$ is continuous and derivable in $[0,3]$ and has maximum value 1 in $[0,3]$.

46.(AC) We have, $f(x)=\begin{cases}\dfrac{\tan^2\{x\}}{x^2-[x]^2} & , \quad x>0\\ 1 & , \quad x=0\\ \sqrt{\{x\}\cot\{x\}} & , \quad x<0\end{cases}$

$$\text{RHL} = \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{\tan^2\{h\}}{h^2-[h]^2}=\lim_{h\to 0}\frac{\tan^2 h}{h^2}=1$$

$$\text{LHL} \quad =\lim_{h\to 0} f(x)=\lim_{h\to 0}\sqrt{\{-h\}\cot\{-h\}}=\lim_{h\to 0}\sqrt{(1-h)\cot(1-h)}=\sqrt{\cot 1}$$

$$\therefore \qquad \cot^{-1}\left(\lim_{x\to 0^-} f(x)\right)^2=\cot^{-1}\left(\sqrt{\cot 1}\right)^2=1$$

47.(ACD) Given, $f(x)=\lim\limits_{n\to\infty}\sum\limits_{r=0}^{n}\dfrac{x}{(rx+1)\{(r+1)x+1\}}=\lim\limits_{n\to\infty}\sum\limits_{r=0}^{n}\dfrac{[(r+1)x+1]-(rx+1)}{(rx+1)[(r+1)x+1]}$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\left(\frac{1}{rx+1}-\frac{1}{(r+1)x+1}\right)=\lim_{n\to\infty}\left[\begin{array}{l}\left(\frac{1}{1}-\frac{1}{x+1}\right)+\left(\frac{1}{x+1}-\frac{1}{2x+1}\right)\\ +\dots\dots\dots+\left(\frac{1}{nx+1}-\frac{1}{(n+1)x+1}\right)\end{array}\right]=\lim_{n\to\infty}\left[1-\frac{1}{(n+1)x+1}\right]=\lim_{n\to\infty}\frac{(n+1)x}{(n+1)x+1}$$

48.(AB) RHL $=\lim\limits_{x\to 0^+}\dfrac{\cot^{-1}(1/x)}{x}=\lim\limits_{x\to 0^+}\dfrac{\tan^{-1}}{x}=1$

LHL $\quad =\lim\limits_{x\to 0^-}\dfrac{\cot^{-1}(1/x)}{x}=\lim\limits_{x\to 0^-}\dfrac{\pi+\tan^{-1}x}{x}=-\infty \ \therefore$ RHL exists and LHL does not exists.

49.(BCD) Put $x=\dfrac{\pi}{2}-h$

$$l=\lim_{h\to 0}\frac{a^{\tanh}-a^{\sinh}}{\tanh-\sin\mathrm{h}}=\lim_{h\to\infty}\frac{a^{\sinh}\left(a^{\tanh-\sinh}-1\right)}{\tanh-\sinh}=\log_e a$$

and $m=\lim\limits_{x\to-\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2-ax}\right)=\lim\limits_{x\to-\infty}\dfrac{\left(x^2+ax\right)-\left(x^2-ax\right)}{\sqrt{x^2+ax+\sqrt{x^2-ax}}}=\lim\limits_{x\to-\infty}\dfrac{2ax}{|x|\left(\sqrt{1+\frac{a}{x}}+\sqrt{1-\frac{a}{x}}\right)}=-a$

50.(BCD) $f(x)=\lim\limits_{x\to\infty}\left[1+\dfrac{ax+1}{bx+2}-1\right]^x=\lim\limits_{x\to\infty}\left[1+\dfrac{(a-b)x-1}{bx+2}\right]^x=e^{\lim\limits_{x\to\infty}\left[\frac{(a-b)x-1}{bx+2}\right]^x}$

When $a=b, f(x)=e^{\lim\limits_{x\to\infty}\frac{x}{bx+2}}=e^{\frac{1}{b}}$ or $e^{\frac{1}{a}}$.

When $a>b, f(x)=e^{\infty}=\infty,$ does not exist.

When $a<b, f(x)=e^{-\infty}=0.$

51.(BC) (A) $\lim\limits_{x\to 3}[[x]]-[2x-1]$

RHL $=\lim\limits_{h\to 0}[[3+h]]-[6+2h-1]=\lim\limits_{h\to 0}3-5=-2$

LHL $\lim\limits_{h\to 0}[[3-h]]-[6-2h-1]=\lim\limits_{h\to 0}2-4=-2$ $\quad\therefore$ limit exists

(B) $\lim\limits_{x\to 1}[x]-x$

RHL $=\lim\limits_{h\to 0}[1+h]-(1+h)=\lim\limits_{h\to 0}(1-1-h)=0$

LHL $=\lim\limits_{h\to 0}[1-h]-(1+h)=\lim\limits_{h\to 0}0-1+h=-1$ $\quad\therefore$ Limit does not exist.

(C) $\lim\limits_{x\to 0}\{x\}^2-\{-x\}^2$

RHL $=\lim\limits_{x\to 0}\{h\}^2-\{-h\}^2=\lim\limits_{h\to 0}h^2-1(1-h)^2=-1$

LHL $=\lim\limits_{h\to 0}\{-h\}^2-\{h\}^2=\lim\limits_{h\to 0}(1-h)^2-h^2=1$ $\quad\therefore$ Limit does not exist.

(D) $\lim\limits_{x\to 0}\dfrac{\tan(\operatorname{sgn} x)}{\operatorname{sgn} x}$; RHL $=\lim\limits_{h\to 0}\dfrac{\tan(1)}{-1}=\tan 1$

LHL $=\lim\limits_{h\to 0}\dfrac{\tan(-1)}{-1}=\tan 1$ $\quad\therefore$ Limit exists.

52.(BC) We have $f(1)=f(-1)=1$

Let $x>1$ then $0<\dfrac{1}{x^2}<1 \Rightarrow \left[\dfrac{1}{x^2}\right]=0\Rightarrow f(x)=0\,\forall x>1$

Also, if $x<-1$, then $x^2>1\Rightarrow 0<\dfrac{1}{x^2}<1 \Rightarrow f(x)=0\,\forall x<-1$

Hence $f(1)=1$, $f(-1)=1$ and $f(x)=0$ if $|x|>1$ $\quad\therefore$ $f(x)$ cannot be continuous at $x=1$ and $x=-1$

Again Let $\dfrac{1}{2}<x^2<1\Rightarrow 1<\dfrac{1}{x^2}<2 \Rightarrow \left[\dfrac{1}{x^2}\right]=1\Rightarrow \dfrac{1}{2}<x^2\left[\dfrac{1}{x^2}\right]<1 \quad\therefore\quad \left[x^2\left[\dfrac{1}{x^2}\right]\right]=0$

$\Rightarrow \quad f(x)=0$ if $x\in\left(-1,\dfrac{1}{\sqrt{2}}\right)\cup\left(\dfrac{1}{\sqrt{2}},1\right)$

Next, Let $\dfrac{1}{3}<x^2<\dfrac{1}{2}\Rightarrow 2<\dfrac{1}{x^2}<3 \Rightarrow \left[\dfrac{1}{x^2}\right]=2\Rightarrow \dfrac{2}{3}<x^2\left(\dfrac{1}{x^2}\right)<1 \quad\therefore\quad \left[x^2\left[\dfrac{1}{x^2}\right]\right]=0\Rightarrow f(x)=0$ if

$x\in\left(-\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{3}}\right)\cup\left(\dfrac{1}{\sqrt{3}},\dfrac{1}{2}\right)$

At $x=\pm\dfrac{1}{\sqrt{2}}, x^2=\dfrac{1}{2}.\dfrac{1}{x^2}=2\Rightarrow f(x)=1$

Similarly, at $x=\dfrac{1}{\sqrt{3}},\dfrac{1}{2},\dfrac{1}{\sqrt{5}}$...............

$\therefore \quad f(x)$ is discontinuous at infinite number of points

Given by $x\in\left\{\pm\dfrac{1}{\sqrt{n}},n\in N\right\}$.

53.(AC) $f(x)=[\tan x]+\sqrt{\tan x-[\tan x]}=[t]+\sqrt{t-[t]}$, where $t=\tan x$

Clearly $0\le t<\infty$ as $0\le x<\frac{\pi}{2}$. Possible points of discontinuities may be, at which $t\in N$.

Let $t=k\in N$

L.H.L. (at $t=k$) $=\lim\limits_{t\to k^-}[t]+\sqrt{t-[t]}$ $=\lim\limits_{h\to 0}[k-h]+\sqrt{k-h-[k-h]}$ $=\lim\limits_{h\to 0}\left\{k-1+\sqrt{k-h-k+1}\right\}=k$

R.H.L. (at $t=k$) $=\lim\limits_{t\to k^+}[t]+\sqrt{t-[t]}$ $=\lim\limits_{h\to 0}k+\sqrt{k+h-[k+h]}$ $=\lim\limits_{h\to 0}k+\sqrt{k+h-k}=k$

$\therefore$ The function is continuous if $t=k\in N$ Thus function $f(x)$ is continuous for all $x\in\left[0,\frac{\pi}{2}\right)$.

54.(BC) If $0\le\sin^2 x<1$ then $f(x)=\sec^{-1}(-1)=0$

If $\sin^2 x=1$ then $f(x)=\sec^{-1}(2)=\frac{\pi}{3}$

Thus, $f(x)$ is not continuous if $\sin x=\pm 1$ i.e., $x=$ Odd multiple of $\frac{\pi}{2}$

55.(BD) For $x>2, \int_0^x\{1+|1-t|\}dt$ $=\int_0^1(2-t)dt+\int_1^x dt=1+\frac{x^2}{2}$

Thus, $f(x)=\begin{cases}1+\frac{x^2}{2}, & x>2\\ 5x-7, & x\le 2\end{cases}$

$\lim\limits_{x\to 2+}f(x)=1+\frac{4}{2}=3=f(2)=\lim\limits_{x\to 2-}f(x)$

$f'(2+)=\lim\limits_{x\to 0+}\frac{1+(1/2)(2+h)^2-3}{h}$ $=\lim\limits_{h\to 0}\frac{(1/2)h^2+2h}{h}=2$

$f'(2-)=\lim\limits_{x\to 0-}\frac{5(2+h)-7-3}{h}=5$ Hence f is continuous but not differentiable at $x=2$

56.(ABC) Given $F(x)=f(x).g(x)$.......(1)

Differentiating both sides w.r.t, x we get

$F'(x)=f'(x).g(x)+g'(x).f(x)$ $\Rightarrow F'(x)=f'(x)g'(x)\left[\frac{f(x)}{f'(x)}+\frac{g(x)}{g'(x)}\right]$ $\Rightarrow F'=c\left[\frac{f}{f'}+\frac{g}{g'}\right]\Rightarrow$ (a) is correct

Again, differentiating both sides w.r.t., x we get

$F''(x)=f''(x).g(x)+g''(x).f(x)+2f'(x).g'(x)$ $\Rightarrow F''(x)=f''(x).g(x)+g''(x).f(x)+2c$......(2)

Dividing both sides by $F(x)=f(x).g(x)$

$(\because f'(x).g'(x)=c)$ then $\frac{F''(x)}{F(x)}=\frac{f''(x)}{f(x)}+\frac{g''(x)}{g(x)}+\frac{2c}{f(x)g(x)}$ or $\frac{F''}{F}=\frac{f''}{f}+\frac{g''}{g}+\frac{2c}{fg}\Rightarrow$ (b) is correct

Again given $f(x)g'(x)=c$

Differentiating both sides w.r.t., x we get $f'(x)g''(x)+g'(x)f''(x)=0$

From (2), $F''(x) = f''(x).g(x) + g''(x).f(x) + 2c$

Differentiating both sides w.r.t., x we get

$F'''(x) = f''(x).g'(x) + f'''(x).g(x) + g''.f'(x) + f(x).g'''(x) + 0 = f'''(x).g(x) + g'''(x).f(x) + 0$ [from (3)]

Now dividing both sides by $F(x) = f(x)g(x)$

Then $\dfrac{F'''(x)}{F(x)} = \dfrac{f'''(x)}{f(x)} + \dfrac{g'''(x)}{g(x)}$ or $\dfrac{F'''}{F} = \dfrac{f'''}{f} + \dfrac{g'''}{g}$

57.(ABCD) $1+x$ is never zero, so $1+f(x)$ is never zero. It is 1 for $x=0$, so it is always positive.

Hence $f''(x)$ is always positive. $f'(x) > 0$, for all $x > 0$ and hence f is strictly increasing.

So, in particular, $1+f(x) \ge 2$ for all x.

We have $f''(x) \le \dfrac{(1+x)}{2}$.

Integrating, $f'(x) \le f'(0) + \dfrac{x}{2} + \dfrac{x^2}{4} = \dfrac{x}{2} + \dfrac{x^2}{4}$.

Integrating, again, $f(x) \le f(0) + \dfrac{x^2}{4} + \dfrac{x^3}{12}$. Hence $f(1) \le 1 + \dfrac{1}{4} + \dfrac{1}{12} = \dfrac{4}{3}$.

58. **[A-p, q, r, s]** **[B-p, q, r, s]** **[C-p, q, r, s]** **[D-r]**

(A) $f(x) = \begin{cases} \dfrac{2}{x^3} e^{-1/x^2}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$f'(0) = \lim\limits_{h \to 0} \dfrac{e^{-\frac{1}{h^2}} - 0}{h} = \lim\limits_{h \to 0} \dfrac{1}{h.e^{1/h^2}} = \lim\limits_{t \to \infty} t.e^{-t^2}$ let $\dfrac{1}{h} = t$

Applying L.H. Rule $= \lim\limits_{t \to \infty} \dfrac{1}{2t.e^{t^2}} = 0$

Similarly $= \lim\limits_{t \to -\infty} \dfrac{t}{e^{-t^2}} = 0$

$f''(0) = \lim\limits_{h \to 0} \dfrac{2}{h^4.e^{1/h^2}} = \lim\limits_{t \to \infty} \dfrac{2t^4}{e^{t^2}}$ let $\dfrac{1}{h} = t$

Applying L.H. Rule $= \lim\limits_{t \to \infty} \dfrac{8t^3}{2t.e^{t^2}} == \lim\limits_{t \to \infty} \dfrac{4t^2}{e^{t^2}} == \lim\limits_{t \to \infty} \dfrac{8t}{2te^{t^2}} = 0$

$f(x)$ is infinitely differentiable

$f^{(k)}(0) = 0$ for all k $\quad$ A $\to$ p, q, r, s

(B) Similar to (A) $\quad$ B $\to$ p, q, r, s

(C) $f_1(x) = g(x-e)$

$f_2(x) = g(\pi - x)$ where $g(x) = e^{-\frac{1}{x}}$ So $f(x) = f_1(x) f_2(x)$

As f_1, f_2 have both 1st and 2nd derivatives existing and continuous, the function f also will.

(D) Only 1st derivative exists and its not continuous.

59. **[A-s]** **[B-p]** **[C-p]** **[D-p]**

(A) Use sandwich theorem $x\left(\frac{1+2+3...+8}{x}-8\right)<\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+......+\left[\frac{8}{x}\right]\le x\left(\frac{1+2+3+...+8}{x}\right)$

Taking limits find that $\lim_{x\to0} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+....+\left[\frac{8}{x}\right]\right)=36$

(B) 1^∞ form

$k=e^{\lim_{x\to0} xe^{-1/x^2}\sin\frac{1}{x^4}.e^{1/x^2}}=e^{\lim_{x\to0} x\cdot\sin\frac{1}{x^4}}=e^0=1$ $\quad$ B $\to$ p

(C) $u=\left(2\sin\sqrt{x}+\sqrt{x}\sin 1/x\right)^x$

$\ell nu=x\ell n\left(2\sin\sqrt{x}+\sqrt{x}\sin\frac{1}{x}\right)=x\ell n\left(\frac{\left(2\sin\sqrt{x}+\sqrt{x}\sin\frac{1}{x}\right).\sqrt{x}}{\sqrt{x}}\right)$

$=x\ell n\sqrt{x}+x\ell n\left(\frac{2\sin\sqrt{x}}{\sqrt{x}}+\sin\frac{1}{x}\right)=x\ell n\sqrt{x}+xg(x)$

$\lim_{x\to0} xg(x)=0$ $\quad$ $g(x)$ is bounded

$\lim_{x\to0}\frac{x}{2}\ell nx=\lim_{x\to0}\frac{\ell nx}{2/x}$

Using L.H. rule $=\lim_{x\to0}\frac{1/x}{-2/x^2}=\lim_{x\to0}-\frac{x}{2}=0$ $=\lim_{x\to0}(\ell nu)=0$ $\quad$ $u=e^0=1$ $\quad$ $C\to p$

(D) $k=\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{4\left(\frac{\pi}{2}-x^2\right)}\right).\frac{\ln\sin x}{\ln(1+\pi^2-4\pi x+4x^2)}$

$=\lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{4(\pi-x)^2}\right)\cdot\lim_{x\to\frac{\pi}{2}}\frac{\ln\sin x}{\ln(1+\pi^2-4\pi x+4x^2)}=\lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{4\left(\frac{\pi}{2}-x\right)^2}\right)\cdot\lim_{x\to\frac{\pi}{2}}\frac{(1+\pi^2-4\pi x+4x^2)\cos x}{\sin x\cdot(8x-4\pi)}$

Let $x-\frac{\pi}{2}=t$

$\lim_{t\to0}\left(\frac{1-\cos t}{4t^2}\right)\cdot\frac{-\sin t}{-8t}\cdot\lim_{x\to\frac{\pi}{2}}\left(\frac{1+\pi^2-4\pi x+4x^2}{\sin x}\right)=\frac{1}{8}\lim_{t\to0}\frac{2\sin^2 t/2}{16\cdot t^2/4}\cdot1$; $K=\frac{1}{64}$; $8\sqrt{|k|}=1$ $\quad$ $D\to p$

60. **[A-s]** **[B-r]** **[C-q, r]** **[D-p]**

$F'(x)=kx(x^2-1)$

$F''(x)=k(3x^2-1)\to F''\left(\frac{1}{2}\right)=k\left(\frac{3}{4}-1\right)\Rightarrow k=4$

$F'(x)=4x(x^2-1)$ $\quad$ $F(x)=x^4-2x^2+\mu$

$f(0)=4\Rightarrow\mu=4$ $\quad$ $F(x)=x^4-2x^2+4=(x^2-1)^2+3$

No real roots $\quad$ A $\to s$; B $\to r$; C $\to q, r$; D $\to p$

61. **[A-p, q]** **[B-p, s]** **[C-q]** **[D-r, t]**

$$f(x)=x^2+ax+b$$

$$f'(x)|_{\left(\frac{5}{2},p\right)}=0 \Rightarrow 2x+a|_{\left(\frac{5}{2},p\right)}=0 \Rightarrow 2\times\frac{5}{2}+a=0$$

$$a=-5 \qquad \therefore \qquad f(x)=x^2-5x+b$$

Let its roots are $\alpha \& \beta,\ \alpha,\beta\in I$

$$f(0)>0 \qquad b>0$$

$$\alpha+\beta=5 \qquad \alpha\beta=b>0$$

Both $\alpha \& \beta$ are +ve (α,β) can be (1, 4) or (2, 3)

$\therefore \qquad b=4$ or 6

$$f(x)=x^2-5x+4 \text{ or } f(x)=x^2-5x+6$$

62.(1) $k=\lim\limits_{x\to 0}\dfrac{\sin(\pi\cos^2(\tan(\sin x)))}{\pi x^2}=\lim\limits_{x\to 0}\dfrac{\sin(\pi-\pi\cos^2(\tan(\sin x)))}{\pi x^2}=\lim\limits_{x\to 0}\dfrac{\sin(\pi\sin^2(\tan(\sin x)))}{\pi x^2}=1$

63.(1) $\dfrac{d^2x}{dy^2}=-\left(\dfrac{dx}{dy}\right)^3\dfrac{d^2y}{dx^2}$

$\Rightarrow \qquad \dfrac{d^2y}{dx^2}=\dfrac{-d^2x}{dy^2}\dfrac{1}{\left(\dfrac{dx}{dy}\right)^3} \qquad \therefore \qquad x\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2-\dfrac{dy}{dx}=0$

Becomes $-x\dfrac{d^2x}{dy^2}\dfrac{1}{\left(\dfrac{dx}{dy}\right)^3}+\dfrac{1}{\left(\dfrac{dx}{dy}\right)^2}-\dfrac{1}{\dfrac{dx}{dy}}=0$

$\Rightarrow \qquad x\dfrac{d^2x}{dy^2}+\left(\dfrac{dx}{dy}\right)^2=\dfrac{dx}{dy} \qquad \therefore \qquad \lambda=1$

64.(6) $g(x)=f(-x+f(f(x)))$

$g'(x)=f'(-x+f(f(x))\times(-1+f'(f(x))\times f'(x))$

$g'(0)=6$

65.(1). In the vicinity of $x = 0$, we have $x^2\sum\limits_{r=0}^{\left[\frac{1}{|x|}\right]}r \quad =x^2\left(1+2+3+...\left[\dfrac{1}{|x|}\right]\right)$

Use sandwich theorem $P=x^2\left(1+2+3+......+\left[\dfrac{1}{|x|}\right]\right)=\dfrac{x^2\left(1+\left[\dfrac{1}{|x|}\right]\right)}{2}\left[\dfrac{1}{|x|}\right]$

So $\qquad \dfrac{1}{2}(1-|x|)<P\le\dfrac{1}{2}(1+|x|)$

Then the limit is $\dfrac{1}{2}$

66.(8) $1+f(x+y)=f(x)+f(y)+f(x).f(y)+1$

$1+f(x+y)=(1+f(x))(1+f(y))$

Let $\quad 1+f(x)=g(x)$

$g(x+y)=g(x).g(y)$

From this functional equation $\quad g(x)=e^{\lambda x}$

$f(x)=e^{\lambda x}-1\ ,\quad f'(x)=\lambda.e^{\lambda x},\ f'(0)=\lambda=1\ ,\quad f(x)=e^{x}-1$

$$\left[\frac{e^4-1}{e^2-1}\right]=\left[e^2+1\right]=[7.29+1]=8$$

67.(2) $f'(1^+)=\lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h}=\lim\limits_{h\to 0}\dfrac{\frac{\pi}{4}+\frac{1+h-1}{2}-\frac{\pi}{4}}{h}=\dfrac{1}{2}$

68.(2) Let $x=K\sin\theta$

$\lambda=\lim\limits_{\theta\to 0}\dfrac{1}{K^2}\left(\dfrac{1-4K\cos\theta}{\sin^2\theta\cos\theta}\right)\qquad \lambda$ is finite $K=\dfrac{1}{4}\ \Rightarrow\ \lambda=\dfrac{2}{K}$

69.(4) $\lim\limits_{x\to 1}\dfrac{x^2\sin(f(x^2))\times 2x+2x\displaystyle\int_1^{x^2}\sin f(t)dt}{2(x-1)}$

Applying L – H rule again. We will get $2f'(1)=4$

70.(8) $\left(f^{-1}(x)\right)'_{at\ x=0}=\dfrac{1}{f'(0)}$

71.(5) We know, $\sin 3\theta=3\sin\theta-4\sin^3\theta=\sin\theta\left(1+2\cos 2\theta\right)\qquad\therefore\quad \sin 3\theta=\sin\theta\left(1+2\cos 2\theta\right)\qquad$ …(i)

Now, on putting $3\theta=x,\dfrac{x}{3},\dfrac{x}{3^2},......$ one by one in Eq. (i), we get $\Rightarrow\quad \sin x=\sin\left(\dfrac{x}{3}\right).\left(1+2\cos\dfrac{2x}{3}\right)$

$$\Rightarrow\quad \frac{\sin x}{x}=\lim_{n\to\infty}\frac{\sin x/3^n}{x/3^n}\prod_{k=1}^{n}\frac{\left(1+2\cos\left(\frac{2x}{3^k}\right)\right)}{3^n}\quad\Rightarrow\quad \frac{\sin x}{x}=\prod_{k=1}^{\infty}\left(\frac{1+2\cos\left(\frac{2x}{3^k}\right)}{3}\right)=f(x)$$

Thus, $\left|xf(x)\right|+\left|\left|x-2\right|-1\right|=\dfrac{|x||\sin x|}{|x|}+\left|\left|x-2\right|-1\right|$ is not

Differentiable at $\{\pi,2\pi,1,2,3\}\qquad\therefore$ Number of points are 5.

72.(3) Here, $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{f\left(\frac{2x+2h}{2}\right)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{f\left(\frac{2x(1+h/x)}{2}\right)-f\left(\frac{2x.1}{2}\right)}{h}$

$=\dfrac{f(2x)}{2x}.\lim\limits_{h\to 0}\dfrac{f(1+h/x)-f(1)}{h/x}=\dfrac{f(2x)}{2x}.f'(1)=\dfrac{f(2x)}{2x}.f(1)$, given $f(1)=f'(1)$

$=\dfrac{f\left(\frac{2x.1}{2}\right)}{x}$, using $f\left(\dfrac{xy}{2}\right)=\dfrac{f(x).f(y)}{2}\qquad\therefore\qquad f'(x)=\dfrac{f(x)}{x}\quad\Rightarrow\quad \dfrac{f(x)}{f'(x)}=x\Rightarrow\dfrac{f(3)}{f'(3)}=3$

73.(4) Given, $f(x) = f(y)f(x-y)$

On replacing x by $(x+y)$,

$f(x+y) = f(y).f(x) \quad \Rightarrow \quad f(x) = e^{kx} \Rightarrow f'(x) = ke^{kx}$

But, $f'(0) = \int_0^4 \{2x\}dx = 2 \quad \Rightarrow \quad f'(0) = k = 2 \Rightarrow f'(x) = 2e^{2x}$

$f'(-3) = 2e^{-6} \Rightarrow |\alpha+\beta| = 4$

74.(1) Here, $(f(\alpha))^2 + (f'(\alpha))^2 = 0 \Rightarrow f(\alpha) = f'(\alpha) = 0 \quad \therefore \quad \alpha$ is repeated root of $f(x)$

Now, $\lim_{x\to\alpha} \frac{f(x)}{f'(x)}\left[\frac{f'(x)}{f(x)}\right] = \lim_{x\to\alpha} \frac{f(x)}{f'(x)}\left(\frac{f'(x)}{f(x)} - \left\{\frac{f'(x)}{f(x)}\right\}\right) = \lim_{x\to\alpha}\left(1 - \frac{f(x)}{f'(x)}.\left\{\frac{f'(x)}{f(x)}\right\}\right)$

[as, $x = \alpha$ is the repeated root and $\left\{\frac{f(x)}{f'(x)}\right\}$ is bounded; $\therefore \lim_{x\to\alpha}\frac{f(x)}{f'(x)} = 0$] $= 1 - 0 = 1$

75.(9) Since, $f(10-x) = f(x) = f(4-x) \quad \Rightarrow \quad f(10-x) = f(4-x)$

Say, $4 - x = t \Rightarrow f(6+t) = f(t) \quad \Rightarrow \quad f(x)$ is periodic function with period 6.

So, for $x \in [0,25]$

$f(x) = 101$ at $x = 0, 6, 12, 18, 24$

Total number = 5

Since $f(2-x) = f(2+x) \Rightarrow f(x)$ is symmetric about $x = 2$ line

Due to symmetry in one period length.

$f(x) = 101$ has one solution at $x = 4$ other than 0 and 6.

Now, $f(x) = 101$ at $x = 4, 10, 16, 22$

Total numbers = 4 Hence, at least minimum possible number of values of $x = 9$.

76.(1.41) RHL $= \lim_{h\to0}\frac{\left(\frac{\pi}{2} - \sin^{-1}(1-h^2)\right)\sin^{-1}(1-h)}{\sqrt{2}(h-h^2)} = \lim_{h\to0}\frac{\cos^{-1}(1-h^2)\sin^{-1}(1-h)}{\sqrt{2}h(1-h^2)} = \lim_{h\to0}\frac{\sin^{-1}\sqrt{2h^2-h^4}.\sin^{-1}(1-h)}{\sqrt{2}h(1+h)(1-h)}$

$$= \lim_{h\to0}\frac{\sin^{-1}\left(h\sqrt{2-h^2}\right)\sin^{-1}(1-h)}{\sqrt{2}h(1+h)(1-h)} = \lim_{h\to0}\frac{1}{\sqrt{2}}.\frac{\sin^{-1}\left(h\sqrt{2-h^2}\right)}{h.\sqrt{2-h^2}}.\frac{\sqrt{2-h^2}}{1+h}.\frac{\sin^{-1}(1-h)}{1-h}$$

$$= \frac{1}{\sqrt{2}}.\sqrt{2}.\frac{\sin^{-1}(1)}{1} = \frac{\pi}{2} \Rightarrow K = \frac{\pi}{2} \qquad \text{....(1)}$$

LHL $= \lim_{h\to0}\frac{A\sin^{-1}(1-(1-h)).\cos^{-1}(1-(1-h))}{\sqrt{2(1-h).(1-(1-h))}}$

$$\lim_{h\to0}\frac{A.\sin^{-1}(h).\cos^{-1}(h)}{\sqrt{2}(1-h).h} = \frac{A.\pi/2}{\sqrt{2}} = \frac{A\pi}{2\sqrt{2}} \quad \Rightarrow \quad \frac{A\pi}{2\sqrt{2}} = \frac{\pi}{2} \Rightarrow A = \sqrt{2}$$

77.(0) A, B, C are in AP. $\quad \therefore \quad 2B = A + C$ and $A + B + C = 180° \quad \therefore \quad B = 60°$

$$\therefore \quad \cos B = \frac{1}{2} = \frac{a^2 + c^2 - b^2}{2ac} \quad \Rightarrow \quad a^2 + c^2 = b^2 + ac \quad \Rightarrow \quad (a - c)^2 = b^2 - ac$$

Or $\quad |\sin A - \sin C| = \sqrt{\sin^2 B - \sin A \sin C}$

$$\Rightarrow \quad 2\cos\left(\frac{A+C}{2}\right)\left|\sin\left(\frac{A-C}{2}\right)\right| = \sqrt{\frac{3}{4} - \sin A \sin C} \quad \Rightarrow \quad 2\left|\sin\left(\frac{A-C}{2}\right)\right| = \sqrt{3 - 4\sin A \sin C}$$

$$\therefore \quad \lim_{A \to C} \frac{\sqrt{3 - 4\sin A \sin C}}{|A - C|} = \lim_{A \to C} \frac{2\left|\sin\left(\frac{A-C}{2}\right)\right|}{|A - C|}$$

$$\lim_{A \to C} \left|\frac{\sin\left(\frac{A-C}{2}\right)}{\left(\frac{A-C}{2}\right)}\right| = |1| = 1 \Rightarrow f(x) = 1 \quad \therefore \quad f'(x) = 0$$

78.(3) $F(x) = 3e^x$ and $G(x) = e^{-x}$

The equation $9x^4 = (F(x))^2 G(x)$ becomes $x^4 = e^x$

Hence, number of solutions = 3

79.(2) Let $g(x) = x\tan^{-1}(x^2)$. It is an odd function. So, $g^{2m}(0) = 0.$

Let $h(x) = x^4$ $\quad$ So, $f(x) = g(x) + h(x) \quad \Rightarrow \quad f^{2m}(0) = g^{2m}(0) + h^{2m}(0) = h^{2m}(0) \neq 0$

It happens when $2m = 4 \Rightarrow m = 2$

80.(3) Let $\left[\sqrt{n} + \frac{1}{2}\right] = K \Rightarrow K \le \sqrt{n} + \frac{1}{2} < (K+1) \quad \Rightarrow \quad (K - 1/2)^2 \le n < (K + 1/2)^2$

$$\Rightarrow \quad K^2 - K + 1/4 \le n < K^2 + K + 1/4, K \in N, n \in N \quad \Rightarrow \quad K^2 - K + 1 \le n \le K^2 + K$$

So, $S = \displaystyle\sum_{n=1}^{\infty} \frac{2^{f(n)} + 2^{-f(n)}}{2^n} = \sum_{n=1}^{\infty} \frac{2^K + 2^{-K}}{2^n} = \left(\frac{2 + 2^{-1}}{2} + \frac{2^1 + 2^{-1}}{2^2}\right) + \left(\frac{2^2 + 2^{-2}}{2^3} + \frac{2^2 + 2^{-2}}{2^4} + \frac{2^2 + 2^{-2}}{2^5} + \frac{2^2 + 2^{-2}}{2^6}\right)$

$$\text{As, } \begin{cases} K = 1 \quad \Rightarrow \quad 1 \le n \le 2 \\ K = 2 \quad \Rightarrow \quad 3 \le n \le 6 \\ K = 3 \quad \Rightarrow \quad 7 \le n \le 12 \\ \quad \vdots \qquad\qquad\qquad \vdots \end{cases}$$

$$\therefore \quad S = \sum_{K=1}^{\infty} \sum_{n=K^2-K+1}^{K^2+K} \frac{2^K + 2^{-K}}{2^n} = \sum_{K=1}^{\infty} \left(2^K + 2^{-K}\right). \sum_{n=K^2-K+1}^{K^2+K} \frac{1}{2^n}$$

$$= \sum_{K=1}^{\infty} \left(2^K + 2^{-K}\right).\left[\frac{1}{2^{K^2-K+1}} + \frac{1}{2^{K^2-K+2}} + ... + \frac{1}{2^{K^2+K}}\right] = \sum_{K=1}^{\infty} \left(2^K + 2^{-K}\right) \frac{\frac{1}{2^{K^2-K+1}}\left(1 - \frac{1}{2^{2K}}\right)}{1 - 1/2}$$

$$= \left(2^1 - 2^{-3}\right) + \left(2^0 - 2^{-8}\right) + \left(2^{-3} - 2^{-15}\right) + \left(2^{-8} - 2^{-24}\right) + ...\infty = 2 + 1 = 3$$

81.(1) We have, $\lim_{x\to\frac{\pi}{2}}\sqrt{\dfrac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}$

Now, RHL at $x=\dfrac{\pi}{2}$,

$$=\lim_{x\to\frac{\pi^+}{2}}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}=\lim_{x\to\frac{\pi^+}{2}}\sqrt{\frac{\tan x-\sin(x-\pi)}{\tan x+\cos^2(\tan x)}}=1 \qquad \left[\because\ \tan^{-1}(\tan x)=x-\pi,\text{when } x>\frac{\pi}{2}\right]$$

Again, LHL at $x=\dfrac{\pi}{2}$,

$$=\lim_{x\to\frac{\pi-}{2}}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}=\lim_{x\to\frac{\pi-}{2}}\sqrt{\frac{\tan x-\sin(x)}{\tan x+\cos^2(\tan x)}}\left[\because \tan^{-1}(\tan x)\right]=x,\text{when } x<\frac{\pi}{2}$$

$$=\lim_{x\to\frac{\pi-}{2}}\sqrt{\frac{1-\frac{\sin x}{\tan x}}{1+\frac{\cos^2(\tan x)}{\tan x}}}=\sqrt{\frac{1+0}{1+0}}=1=\lim_{x\to\frac{\pi}{2}}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}=1$$

82.(1.5) We have, $A(t)=\int_0^t \sin x^2.dx; B(t)=\dfrac{t\sin t^2}{2}$ $\quad\therefore\quad$ $\lim_{t\to0}\dfrac{A(t)}{B(t)}=\lim_{t\to0}\dfrac{2\int_0^t\sin x^2.dx}{t\sin t^2}=\lim_{t\to0}\dfrac{2\int_0^t\sin x^2dx}{t^3.\frac{\sin^2 t}{t^2}}$

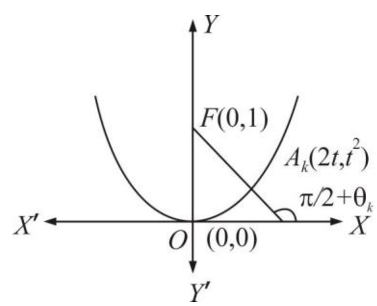
$$\lim_{t\to0}\frac{2\int_0^t\sin x^2.dx}{t^3}=\lim_{t\to0}\frac{2.\sin(t^2)}{3t^2}$$

[applying Newton –Leibnitz's rule followed by L' Hospital's rule]

$\therefore\quad \lim_{t\to0}\dfrac{A(t)}{B(t)}=\dfrac{2}{3}$

Then, $m+n=2+3=5$

83.(4) Let $A_k=(2t,t^2)$ $\therefore$ Slope of $FA_k=\dfrac{t^2-1}{2t-0}=\tan\left(\dfrac{\pi}{2}+\theta_k\right)$



$\tan(\theta_k)=\dfrac{2t}{1-t^2}=\tan(2\phi)$, where $t=\tan\phi$

$\therefore\quad \phi=\dfrac{\theta_k}{2}=\dfrac{k\pi}{4n}$, where $\tan\phi=t.$

Also, $FA_k=\sqrt{(t^2-1)^2+(2t)^2}=t^2+1=1+\tan^2\phi=\sec^2\left(\dfrac{k\pi}{4n}\right)$

$$\therefore \lim_{n\to\infty}\frac{1}{n}.\sum_{k=1}^{n}FA_k=\lim_{n\to\infty}\frac{1}{n}.\sum_{k=1}^{n}\sec^2\left(\frac{k\pi}{4n}\right)=\int_0^1\sec^2\left(\frac{\pi x}{4}\right)dx=\frac{4}{\pi}.\left[\tan\left(\frac{\pi x}{4}\right)\right]_0^1=\frac{4}{\pi}$$

Hence, $m=4.$

84.(2) Let $f(x)=\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+.....+\frac{1}{\sqrt{n^2+2n}}\right)$

As, $\frac{1}{\sqrt{n^2+2n}}<\frac{1}{\sqrt{n^2}}=\frac{1}{\sqrt{n^2}}$

$\frac{1}{\sqrt{n^2+2n}}<\frac{1}{\sqrt{n^2}+1}<\frac{1}{\sqrt{n^2}}$

On adding, we get $\frac{2n+1}{\sqrt{n^2+2n}}<\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2}+1}+.....+\frac{1}{\sqrt{n^2+2n}}<\frac{2n+1}{\sqrt{n^2}}$

$2<\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+.....+\frac{1}{\sqrt{n^2+2n}}\right)<2 \qquad \therefore \qquad \lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+....+\frac{1}{\sqrt{n^2+2n}}\right)=2$

85.(3.14) Here, $\sin(x_{n+1}-x_n)+2^{-(n+1)}\sin x_n.\sin x_{n+1}=0$

$\Rightarrow \qquad \cot x_{n+1}-\cot x_n=2^{-(n+1)} \quad \Rightarrow \qquad \cot x_n-\cot x_{n-1}=2^{-n}$

$\Rightarrow \qquad \cot x_{n+1}-\cot x_1=\frac{1}{2^2}+\frac{1}{2^3}+......+\frac{1}{2^{n+1}} \qquad \Rightarrow \qquad \cot x_{n+1}=\frac{1}{2}+\frac{1}{2^2}+........\frac{1}{2^{n+1}}$

$\Rightarrow \qquad \lim\limits_{n\to\infty}\cot x_{n+1}=1 \quad \therefore \qquad \lim\limits_{n\to\infty}x_{n+1}=\frac{\pi}{4}$

$l=\frac{\pi}{4}\Rightarrow 4l=\pi$

86.(8) We have, $f(x+y)=f(x)+f(y)+xy(x+y)$

$f(0)=0 \qquad \therefore \qquad \lim\limits_{h\to 0}\frac{f(h)}{h}=-1 \quad \therefore \qquad \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim\limits_{h\to 0}\frac{f(x)+f(h)+xh(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\frac{f(h)}{h}+\lim\limits_{h\to 0}x(x+h)=-1+x^2$

$\therefore \qquad f'(x)=-1+x^2 \quad \therefore \qquad f(x)=\frac{x^3}{3}-x+c \qquad \therefore \qquad f(x)$ is a polynomial function.

$f(x)$ is twice differentiable for all $x\in R$ and

$f'(3)=3^2-1=8$.

87.(3) $\lim\limits_{x\to\infty}4x\frac{\tan^{-1}\left(\frac{1}{2x+3}\right)}{\left(\frac{1}{2x+3}\right)}\times\frac{1}{2x+3}=2 \qquad \Rightarrow \qquad y^2+4y+5=2 \quad \Rightarrow \qquad y=-1,-3.$

88.(12.07) Given limit $=\lim\limits_{x\to 0}\frac{a\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)-bx+cx^2+x^3}{2x^2\left(x-\frac{x^2}{2}+\frac{x^3}{3}-.....\right)-2x^3+x^4}=\lim\limits_{x\to 0}\frac{(a-b)x+cx^2+\left(1-\frac{a}{6}\right)x^3+\frac{ax^5}{120}....}{2\frac{x^5}{3}-\frac{x^6}{2}+...}$

For this limit to exist, we must have $a=b, c=0, a=6$ and given limit $=\frac{a}{120}\times\frac{3}{2}=\frac{6\times 3}{120\times 2}=\frac{3}{40}$.

89.(6) $\ln P = \frac{1}{n}\left[\left\{\ln\left(n^3+1\right)+\ln\left(n^3+2^3\right)+..........+\ln\left(n^3+n^3\right)\right\}-3n\,\text{lnn}\right]$

Hence $T_r = \frac{1}{n}\left[\ln\left(n^3+r^3\right)-3\ln n\right] = \frac{1}{n}\left[3\text{lnn}+\ln\left(1+\left(\frac{r}{n^3}\right)^3\right)-3.\ln n\right] = \frac{\ln\left(1+\left(\frac{r}{h}\right)^3\right)}{n}$

Let $S = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\left(\frac{r}{n}\right)^3\right) = \int_0^1 \ln\left(1+x^3\right)dx = \int_0^1 \ln(1+x)dx + \int_0^1 \ln\left(x^2-x+1\right)dx$

$= \ln(1+x).x]_0^1 - \int_0^1 \frac{1+x-1}{1+x}dx + \ln\left(1-x+x^2\right).x]_0^1 - \int_0^1 \frac{x(2x-1)}{x^2-x+1}dx$

$\ln 2-(1-\ln 2)-\int_0^1 \frac{2x^2-x}{x^2-x+1}dx$

$S = \ln 4 - 1 - I_1$

$I_1 = \int_0^1\left(2+\frac{x-2}{x^2-x+1}\right)dx = 2-\frac{\pi}{\sqrt{3}} \quad \Rightarrow \quad S = \ln 4-1-2+\frac{\pi}{\sqrt{3}} = \ln 4+\frac{\pi}{\sqrt{3}}-3. \quad \therefore \quad \ln P = \ln 4+\frac{\pi}{\sqrt{3}}-3$

$\Rightarrow \quad P = 4e^{\frac{\pi}{\sqrt{3}}}.e^{-3}.$

90.(2) $H_n = \frac{n}{\frac{1}{n+1}+\frac{1}{n+2}+.........+\frac{1}{n+n}} \quad \Rightarrow \quad \frac{n}{H_n} = \frac{1}{n+1}+\frac{1}{n+2}+.......+\frac{1}{n+n}$

$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{n}{H_n}\right) = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\frac{1}{n}}{1+\frac{r}{n}} = \int_0^1\frac{dx}{1+x} = \ln 2$

$\therefore \quad \lim_{n\to\infty}\left(\frac{H_n}{n}\right) = \frac{1}{\ln 2}.$

DIFFERENTIAL CALCULUS-2

1.(D) Let $f(x)=x^2+3x+4$

Then we have to find solutions of $f(f(x))=x$

$$f(x)-x>0$$

$\therefore$ $f(f(x))-f(x)>0$

$\therefore$ $f(f(x))>x,$ thus $f(f(x))=x$ has no solution.

2.(B) Let us consider $g(t)=\int_0^t f(x)dx$

Applying LMVT in interval (0, 2)

We get $\dfrac{g(2)-g(0)}{2-0}=g'(c)$

$$\int_0^2 f(x)dx=2f(c) \quad c\in(0,2).$$

3-5. **3.(A)** **4.(B)** **5.(A)**

(3) $f'(x)=1-\sin x$

$\therefore$ $f(x)$ is monotonically increasing

$f(0)=1-a$ (for exactly one positive root)

$f(0)<0 \Rightarrow 1-a<0 \Rightarrow a>1$

(4) $f(x)=x+\cos x-a$

$f'(x)=1-\sin x$

$f'(c)=1-\sin c$

$\alpha<c<\beta$

Apply L.M.V.T for $x\in(\alpha,\beta)$

$$f'(c)=\frac{(\beta+\cos\beta-a)-(\alpha+\cos\alpha-a)}{\beta-\alpha}=\frac{(\beta-\alpha)+(\cos\beta-\cos\alpha)}{\beta-\alpha}$$

$$=1+\frac{\cos\beta-\cos\alpha}{\beta-\alpha}$$

$\alpha<c<\beta$

$\Rightarrow$ $\sin\alpha<\sin c<\sin\beta \Rightarrow -\sin\beta<-\sin c<-\sin\alpha$ $\Rightarrow$ $1-\sin \Rightarrow <1-\sin c<1-\sin\alpha$

$\Rightarrow$ $1-\sin\beta<1+\dfrac{\cos\beta-\cos\alpha}{\beta-\alpha}<1-\sin\alpha$

$$\sin\beta>\frac{\cos\alpha-\cos\beta}{\beta-\alpha}>\sin\alpha \Rightarrow \sin\alpha<\frac{\cos\alpha-\cos\beta}{\beta-\alpha}<\sin\beta$$

(5) $f'(x)=1-\sin x$

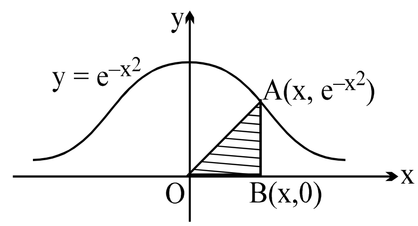
$f'(x)\big|_{x=\frac{\pi}{2}}=0$

$\therefore$ Equation of tangent $y=\dfrac{\pi}{2}-a$

6.(D) $A = \dfrac{xe^{-x^2}}{2}$; $\quad A' = \dfrac{1}{2}\left[e^{-x^2} - 2x^2 \cdot e^{-x^2}\right]$

$= \dfrac{e^{-x^2}}{2}\left[1 - 2x^2\right] = 0 \Rightarrow x = \dfrac{1}{\sqrt{2}}$ gives A_{max}.

$\therefore \quad A_{max} = \dfrac{e^{-1/2}}{2\sqrt{2}} = \dfrac{1}{\sqrt{8e}}$



7.(C) $x^2 y = c^3$

$x^2 \dfrac{dy}{dx} + 2xy \quad \Rightarrow \dfrac{dy}{dx} = 0 = -\dfrac{2y}{x}$

Equation of tangent at (x, y)

$Y - y = -\dfrac{2y}{x}(X - x)$

$Y = 0$, gives, $X = \dfrac{3x}{2} = a$

And $\quad X = 0$, gives, $Y = 3y = b \qquad$ Now $\quad a^2 b = \dfrac{9x^2}{4}.3y = \dfrac{27}{4}x^2 y = \dfrac{27}{4}c^3 \Rightarrow$ (C)

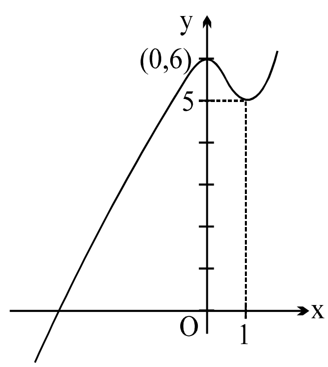
8.(A) $f(x) = 2x^3 - 3x^2 + 6$

$f'(x) = 6x^2 - 6x = 6(x^2 - x) = 0$ gives $x = 0$ or $x = 1$

For inverse to exist function must be Bijective

Hence Domain is $[1, \infty)$ for one-one function

$a \geq 1$



9.(C) Let $\quad A = (t, t^2)$; $\quad m_{OA} = t$; $\quad m_{AB} = -\dfrac{1}{t}$

Equation of AB, $y - t^2 = -\dfrac{1}{t}(x - t^2)$

Put $\quad x = 0$

$h = t^2 + 1 \qquad$ (as $x \to 0$ then $t \to 0$)

Now $\quad \lim\limits_{t \to 0}(h) = \lim\limits_{t \to 0}(1 + t^2) = 1$

10.(D) $F(x) = \dfrac{4}{\sin x} + \dfrac{1}{1 - \sin x}$

$F'(x) = \dfrac{-4\cos x}{\sin^2 x} + \dfrac{\cos x}{(1 - \sin x)^2}$

For maxima or minima

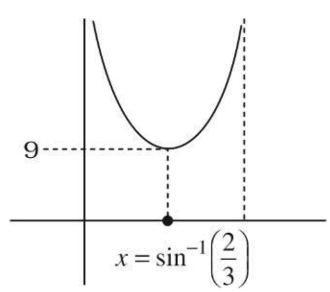
$F'(x) = 0 \Rightarrow \sin x = \dfrac{2}{3}$

$f(x) \to \infty$ for $x \to 0^+$

$f(x) \to \infty$ for $x \to \dfrac{\pi}{2}^-$

$F(x)_{\min}$ when $\sin x = \dfrac{2}{3}$ (Point of minima)

$F(x)_{\min} = a_{\min} = 9$



11.(D) $f'(x) = 12x^2 - 2x - 2 = 2[6x^2 - x - 1] = 2(3x+1)(2x-1)$

Hence $g(x) = \begin{cases} f(x) & if\ 0 \le x < \frac{1}{2} \\ f\left(\frac{1}{2}\right) & if\ \frac{1}{2} \le x \le 1 \\ 3 - x & if\ 1 < x \le 2 \end{cases}$

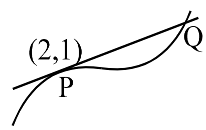
12.(A) f is not differentiable at $x = \frac{1}{2}$

g is not continuous in [0, 1] at x = 0 & 1 ; h is not continuous in [0, 1] at x = 1

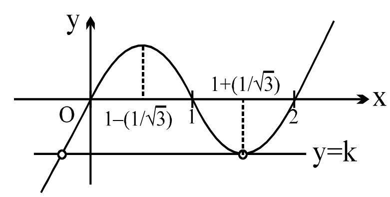
$k(x) = (x+3)^{\ln_2 5} = (x+3)^p$ where $2 < p < 3$

13.(D) Eliminating t gives $y^2(x-1) = 1$.
Equation of tangent at P(2, 1) is $x + 2y = 4$.

Solving with curve $x = 5$ & $y = -1/2 \Rightarrow Q(5, 1/2) \Rightarrow PQ = \frac{3\sqrt{5}}{2}$]



14.(A) $f(x) = x(x^2 - 3x + 2)$, $f(x) = x(3x^2 - 1) = x(\sqrt{3}x - 1)(\sqrt{3}x + 1)$
$f(x) = x(x-2)(x-1)$
Graph of $y = f(x)$ is as shown
Now $f(x) = k$ to have exactly one positive and negative solution



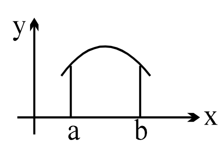
We have, $k = f\left(1 + \frac{1}{\sqrt{3}}\right)$

$\therefore \quad k = \left(1 + \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}} - 1\right)\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{3} - 1\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{-2}{3\sqrt{3}}$

15.(D) Given $g(x) = \ln(h(x))$

$g'(x) = \frac{h'(x)}{h(x)}$

$g''(x) = \frac{h(x)h''(x) - (h'(x))^2}{h^2(x)} < 0$ (given)



$\therefore \quad g''(x) < 0 \Rightarrow g(x)$ is concave down

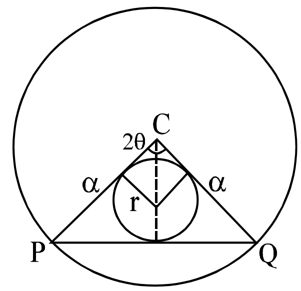
16.(B) $F'(x) = e^{(1+\sin^{-1}(\cos x))^2} \cdot (-\sin x) - e^{(1+x)^2} \cdot \cos x$
$F'(0) = 0 - e = -e$
$\left.F'\left(\frac{\pi}{2}\right) = -e - 0 = -e\right]$ Hence Rolle's theorem is applicable for the function $F'(x)$

$\Rightarrow$ there is some c in $\left(0, \frac{\pi}{2}\right)$ for which $F''(c) = 0$ as Rolle's theorem is applicable for $F'(x)$ $\left[0, \frac{\pi}{2}\right]$ in also F

$(0) = \int_0^1 f(t)dt$ and $F\left(\frac{\pi}{2}\right) = \int_1^0 f(t)dt$, hence F (0) & $F\left(\frac{\pi}{2}\right)$ have opposite signs

$\Rightarrow$ F (c) = 0 for some $c \in \left(0, \frac{\pi}{2}\right)$

17.(B) $r = \dfrac{\Delta}{s}$ where Δ = Area of triangle CPQ and s = semiperimeter of ΔCPQ.



$$r = \frac{\alpha^2 \sin 2\theta}{2s} = \frac{\alpha^2 \sin 2\theta}{2\alpha + 2\alpha \sin\theta} = \frac{\alpha}{2} \cdot \frac{\sin 2\theta}{1 + \sin\theta}$$

Consider $f(\theta) = \dfrac{\sin 2\theta}{1 + \sin\theta}$

$$f'(\theta) = \frac{(1+\sin\theta) 2\cos 2\theta - \sin 2\theta . \cos\theta}{(1+\sin\theta)^2} = 0$$

$2(1 + \sin\theta)(1 - 2\sin^2\theta) - 2\sin\theta\,(1 - \sin^2\theta) = 0$

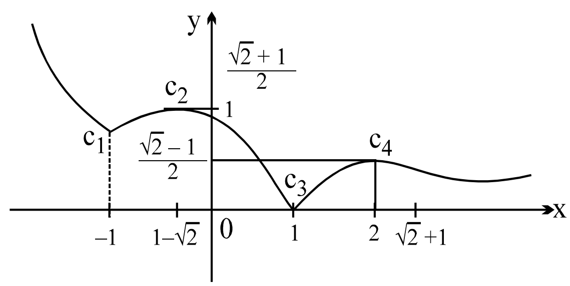
$2(1 - 2\sin^2\theta) = 2\sin\theta\,(1 - \sin\theta)$

$1 - 2\sin^2\theta = \sin\theta - \sin^2\theta$

$\sin^2\theta + \sin\theta - 1 = 0$

$sin\theta = \dfrac{-1 \pm \sqrt{1+4}}{2}$ $\qquad \therefore \quad sin\theta = \dfrac{\sqrt{5}-1}{2}$]

18.(A) $f(x) = \begin{cases} \dfrac{x-1}{1+x^2} & if \quad x \ge 1 \\ \dfrac{1-x}{1+x^2} & if \quad -1 < x < 1 \\ x^2 & if \quad x \le -1 \end{cases}$



$f'(x) = 0$ gives $x = \sqrt{2} + 1$ or

$1 - \sqrt{2}$. The function has a continuity at x = – 1.

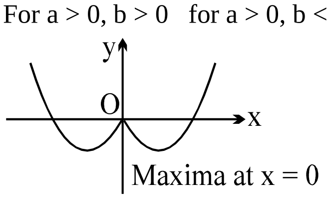
As $x \to \infty$, $f(x) \to 0$.

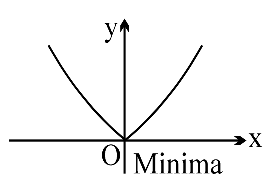
The graph is as shown .

Note that critical points are those where **dy/dx = 0** if it exists or **dy/dx** is non existent .

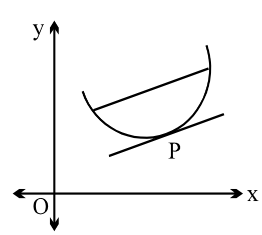
The points x = –1, $1 - \sqrt{2}$, 1 and $1 + \sqrt{2}$ are the four critical points on the graph of this function $\Rightarrow$ A

19.(A) For a > 0, b > 0 for a > 0, b < 0





20.(D) $f(x) = \displaystyle\int_1^x \left(t + \frac{1}{t}\right) dt$ $\qquad \Rightarrow \quad f'(x) = x + \dfrac{1}{x}$



$\therefore$ $g(x) = x + \dfrac{1}{x}$ for $x \in \left[\dfrac{1}{2}, 3\right]$

$g\left(\dfrac{1}{2}\right) = 2 + \dfrac{1}{2} = \dfrac{5}{2}$, $\qquad g(3) = 3 + \dfrac{1}{3} = \dfrac{10}{3}$

Let $P \equiv (c, g(c))$, $c \in \left[\dfrac{1}{2}, 3\right]$

By LMVT, g' (c) = $\dfrac{g(3)-g\left(\frac{1}{2}\right)}{3-\frac{1}{2}}$ $\quad\therefore\quad$ $1-\dfrac{1}{c^2}=\dfrac{\frac{10}{3}-\frac{5}{2}}{3-\frac{1}{2}}$

$\Rightarrow$ $c^2 = \dfrac{3}{2}$ $\Rightarrow$ c = $\sqrt{\dfrac{3}{2}}$ $\quad\therefore\quad$ g (c) = $\sqrt{\dfrac{3}{2}}+\dfrac{1}{\sqrt{\frac{3}{2}}}=\dfrac{5}{\sqrt{6}}$ $\quad\therefore$ P $\equiv\left(\sqrt{\dfrac{3}{2}},\dfrac{5}{\sqrt{6}}\right)$

21.(C) $f(x)=12x^5-15x^4+20x^3-30x^2+60x+1$

$f'(x)=60x^4-60x^3+60x^2-60x+60$

$=60(x^4-x^3+x^2-x+1)$ $=60(x^3(x-1)+x(x-1)+1)$ $=60((x^3+x)(x-1)+1)=60(x(x^2+1)(x-1)+1)$

$f'(x)>0$ for $x\in R$

So $f(x)$ is monotonically increasing

But is not necessary that if a function is monotonic then it will have range R.

22.(C) a = 1

$f(x)=8x^3+4x^2+2bx+1$

$f'(x)=24x^2+8x+2\text{b}=2(12x^2+4x+\text{ b})$

For increasing function, $f'(x)\geq 0\ \forall\ x\in R$ $\quad\therefore\quad$ $D\leq 0\Rightarrow 16-48b\leq 0\Rightarrow b\geq\dfrac{1}{3}$ $\Rightarrow$ **(C)**

23.(B) If $b = 1$

$f(x)=8x^3+4ax^2+2x+a$

$f'(x)=24x^2+8ax+2$ or $f'(x)=2(12x^2+4ax+1)$

For non monotonic $f'(x) = 0$ must have distinct roots

Hence D > 0 i.e. $16a^2-48>0$ $\Rightarrow$ $a^2>3;$ $\quad\therefore\quad$ $a>\sqrt{3}$ or $a<-\sqrt{3}$

$\therefore$ $a\in\{2, 3, 4, ...100\}$

Sum = 5050 – 1 = 5049

24.(D) If x_1, x_2 and x_3 are the roots then $\log_2 x_1+\log_2 x_2+\log_2 x_3=5$

$\log_2(x_1x_2x_3)=5$

$x_1x_2x_3=32$

$-\dfrac{a}{8}=32$ $\Rightarrow$ $a=-256$

25.(AB) $8x\dfrac{dx}{dt}+18y\dfrac{dy}{dt}=0$ **. . . .(i)**

Given $\dfrac{dx}{dt}=\dfrac{-dy}{dt}$ **. . . .(ii)**

From (1) and (2) $4x_1=9y_1$

$4x_1^2+\dfrac{9.16.x_1^2}{81}=13\Rightarrow x_1=\pm\dfrac{3}{2}$

Points are $\left(\dfrac{3}{2},\dfrac{2}{3}\right),\left(-\dfrac{3}{2},-\dfrac{2}{3}\right)$

26.(ABCD) Since $f(x)$ is even, so we check in $[0, 2]$

$$f(x)=\begin{cases} ax^2+b, & 0\le x<1 \\ 1, & x=1 \\ \dfrac{\lambda}{x} & 1<x\le 2 \end{cases}$$

For continuity, a + b = 1 = λ (1)

$$f'(x)=\begin{cases} 2ax, 0<x<1 \\ -\dfrac{\lambda}{x^2}, 1<x<2 \end{cases} \Rightarrow \quad \text{For continuity, } 2a=-\lambda \quad ..(2)$$

From (1) and (2), $a=-\dfrac{1}{2}, b=\dfrac{3}{2}, \lambda=1$

27.(AC) $h(x)=\dfrac{1-x^2}{1+x^2}=\dfrac{2}{1+x^2}-1$, $h'(x)=\dfrac{-4x}{(1+x^2)^2}$, $h'(x)>0$ for $x\in(-\infty, 0)$

$\cos^{-1}$ is a decreasing function

f decreases when h increases

i.e., when $x\in(-\infty, 0)$

28.(ABCD) $y=x^{1/3}(x-1)$

$\dfrac{dy}{dx}=\dfrac{4}{3}x^{1/3}-\dfrac{1}{3}\cdot\dfrac{1}{x^{2/3}}=\dfrac{1}{3x^{2/3}}[4x-1]$

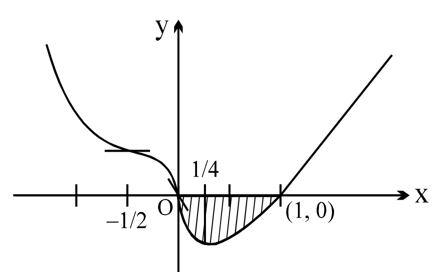
Hence f is $\uparrow$ for $x>\left.\dfrac{1}{4}\right]$, ($x^{2/3}$ is always positive) and $x=\dfrac{1}{4}$ the curves has a local minima

And $f\downarrow$ for $x<\dfrac{1}{4}$

Now $f'(x)=\dfrac{4}{3}x^{1/3}-\dfrac{1}{3}\cdot x^{-2/3}$ (non existent at x = 0, vertical tangent)

$f''(x)=\dfrac{4}{9}\cdot\dfrac{1}{x^{2/3}}+\dfrac{1}{3}\cdot\dfrac{2}{3}\cdot\dfrac{1}{x^{5/3}}$

$=\dfrac{2}{9x^{2/3}}\left[2+\dfrac{1}{x}\right]=\dfrac{2}{9x^{2/3}}\left[\dfrac{2x+1}{x}\right]$



$\therefore$ $f''(x)=0$ at $x=-\dfrac{1}{2}$ (Inflection point), $x=0$ is also point of inflation.

Graph of $f(x)$ is as A = $\left|\int_0^1\left(x^{4/3}-x^{1/3}\right)dx\right|=\left|\left[\dfrac{3}{7}x^{3/7}-\dfrac{3}{4}x^{4/3}\right]_0^1\right|=\left|\dfrac{3}{7}-\dfrac{3}{4}\right|=3\left|\dfrac{4-7}{28}\right|=\dfrac{9}{28}\Rightarrow$ **(D)**

29.(AD) $\dfrac{dy}{dx}=\dfrac{-b}{a}=\dfrac{-1}{t^2}\Rightarrow ab>0$

30.(ABCD) $f'(x) = \sqrt{1-x^4} > 0$ in $(-1, 1) \Rightarrow f$ is $\uparrow$

Now $f(x) + f(-x) = \int_0^x \sqrt{1-t^4}\,dt + \int_0^{-x} \sqrt{1-t^4}\,dt \Rightarrow \int_0^x \sqrt{1-t^4}\,dt + \left(-\int_0^x \sqrt{1-y^4}\,dy\right)$ $(t = -y)$

$= 0 \Rightarrow f(x)$ is odd

Again $f''(x) = \dfrac{-4x^3}{2\sqrt{1-x^4}}$ which vanished at $x = 0$. (0, 0) is inflection since f is well defined in $[-1, 1] \Rightarrow$ A,B,C,D]

31.(AB) Since intercepts are equal in magnitude but opposite in sign $\Rightarrow \left.\dfrac{dy}{dx}\right|_P = 1$

Now $\dfrac{dy}{dx} = x^2 - 5x + 7 = 1 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x = 2$ or 3

32.(BD) $h(x) = \dfrac{\ell n\left(f(x) \cdot g(x)\right)}{\ell n\, a} = \dfrac{\ell n\, a^{\left\{a^{|x|} \cdot \operatorname{sgn} x\right\} + \left[a^{|x|} \cdot \operatorname{sgn} x\right]}}{\ell n\, a}$

$\left(\left\{a^{|x|} \operatorname{sgn} x\right\} + \left[a^{|x|} \operatorname{sgn} x\right] = a^{|x|} \operatorname{sgn} x\right)$ $(\because \{y\} + [y] = y)$

$= \begin{bmatrix} a^x & for & x > 0 \\ 0 & for & x = 0 \\ -a^{-x} & for & x < 0 \end{bmatrix} \Rightarrow h(x)$ is an odd function.

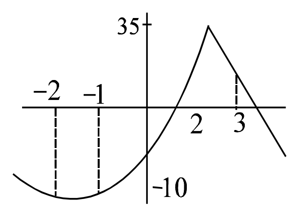
33.(BCD) $f'(x) = 100x^{99} + \cos x$

For $x \in (0,1)$ and $\left(0, \dfrac{\pi}{2}\right)$, cos x and x are both +ve $\Rightarrow f'(x)$ is increasing

For $x \in \left(\dfrac{\pi}{2}, \pi\right)$, $x > 1$ hence $100x^{99}$ obviously $> \cos x \Rightarrow \uparrow$]

34.(ABCD) Note that $f(x)$ is continuous at $x = 2$ and f is decreasing for (2, 3) and increasing for [–1, 2].

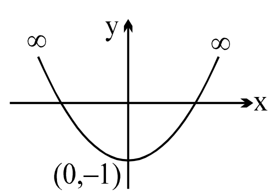
At $x = 2$, f has a maxima



35.(ABC) Graph of y = f (x) $\Rightarrow$ (A) and (C)

$f'(x) = x(2 - \cos x)$

$f(x)$ is even function



36.(ACD) If f and g are inverse then (fog)(x) = x

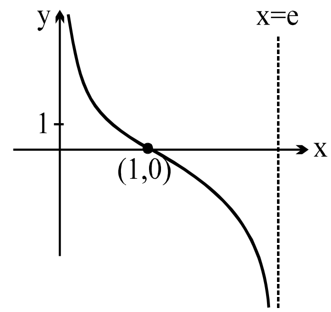
$f'[g(x)]\, g'(x) = 1$

If f is increasing $\Rightarrow$	$f' > 0$	$\Rightarrow$	sign of g' is also + ve	$\Rightarrow$	(A) is correct
If f is decreasing $\Rightarrow$	$f' < 0$	$\Rightarrow$	sign of g' is – ve	$\Rightarrow$	(B) is false
Since f has an inverse	$\Rightarrow$	f is bijective	$\Rightarrow$	f is injective	$\Rightarrow$	(C) is correct

Inverse of a bijective mapping is bijective

$\Rightarrow$ g is also bijective $\Rightarrow$ g is onto $\Rightarrow$ (D) is correct

37.(ABC) $f(x) = \ln(1 - \ln x)$



Domain (0, e)

$f'(x) = -\dfrac{1}{(1-\ln x)} \cdot \dfrac{1}{x} < 0 \;\Rightarrow$ decreasing $\forall$ x in its domain

$\Rightarrow$ (A) & (B) are incorrect

$f'(1) = -1 \;\Rightarrow$ (C) is also incorrect

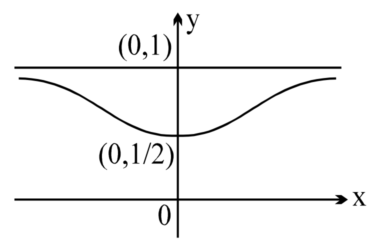
Also $f(1) = 0$; $\lim\limits_{x \to e^{-1}} f(x) \to -\infty$; $\lim\limits_{x \to 0^+} f(x) \to \infty$

$f''(x) = \dfrac{-\ln x}{x^2(1-\ln x)^2}$

f '' (1) = 0, $x = 1$ which is a point of inflection graph is as shown y axis and x = e are two asymptotes

38.(BCD)

Domain is x $\in$ R



Also f (x) = $\left[\cos\left(\tan^{-1}(\sin\theta)\right)\right]^2$ where $\cot\theta = x$

$= \left[\cos\left(\tan^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)\right)\right]^2 = (\cos\phi)^2$ where $\tan\phi = \dfrac{1}{\sqrt{1+x^2}}$

$= \left(\dfrac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right)^2$

g(x) = $\dfrac{1+x^2}{2+x^2} = 1 - \dfrac{1}{2+x^2}$

Range is $\left[\dfrac{1}{2}, 1\right)$; f ' (x) = $\dfrac{2x}{(2+x^2)^2}$

Hence f ' (0) = 0

Also $\lim\limits_{x \to \infty} f(x) = 1$. Hence (B), (C), (D)

39.(AC) Let the common tangent is $y = mx + c$

Solving with first curve we get, $x^2 + 1 = mx + c \;\Rightarrow\; x^2 - mx + 1 - c = 0$

D = 0

$\boxed{m^2 = 4(1-c)}$ ....(A)

Solving with second curve

$mx + c = -x^2 \;\Rightarrow\; x^2 + mx + c = 0$

D = 0

$\boxed{m^2 = 4c}$ ....(B)

From A and B : $8c = 4 \Rightarrow c = \dfrac{1}{2}$. Hence $m = \pm\sqrt{2}$

The tangent lines are $y = \sqrt{2}x + \dfrac{1}{2}$ and $y = -\sqrt{2}x + \dfrac{1}{2}$

40.(CD) $f(x)=\int_0^{\pi}\cos t\cos(x-t)dt$ ....(1)

$=\int_0^{\pi}-\cos t\cdot\cos(x-\pi+t)dt$

$f(x)=\int_0^{\pi}\cos t\cdot\cos(x+t)dt$ ....(2)

(1) + (2) gives

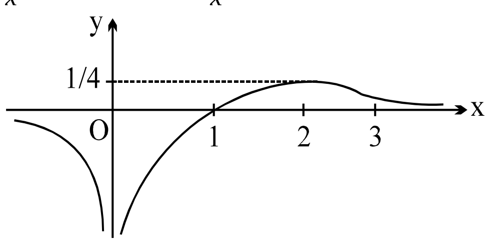
$2\ f(x)=\int_0^{\pi}\cos t(2\cos x\cdot\cos t)dt$

$\therefore$ $f(x)=\cos x\int_0^{\pi}\cos^2 t\,dt\ =\ 2\cos x\int_0^{\pi/2}\cos^2 t\,dt$

f (x) = $\dfrac{\pi\cos x}{2}$

Alternatively: Convert the integrand into sum of two cosine functions.]

41.(BCD) $f'(x)$ = $\dfrac{2-x}{x^3}$ and $f''(x)$ = $\dfrac{x-3}{x^4}$. $x=2$ is point of maxima. $x=3$ is point of inflection



42.(BCD)

(A) $f(x)$ has no relative minimum on (–3, 4)

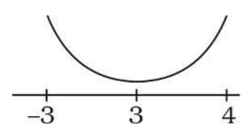
(B) $f(x)$ is continuous function on [–3, 4]

$\Rightarrow$ $f(x)$ has min. and max. on [–3, 4] by IVT

(C) $f''(x) > 0$ $\Rightarrow$ f (x) is concave upwards on [–3, 4]

(D) $f(3) = f(4)$

By Rolle's theorem $\exists\ c \in (3, 4)$, where $f'(c) = 0$ $\Rightarrow$ $\exists$ critical point on [– 3, 4]



43.(BCD) **(A)** False, e.g. $f(x) = \sin\sqrt{x}$ **(B)** True, from IVT

(C) True as $\lim\limits_{x\to\infty}\sin^{-1}\left(1+\dfrac{1}{x}\right)=\sin^{-1}$ (a quantity greater than one) $\Rightarrow$ not defined

(D) True, as the line passes through the centre of the circle.

44.(ABC) Because f is continuous for all x, the intermediate value theorem implies that the graph of f must intersect the $x-$axis. The graph must also intersect the y axis since f is defined for all x, in particular at $x=0$, as f need not be differentiable. Hence A, B and C need not be correct.

45.(AC) (A) $f''(x) > 0 \forall \times \in R \Rightarrow f'(x)$ is a strictly increasing function hence $f'(1) > f'(0)$, but given $f'(0) = f'(1) = 1$ which is not possible.

(B) $f''(x) > 0,\ \forall x \in R \Rightarrow f'(x)$ is increasing $f'(1) > f'(0)$, which is true.

46.(BCD) $a > (2-x)x^2,\ \forall x > 0$

Let $f(x) = (2-x)x^2$

$\therefore$ $a >$ maximum value of $f(x)$ for $x > 0$

$\therefore$ $a > \dfrac{32}{27} \Rightarrow$ Least natural number is 2.

47.(BC) Since $x = 1, 2, 3$ are normal

$\Rightarrow$ $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$\Rightarrow$ $b = -8a,\ C = 22a,\ d = -24a$

Also $f(2) > f(0)$

$\therefore$ $x = 1\ \&\ 3$ are point of local maxima

Also $f(k) = f(0)$

$\Rightarrow$ $ak^4 + bk^3 + ck^2 + dk = 0$

$\Rightarrow$ $ak(k-4)\left(k^2 - 4k + 6\right) = 0$ $\Rightarrow$ $k = 4$

48.(ABC) $f'(x) = n(1-x)^{-(n+1)} + n(1+x)^{-(n+1)} - 2n$

$f''(x) = n(n+1)(1-x)^{-(n+2)} - n(n+1)(1+x)^{-(n+2)}$

for $0 < x < 1$

$\dfrac{1}{1-x} > \dfrac{1}{1+x}$ $\Rightarrow$ $f''(x) > 0$

$\Rightarrow$ $f'(x)$ is increasing $\Rightarrow$ $f'(x) > f'(0) = 0$

$\Rightarrow$ $f(x)$ is increasing

$f(x) > f(0) = 0.$

49.(ABC)

50.(ACD) $f'(x) - \sin x\ f(x) \le -\sin x$

$\Rightarrow$ $f(x) \le 1$ $\Rightarrow$ $(-\infty, 1]$

51.(ABC)

52.(AC) $2f(x)f(y) = f(x+y) + f(x-y)$

Inter changing x, y

$2f(y)f(x) = f(y+x) + f(y-x)$

then $f(x) = f(-x)$

$f'(x) = -f'(-x)$

53.(AC) $\lim\limits_{x\to\infty} \dfrac{n\log x}{[x]} - 1 = 0 - 1 = -1$

$f(x) = (-1)^n = \pm 1$

54.(BD) $x = r\cos\theta, y = r\sin\theta \to$

$\dfrac{4}{r^2} = 5(1+\cos 2\theta) + 3\sin 2\theta + 1 - \cos 2\theta$

$= 6 + 4\cos 2\theta + 3\sin 2\theta,$ which has maximum 11 and minimum 1

$\therefore$ OP has minimum $\dfrac{2}{\sqrt{11}}$ and maximum $2.$

55.(ABCD) $f'(x) = 3\left(1 - \sqrt{\dfrac{21-4b-b^2}{b+1}}\right)x^2 + 5 > 0$

$\dfrac{\sqrt{21-4b-b^2}}{b+1} \le 1$

$21 - 4b - b^2 \ge 0 \Rightarrow -7 \le b \le 3$

Case-I

$b+1<0$

then $-7 \le b < -1$ …(i)

Case-II

If $b+1>0$

$b \in [2,\infty)$ …(ii)

$\Rightarrow$ $b \in [-7,-1) \cup [2,3]$

56.(AD) Differentiating partially w.r.t. x

$g'(x) = g(y)[g'(x-y)]$

Put $y = x$

$g'(x) = g(x).g'(0) = a.g(x) \Rightarrow g(x) = e^{ax}$

57.(AD) $\lambda^3 = -a\lambda^2 - b\lambda - c$

$\Rightarrow$ $\lambda^4 = (1-a)\lambda^3 + (a-b)\lambda^2 + (b-c)\lambda + c$

If $|\lambda| > 1 \Rightarrow |\lambda|^4 \le |\lambda|^3$ (not possible)

for $|\lambda| \le 1$

$|\lambda|^4 \le (1-a) + (a-b) + (b+c) + c$ $\Rightarrow$ $|\lambda| \le 1.$

58.(B) Put $x = \sin\theta, y = \cos\theta$

$\therefore$ $k\tan^2\theta + \dfrac{1}{k}\cot^2\theta + 1$

$k\tan^2\theta + \dfrac{1}{k\tan^2\theta} \ge 2$

59.(D) $\frac{-x^2}{2} \le f(x) \le \frac{x^2}{2}$

60.(D) $f'(x) < 0 \forall x \in [-1, 0)$

and $f'(x) > 0 \forall x \in [0,1]$

hence for $f(x)$ to have a minima at $x = 0$

LHL $\ge$ RHL

$\Rightarrow \quad \lim_{x \to 0^-} f(x) \ge \lim_{x \to 0^+} f(x) \quad \Rightarrow \quad 3 + \log_{0.5} \log_2 [k+3] \ge 2 \quad \therefore \quad k \in [-1, 2).$

61.(ABD) Clearly if $f(x) = a(x-\alpha)(x-\beta)$ then $g(x) = b(x-\alpha)(x-\beta)$

$\Rightarrow \quad h(x) = k(x-\alpha)^2 (x-\beta)^2 \Rightarrow h(x)h'(x) = 2k(x-\alpha)^3 (x-\beta)^3 (2x - \alpha - \beta)$

So distinct roots of $\frac{d}{dx}(h(x)h'(x)) = 0$, are 4.

62.(BD) In sine curve chord joining two points on the curve lies below the curve, hence (A) cannot be true. In log curve also, the same pattern is following but in tan curve and x^2 curves opposite pattern is followed.

63.(BC) (B) $\frac{df(x)}{dx} - f(x) > 0$

$\Rightarrow \quad \frac{e^{-x} df(x)}{dx} - e^{-x} f(x) > 0 \qquad (\because e^{-x} > 0)$

$\Rightarrow \quad \frac{d}{dx}\left(e^{-x} f(x)\right) > 0 \quad \Rightarrow \quad e^{-x} f(x)$ is an increasing function.

$e^{-x} f(x) > e^{-1} f(1) = 0 \quad \Rightarrow \quad e^{-x} f(x) > 0 \Rightarrow f(x) > 0$ for all $x > 1$.

64. **[A-p, q, r, s]** **[B-p, r]** **[C-p, q, r, s]** **[D-p, q, r, s]**

Let $f(x) = \ln(1+x) - \frac{x}{1+x}$, $f(0) = 0$

$f'(x) = \frac{x}{(1+x)^2}$

$\therefore$ For $x > 0$ $f(x)$ is $\uparrow$ $\Rightarrow$ $f(x) > f(0)$

$\ln(1+x) > \frac{x}{1+x}$

For $x < 0$ $f(x)$ is $\downarrow$ $\Rightarrow$ $f(x) > f(0)$

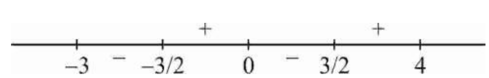
$\ln(1+x) > \frac{x}{1+x}$ for $\forall\ x \in (-1, \infty)$ $\therefore$ [A-p, q, r, s]

Similarly other parts can be done.

65. **[A-p]** **[B-q]** **[C-r]** **[D-s]**

$h'(x) = 3f'\left(\frac{x^2}{3}\right) \times \frac{2x}{3} - 2f'(3-x^2)x = 2x\left(f'\left(\frac{x^2}{3}\right) - f'(3-x^2)\right) = 0$

$\Rightarrow \quad x = 0$ or $\frac{x^2}{3} = 3 - x^2 \Rightarrow x = 0$ or $\pm\frac{3}{2}$ as $f'(x)$ is one-one



[A-p] [B-q]

$x^2+4y^2=4 \quad \Rightarrow \quad x=2\cos\theta,\ y=\sin\theta$

$x^2+y^2-xy=4\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta$

$$=3\left(\frac{1+\cos 2\theta}{2}\right)+1-\sin 2\theta=\frac{5}{2}+\frac{3}{2}\cos 2\theta-\sin 2\theta$$

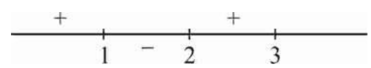
$$\in\left[\frac{5-\sqrt{13}}{2},\frac{5+\sqrt{13}}{2}\right]$$

[C-r]

$f(x)=2x^3-9x^2+12x+6$

$f'(x)=6x^2-18x+12$

$=6\left(x^2-3x+2\right)=6(x-1)(x-2)$



Global minimum of $f(x)$ occur at $x=2$ in $(1, 3)$ [D-s]

66. **[A-q]** **[B-r]** **[C-p]** **[D-t]**

(A) $f'(x)=3x^2+6x(a-7)+3(a^2-9)$

$D>0 \Rightarrow 36(a-7)^2-36(a^2-9)>0$

$\Rightarrow \quad a^2-14a+49-a^2+9>0 \quad \Rightarrow \quad -14a>-58$

$14a<58$

$a<\frac{29}{7} \qquad \therefore \qquad a\in(-\infty,-3)\cup\left(3,\frac{29}{7}\right)$

(B) Let $\alpha,\beta,\gamma=\lambda \quad \Rightarrow \quad \alpha\beta+\beta\gamma+\gamma\alpha=\mu \quad \Rightarrow \quad \alpha\beta\gamma=6$

$\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{3}>(\alpha\beta\gamma)^{\frac{2}{3}} \quad \Rightarrow \quad \mu>3(6)^{\frac{2}{3}}$

(C) $f'(x)=\cos x-2\sin 2x=\cos x(1-4\sin x)=0$

$x=\frac{\pi}{2},\sin x=\frac{1}{4}$

$f''(x)=-\sin x-4\cos 2x=-\sin x-4(1-2\sin^2 x)=8\sin^2 x-\sin x-4$

For $x=\frac{\pi}{2}, f''(x)=3>0$

For $\sin x=\frac{1}{4}, f''(x)=2-\frac{1}{4}-4<0$

Maximum value is $\frac{1}{4}+1-2\frac{1}{16} \quad \Rightarrow \quad \frac{5}{4}-\frac{1}{8}=\frac{9}{8}$

(D) $f(x)=\log(1+x)-x$

$f(x)$ is defined for $x>-1$

$f'(x)=\frac{1}{1+x}-1=\frac{-x}{1+x}$

$f'(x)\le 0$ for $x\ge 0$

$f'(x)\ge 0$ for $-1<x\le 0$

$\therefore \quad \log(1+x)\le x$ for $-1<x\le 0$

67.(9) Let $t=\sqrt{4-x^2}$ $|x|\le 2 \therefore t\in[0,2]$.

Let $g(t)=(3-t)^2+(1+t)^3$ for $t\in[0,2]$.

$g'(t)=(t+3)(3t-1)=0$

$\therefore \quad t=-3,\frac{1}{3}$

Also $g''(t)=2(3t+4)$

$$g(0)=10,\ g(2)=28,\ g\left(\frac{1}{3}\right)=\frac{256}{27}.$$

$$\therefore \quad p=28-\frac{256}{27}=\frac{500}{27}$$

68.(0) Let $f(x)=a(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)$

Take log on both sides & Differentiate $\frac{f'}{f}=\frac{1}{x-x_1}+\frac{1}{x-x_2}+\frac{1}{x-x_3}+\frac{1}{x-x_4}+\frac{1}{x-x_5}$

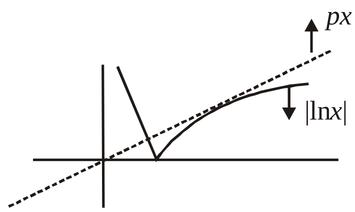
Differentiate again were x

$$\frac{f.f''-(f')^2}{(f)^2}=\frac{1}{(x-x_1)^2}+\ldots+\frac{1}{(x-x_5)^2}<0$$

69.(2) $f'(x)=-2-3x^2<0 \Rightarrow f(x)$ is decreasing function

$\Rightarrow f(x)>-x \Rightarrow (x-3)(x^2+3x+10)<0 \Rightarrow x<3$

70.(0)



$y=px$ touches

$y=|\ln x|$ for $p=\frac{1}{e}$.

$\therefore$ The equation has 3 distinct solution if $p\in\left(0,\frac{1}{e}\right)$, therefore $[p]=0$.

71.(2) Since $f(x)$ is monotonic, therefore, $f^{-1}(x)$ exists.

Let $\quad f^{-1}(x)=z$,then $x=f(z)$

$x=f(a)\Rightarrow z=f^{-1}(f(a))=a, x=f(b) \quad \Rightarrow \quad z=f^{-1}(f(b))=b$ and $dx=f'(z)dx$

$$\therefore \quad \int_{f(b)}^{f(a)} x(b-f^{-1}(x))\,dx=\int_a^b f(z)(b-z)f'(z)dz$$

$$=-\frac{1}{2}(b-a)(f(a))^2+\frac{1}{2}\int_a^b (f(z))^2dz=\frac{1}{2}\int_a^b f^2(z)-f^2(a))dz=\frac{1}{2}\int_a^b (f(x)-f(a))(f(x)+f(a))\,dx$$

$\therefore \quad k=2$

72.(1) Equation of the normal is $y=mx-2m-m^3$ $(a=1)$

If it passes through $(21,30)$ we have $30=21m-2m-m^3 \Rightarrow m^3-19m+30=0$

Then $m=-5,2,3$

But if $m=2$ or 3 the point $(am^2,-2am)$ do not lie on the parabola.

Therefore $m=-5 \qquad \therefore m+6=1$

73.(3) $Q^3 = P - 3Q$

$\dfrac{P}{Q^2} = Q + \dfrac{3}{Q} \Rightarrow f(Q) = Q + \dfrac{3}{Q} \Rightarrow f'(Q) = 1 - \dfrac{3}{Q^2} \Rightarrow Q = \pm\sqrt{3} \Rightarrow f(Q)$ will be minimum at $Q = \sqrt{3}$

So minimum value of $f(Q)$ is $2\sqrt{3}$ i.e. minimum of $\left[\dfrac{P}{Q^2}\right] = \left[2\sqrt{3}\right] = 3$

74.(2) $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

So $y = \left[|\sin x| + |\cos x|\right] = 1$

Now intersection points are $(2,1)$ and $(-2,1)$

Differentiating the equation $x^2 + y^2 = 5, \dfrac{dy}{dx} = -\dfrac{x}{y} \Rightarrow \left(\dfrac{dy}{dx}\right)_{(2,1)} = -2$ and $\left(\dfrac{dy}{dx}\right)_{(-2,1)} = 2$

75.(9) Given $f^3(x) = \int_0^x t f^2(t)dt$ …. (1)

Differentiating both the sides of equation (1) by using leibnitz, we get

$3f^2(x).f'(x) = xf^2(x)$ $\therefore f(x) = \dfrac{x^2}{6}, x \ge 0$

Slope of normal $= \dfrac{-3}{x_1}$ $\Rightarrow x_1 = 6$ and $y = 6$

Hence $y-$ intercepts $= 9$

76.(5) We have $f'(x) = 6p - 4p\cos 4x - 5 - 3\cos 3x$

$= 4p(1 - \cos 4x) + 2(p-4) + 3(1 - \cos 3x)$

Hence $f'(x) \ge 0 \Rightarrow p > 4 \forall x \in R$

Hence least integral value of $p = 5$.

77.(4) $\sqrt{A_1} = \sqrt{A_2} + A_2^{3/2} + 6$

$$\frac{\frac{dA_2}{dt}}{\frac{dA_1}{dt}} = \frac{1}{32}$$

78.(2) Let $F(x) = 2x + \int_0^x f(t)dt - p$ in $(-1, 1)$

$F(0) = -p$ $F(1^-) = 7 - p$

$F(-1^+) = 3 - p$

$F(-1)F(0) < 0$ & $F(0)F(1) < 0$ $\Rightarrow$ $p \in (0,3)$

79.(3) $\phi'(c) = 0 \Rightarrow f''(c) = 2A$

$\phi(a) = \phi(b) \Rightarrow f(b) = f(a) + (b-a)f'(a) + (b-a)^2 A \Rightarrow \lambda = \dfrac{1}{2}$

80.(5) Given curve is : $xy = 1 \Rightarrow y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2}$

$\therefore$ Slope of normal = $x^2 > 0$

$\therefore$ Slope of given line $= \frac{\log_2(1+5a-a^2)}{5} > 0$

$\Rightarrow$ $\log_2\left(1+5a-a^2\right) > 0 \Rightarrow 1+5a-a^2 > 1 \Rightarrow a^2-5a < 0$

Hence $a \in (0,5)$. Slope of $y = 1$ is zero, so $\tan\theta = 2 \Rightarrow |\tan\theta| = 2$

81.(1200) $\because$ Degree of $P(x)$ is 5 with leading coefficient one.

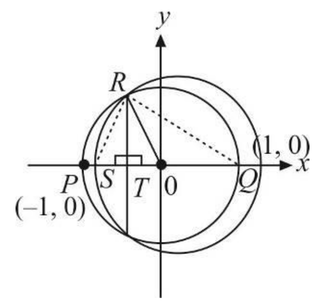
$\therefore$ Degree of $P'(x)$ is 4 with leading coefficient five.

$\therefore$ $P'(x) = 5(x-1)(x-3)(x-2)^2$

$\therefore$ $P'(6) = 5\times5\times3\times4^2 = 1200$

82.(1) Let r the radius of the second circle. The circle are

$S_1 = x^2+y^2-1 = 0$; $S_2 = x^2+y^2-2x+1-r^2 = 0$



The common chord is $S_1 - S_2 = 0 \rightarrow x = 1-\frac{r^2}{2}$

Let it meet x-axis at T

$RT^2 = OR^2 - OT^2 = 1-\left(1-\frac{r^2}{2}\right)^2 = \frac{r^2}{4}\left(4-r^2\right)$

$\Delta QSR = \frac{1}{2}.QS.RT = \frac{r^2}{4}\sqrt{4-r^2} = f(r)$

$f'(r) = 0 \rightarrow r^2 = \frac{8}{3}$

$\therefore$ Maximum area $= \frac{2}{3}\sqrt{4-\frac{8}{3}} = \frac{4}{3\sqrt{3}} = A$ $\qquad [2A] = \left[\frac{4\times2}{3\sqrt{3}}\right] = 1$

83.(0) Let the two parts of the wire be x and z, where x is the circumference of the circle and z is the perimeter of the square.

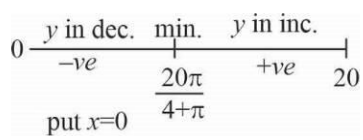
Given, $x+z = 20$ $\qquad \therefore z = 20-x$ $\qquad$ …(i)

Let y be the sum of area of the circle and the square.

Then $y = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{z}{4}\right)^2 = \frac{x^2}{4\pi} + \frac{(20-x)^2}{16}$ $\qquad$ …(ii)

$\therefore$ $\frac{dy}{dx} = \frac{x}{2\pi} - \frac{(20-x)}{8} = \frac{4x-(20-x)\pi}{8\pi} = \frac{(4+\pi)\,x-20\pi}{8\pi}$ $\qquad$ …(iii)

Sign scheme for $\frac{dy}{dx}$ i.e. for $\left[(4+\pi)x-20\pi\right]$ is



Here $0 \le x \le 20$

y has local minimum at $x=\frac{20\pi}{4+\pi}$

y will have maximum (greatest) value at $x=0$ or at $x=20$

From (ii), at $x=0,\ y=\frac{400}{16}=25$ sq. cm. at $x=20,\ y=\frac{400}{4\pi}=\frac{100}{\pi}$ sq. cm

Hence y has greatest value at $x=20$

Thus for greatest area the whole wire should be bent into a circle.

84.(1) $\int_e^x t\ f\ t)\,dt=\sin x-x\cos x-\frac{x^2}{2}$

Differentiating both sides

$xf(x)=\cos x-\cos x+x\sin x-x$

$f(x)=\sin x-1 \qquad \Rightarrow \qquad -2f\left(\frac{\pi}{6}\right)=-2\left(-\frac{1}{2}\right)=1$

85.(2) Given $f(x)=Ax^2+Bx+C,$ so $f'(x)=2Ax+B,$ …(i)

$\therefore \qquad f(x+h)=A(x+h)^2+B(x+h)+C=Ax^2+2Axh+Ah^2+Bx+Bh+C$

$=\left(Ax^2+Bx+C\right)+\left(2Axh+Ah^2+Bh\right).$

Also $f'(x+\theta h)=2A(x+\theta h)+B,$ from (i)

$=2Ax+2A\theta h+B$

$\therefore$ From $f(x+h)=f(x)+hf'(x+\theta h),$ we get

$\left(Ax^2+Bx+C\right)+\left(2Axh+Ah^2+Bh\right)=\left(Ax^2+Bx+C\right)+h\left(2Ax+2A\theta h+B\right)$

or $Ah^2=2A\theta h^2$ or $2\theta=1$ or $\theta=\frac{1}{2}.$

86.(2) From first mean Value Theorem, we know that it $f(x)$ and $\phi(x)$ be functions such that these are continuous in a $\le x\le b$ and $f'(x),\phi'(x)$ exist in $a<x<b,$ then there exists at least one point c in (a,b), such that

$f(b)-f(a)=(b-a)f'(c)$ …(i)

and $\phi(b)-\phi(a)=(b-a)\phi'(c).$ …(ii)

Multiplying (ii) by $f(a)$ and (i) by $\phi(a)$ we get

$\phi(b)f(a)-f(a)\phi(a)=(b-a)\phi'(c)f(a)$ …(iii)

$f(b)\phi(a)-f(a)\phi(a)=(b-a)f'(c)\phi(a)$ …(iv)

Subtracting (iv) from (iii), we get

$\phi(b)f(a)-f(b)\phi(a)=(b-a)[\phi'(c)f(a)-f'(c)\phi(a)]$

or $\begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix}=(b-a)\begin{vmatrix} f(a) & f'(c) \\ \phi(a) & \phi'(c) \end{vmatrix} \qquad \Rightarrow \qquad \lambda^2=1$ so sum of absolute value λ is $2.$

87.(2) $y=\tan^{-1}(\cot x)+\cot^{-1}(\tan x)$

$y=\frac{\pi}{2}-x+\frac{3\pi}{2}-x,\ \frac{\pi}{2}<x<\pi$

$\frac{dy}{dx}=-2$

$-\frac{dy}{dx}=2$

88.(0) $f'''(x)=0$

$f''(x)=c=-1$

$f'(x)=-x+d$

$d=3$

$f(x)=-\frac{x^2}{2}+3x+1$

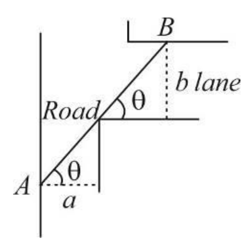
$f'(x)=-x+3$

89.(5) Let AB be the pole $AB=f(\theta)=a\sec\theta+b\cos ec\theta$

$f'(\theta)=0 \Rightarrow a\sec\theta\tan\theta=b\text{cosec}\theta\cot\theta \Rightarrow a\sin^3\theta=b\cos^3\theta \Rightarrow$

$\frac{\cos^2\theta}{a^{\frac{2}{3}}}=\frac{\sin^2\theta}{b^{\frac{2}{3}}}=\frac{1}{a^{\frac{2}{3}}+b^{\frac{2}{3}}}$

$\rightarrow \quad \frac{a}{\cos\theta}=\frac{a\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}}=a^{\frac{2}{3}}\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}}$



Similarly, $\frac{b}{\cos\theta}=b^{\frac{2}{3}}\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}}$

Maximum $AB=\frac{a}{\cos\theta}+\frac{b}{\sin\theta}=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad \Rightarrow \quad L=\left(64^{\frac{2}{3}}+27^{\frac{2}{3}}\right)^{\frac{3}{2}}=(16+9)^{\frac{3}{2}}=125$

90.(9) Put $a=\cos\theta$ and $b=\sin\theta$

Hence, $E=\frac{b+1}{a+b-2}=\frac{\sin\theta+1}{\sin\theta+\cos\theta-2}=f(\theta)$

For maxima/minima, $f'(\theta)=0$

$\Rightarrow \quad (\sin\theta+\cos\theta-2)(\cos\theta)-(\sin\theta+1)(\cos\theta-\sin\theta)=0$

$\Rightarrow \quad \sin\theta\cos\theta+\cos^2\theta-2\cos\theta-\sin\theta\cos\theta+\sin^2\theta-\cos\theta+\sin\theta=0$

$\Rightarrow \quad \cos^2\theta-2\cos\theta+\sin^2\theta-\cos\theta+\sin\theta=0$

$\Rightarrow \quad 1-3\cos\theta+\sin\theta=0$ $\quad\Rightarrow\quad 1+\sin\theta=3\cos\theta$

$\Rightarrow \quad (1+\sin\theta)^2=9\cos^2\theta=9-9\sin^2\theta$ $\quad\Rightarrow\quad 10\sin^2\theta+2\sin\theta-8=0$

$\Rightarrow \quad 5\sin^2\theta+\sin\theta-4=0$ $\quad\Rightarrow\quad 5\sin^2\theta+5\sin\theta-4\sin\theta-4=0$

$\Rightarrow \quad (\sin\theta+1)(5\sin\theta-4)=0$ $\quad\Rightarrow\quad \sin\theta=-1$ or $\sin\theta=\dfrac{4}{5}$

Now value of E, when $\sin\theta=-1$ is 0.

and value of E when $\sin\theta=\dfrac{4}{5}$ is $\dfrac{\sin\theta+1}{\sin\theta+\cos\theta-2}=\dfrac{\frac{4}{5}+1}{\frac{4}{5}+\frac{3}{5}-2}$ or $\dfrac{\frac{4}{5}+1}{\frac{4}{5}-\frac{3}{5}-2}$

i.e., $E=\dfrac{-9}{3}$ or $\dfrac{9}{-9}$

Hence, $E_{\min imum}=-3$ When $\sin\theta=\dfrac{4}{5}$ and $\cos\theta=\dfrac{3}{5}\Rightarrow u=-3$ Hence, $u^2=9.$

91.(9) Given, $f(x)=e^x.\int_0^x e^{-y}.f'(y)dy-(x^2-x+1)e^x$ ....(i)

Differentiating both the sides

$$f'(x)=e^x.e^{-x}.f'(x)+e^x\int_0^x e^{-y}.f'(y)dy-\left[(x^2-x+1)e^x+e^x(2x-1)\right]$$

$$f'(x)=f'(x)+\int_0^x e^{x-y}.f'(y)dy-e^x(x^2+x) \quad\Rightarrow\quad 0=f(x)+(x^2-x+1)e^x-e^x(x^2+x)$$

[Substituting $\int_0^x e^{x-y}f'(y)dy=f(x)+(x^2-x+1)e^x$ from (i)]

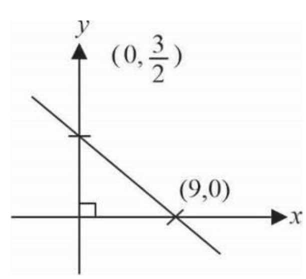
$f(x)=e^x(x^2+x-x^2+x-1)$; $f(x)=e^x.(2x-1)\Rightarrow f(1)=e.$

Also, $f'(x)=e^x.2+(2x-1)e^x\Rightarrow f'(1)=2e+e=3e$ and $f''(x)=2e^x+(2x-1)e^x+2e^x$

$\Rightarrow \quad f''(1)=2e+e+2e=5e$ Hence, $f(1)+f'(1)+f''(1)=9e\Rightarrow k=9$

92.(3) $(x^2-11)(y+1)=-4=-2\times 2$

On comparing we get $(x_0,y_0)\equiv(3,1)$



$$y'=\frac{8x}{(x^2-11)^2}$$

$y'|_{x=3}=6 \quad\therefore\quad m_N=\dfrac{-1}{6}$

Equation of normal $y-1=\dfrac{-1}{6}(x-3)$

$x+6y=9 \quad\therefore$ Area $=\dfrac{1}{2}\times 9\times\dfrac{3}{2}=\dfrac{27}{4}$

93.(48) Given, $f''(x) \geq -2 \forall x \in [-3,3]$

$\therefore \quad \int_{-3}^{3} f''(x)dx \geq -2\int_{-3}^{3} dx \Rightarrow f'(x)]_{-3}^{3} \geq -12 \Rightarrow -12 \geq -12$

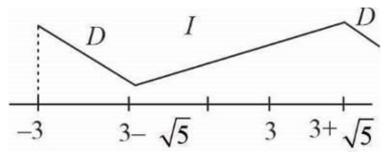
$\therefore \quad f''(x) = -2 \forall x \in [-3,3]$ or apply LMVT in $y = f'(x)$ in $[-3,3]$, we get $f''(x) = -2$

Now, $f'(x) = -2x + C$

$f'(3) = C - 6 = 0 \Rightarrow C = 6$

$f'(x) = 6 - 2x \Rightarrow f(x) = 6x - x^2 + C$

$f(0) = C = -4$



Hence, $f(x) = 6x - x^2 - 4$

Now, $g(x) = \int_{0}^{x} f(t)dt = \int_{0}^{x} \left(-t^2 + 6t - 4\right)dt \quad \therefore \quad g(x) = \dfrac{-x^3}{3} + 3x^2 - 4x$ in $[-3, 3]$

$g'(x) = -x^2 + 6x - 4$

$g(-3) = 48$ which is maximum value in $[-3,3]$.

94.(0) Given, $9 + f''(x) + f'(x) = x^2 + f^2(x)$ or $9 + \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 + y^2$

As, P be the point of maxima of $f(x)$ so $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0 \Rightarrow x^2 + y^2 = 9 + \dfrac{d^2y}{dx^2}$

So, $P(x,y) = 9 + \dfrac{d^2y}{dx^2}$

So, $P(x,y)$ will lie inside the circle $x^2 + y^2 = 9$. Hence, no tangent is possible.

INTEGRAL CALCULUS-1

1.(C) $f(x)=\lim_{n\to\infty} e^{x\tan(1/n)\log(1/n)}$

But $\lim_{n\to\infty}\tan(1/n)\log 1/n=\lim_{n\to\infty}\left[-\frac{\log n}{n}\frac{\tan 1/n}{1/n}\right]=0$

So $f(x)=e^0=1$. Hence

$$\int\frac{f(x)}{\sqrt[3]{\sin^{11}x\cos x}}dx=\int\frac{dx}{\sqrt[3]{\sin^{11}x\cos x}}=\int\frac{1+t^2}{\left(t^{11/3}\right)}dt\left(t=\tan x\right)$$

$$\int\left(t^{-11/3}+t^{-5/3}\right)dt=-\frac{3}{8}t^{-8/3}-\frac{3}{2}t^{-2/3}+C=-\frac{3}{8}\frac{(1+4\tan^2 x)}{\tan^2 x\sqrt[3]{\tan^2 x}}+C$$

2.(D) Let $g_n(x)=1+x^2+x^4+........+x^{2n}=\frac{x^{2n+2}-1}{\left(x^2-1\right)}$

So $2x+4x^3+........+2nx^{2n-1}=h_n(x)=g_n'(x)=\frac{2x\left(nx^{2n+2}-(n+1)x^{2n}+1\right)}{\left(x^2-1\right)^2}$

Now, $f(x)=\lim_{n\to\infty}h_n(x)=\frac{2x}{\left(x^2-1\right)^2}$ as 0 < x < 1

Thus $\int f(x)dx=\int\frac{2x}{\left(x^2-1\right)^2}dx=-\frac{1}{x^2-1}$ +C

3.(B) The given integral is $I=\int\frac{(\sec\theta-1)\sqrt{\sec\theta}}{1+\sec^2\theta}\tan\theta d\theta=\int\frac{(\sec\theta-1)\sec\theta\tan\theta}{\left(1+\sec^2\theta\right)\sqrt{\sec\theta}}d\theta$

Put, $\sqrt{\sec\theta}=t\Rightarrow\frac{1}{2\sqrt{\sec\theta}}.\sec\theta\tan\theta d\theta=dt$

$$I=\int\frac{\left(t^2-1\right)}{1+t^4}.2dt=2\int\frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt=2\int\frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)-2}$$

Put $t+\frac{1}{t}=u\Rightarrow\left(1-\frac{1}{t^2}\right)dt=du$

$$I=2\int\frac{du}{u^2-2}=\frac{2}{2\sqrt{2}}\log_e\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|+C=\frac{1}{\sqrt{2}}\log_e\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|+C$$

4.(A) $f(x)=\lim\limits_{n\to\infty}\dfrac{x^{2n}-1}{x^{2n}+1}=-1(0<x<1)$, so

$$\int \sin^{-1}x\big(f(x)\big)dx=-\int \sin^{-1}xdx=-\left[x\sin^{-1}x+\sqrt{1-x^2}\right]+C$$

5.(A) $I_n=\int \cot^n xdx=\int \cot^{n-2}x\cot^2 xdx=\int \cot^{n-2}x\left(\operatorname{cosec}^2 x-1\right)dx$

Thus $\quad I_n+I_{n-2}=-\dfrac{\cot^{n-1}x}{n-1}$

$$I_0+I_1+2(I_2+.......+I_8)+I_9+I_{10}=(I_0+I_2)+(I_3+I_1)+(I_4+I_2)+(I_5+I_3)+(I_6+I_4)$$
$$+(I_7+I_5)+(I_8+I_6)+(I_9+I_7)+(I_{10}+I_8)$$

$$=-\left(\frac{\cot x}{1}+\frac{\cot^2 x}{2}+........+\frac{\cot^9 x}{9}\right)(\text{using }(i))$$

$$-\sum_{k=1}^{9}\frac{\cot^k x}{k}$$

6.(B) Let $I=\dfrac{\left(\sin^{3/2}\theta+\cos^{3/2}\theta\right)d\theta}{\sqrt{\sin^3\theta\cos^3\theta\sin(\theta+\alpha)}}=\int\dfrac{\sin^{3/2}\theta d\theta}{\sqrt{\sin^3\theta\cos^3\theta\sin(\theta+\alpha)}}+\int\dfrac{\cos^{3/2}\theta d\theta}{\sqrt{\sin^3\theta\cos^3\theta\sin(\theta+\alpha)}}$

Put $(\cos\alpha\tan\theta)+\sin\alpha=t^2$ in the first integral and $\cos\alpha+\cot\theta\sin\alpha=u^2$ in second integral

$\Rightarrow \quad \sec^2\theta d\theta=\dfrac{2tdt}{\cos\alpha}$ and $\operatorname{cosec}^2\theta d\theta=-\dfrac{2udu}{\sin\alpha}$

$$I=\int\frac{2tdt}{(\cos\alpha)t}-\int\frac{2udu}{(\sin\alpha)u}=\frac{2}{\cos\alpha}t-\frac{2}{\sin\alpha}u+c$$

$$=\frac{2}{\cos\alpha}\sqrt{\cos\alpha\tan\theta+\sin\alpha}-\frac{2}{\sin\alpha}\sqrt{\cos\alpha+\cot\theta\sin\alpha}+c$$

7.(D) Let $I=\int\dfrac{\cos^2 x+\sin 2x}{(2\cos x-\sin x)^2}dx=\int\dfrac{(\cos x+2\sin x)\cos x}{(2\cos x-\sin x)^2}dx$

Integrating by parts, taking *cos x* as the first and $\dfrac{(\cos x+2\sin x)}{(2\cos x-\sin x)^2}$ as the second function,

we have $=\cos x\left\{\dfrac{1}{2\cos x-\sin x}\right\}-\int\dfrac{-\sin xdx}{2\cos x-\sin x}=\cos x\left\{\dfrac{1}{2\cos x-\sin x}\right\}+\int\dfrac{\sin xdx}{2\cos x-\sin x}$

$$=\frac{\cos x}{(2\cos x-\sin x)}+\int\frac{\frac{-1}{5}(2\cos x-\sin x)-\frac{2}{5}(-2\sin x-\cos x)}{2\cos x-\sin x}dx$$

8.(A) As m = 9 > 0, hence we can substitute $\sqrt{9x^2+4x+6}=u\pm 3x$

9.(D) Here as per notations given, we can substitute $\sqrt{1+x^2}=(u-x) as\ m=1>0$ and p = 4 > 0

$$\Rightarrow \qquad I=\int \frac{u^{15}}{u}du=\int u^{14}du=\frac{1}{15}u^{15}+c$$

10.(D) Here m = – 1 < 0

P = – 2<0

Also $\quad -x^2+3x-2=-(x-1)(x-2)$

We can use case 3

$\Rightarrow \quad$ Putting $\sqrt{-x^2+3x-2}=u(x-2)$ or $(x-1)u\ or\, u(1-x)$

11.(B) $I_n=\int \frac{dx}{\left(x^2+a^2\right)^n}=\frac{1}{\left(x^2+a^2\right)^n}.x-\int \frac{(-n)2x}{\left(x^2+a^2\right)^{n+1}}x.dx$ [integrating by parts using I as second function]

$$I_n=\frac{x}{\left(x^2+a^2\right)^n}+2n\int \frac{x^2+a^2-a^2}{\left(x^2+a^2\right)^{n+1}}dx=\frac{x}{\left(x^2+a^2\right)^n}+2n\left(I_n-a^2I_{n+1}\right)$$

12.(C) Let $I=\int \frac{\left(ax^2-b\right)dx}{x\sqrt{c^2x^2-\left(ax^2+b\right)^2}}=\int \frac{\left(ax^2-b\right)dx}{x^2\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}=\int \frac{\left(a-\frac{b}{x^2}\right)dx}{\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}$

Put $ax+\frac{b}{x}=c\sin\theta$

Then $I=\int \frac{c\cos\theta d\theta}{c\cos\theta}=\int d\theta=\theta+k, k$ is constant of integration $=\sin^{-1}\left(\frac{ax^2+b}{cx}\right)+k$

13.(C) $\because\ f'(x)=\frac{1}{1+\cos x}=\frac{1}{2\cos^2(x/2)}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

Integrating both sides with respect to *x*, we have $f(x)=\tan\left(\frac{x}{2}\right)+c$

$\therefore \qquad f(0)=0+c=3$ then $f(x)=\tan\left(\frac{x}{2}\right)+3 \qquad \therefore \qquad f\left(\frac{\pi}{2}\right)=\tan\left(\frac{\pi}{4}\right)+3=4$

Now, $3+\frac{\pi}{4}=3+\frac{22}{7\times 4}=3+\frac{11}{14}=\frac{53}{14}$

And $3+\frac{\pi}{4}=3+\frac{22}{14}=3+\frac{11}{7}$

14.(A) $\int \frac{x^2+1}{x^4+1}dx=\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

Put $x-\frac{1}{x}=t\Rightarrow\left(1+\frac{1}{x^2}\right)dx=dt \qquad \therefore\ I=\int \frac{dt}{t^2+2}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+c$

15.(C) $I=\int \frac{x^2-1}{\left(x^4+3x^2+1\right)\tan^{-1}\left(x+\frac{1}{x}\right)}dx=\int \frac{1-\frac{1}{x^2}}{\left(x^2+\frac{1}{x^2}+3\right)\tan^{-1}\left(x+\frac{1}{x}\right)}dx$

Put $x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx=dt$ and $x^2+\frac{1}{x^2}+2=t^2$ $\therefore I=\int \frac{dt}{\left(t^2+1\right)\tan^{-1}t}=\ln\left|\tan^{-1}t\right|+C$

16.(B) $I=\int \frac{x^4-2}{x^2\sqrt{x^4+x^2+2}}dx=\int \frac{x^4-2}{x^3\sqrt{x^2+1+\frac{2}{x^2}}}dx$

Put $x^2+\frac{2}{x^2}+1=t \Rightarrow \left(x-\frac{2}{x^3}\right)dx=\frac{dt}{2}$, we get $I=\int \frac{dt}{2\sqrt{t}}=\sqrt{t}+C$

17.(C) $I=\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx=\int \frac{x^2-1}{(x+1)^2\sqrt{x^3+x^2+x}}dx$

Put $x+\frac{1}{x}+1=t^2 \Rightarrow \left(1-\frac{1}{x^2}\right)dx=2tdt$

$\therefore I=\int \frac{2tdt}{\left(t^2+1\right)t}=2\int \frac{dt}{t^2+1}$

18.(D) Divide numerator and demoninator by x^{10} we get $I=\int \frac{5x^{-6}+4x^{-5}}{\left(1+x^{-4}+x^{-5}\right)^2}dx$

19.(A) $f(xy)=f(x)f(y)$ ....**(i)**

Put $x=y=1$, we get : $f(1)=f^2(1) \Rightarrow f(1)=1$

Differentiating w.r.t. x (partially), we get : $yf'(xy)=f'(x).f(y)$

Putting $x=1$, $yf'(y)=f'(1).f(y) \Rightarrow \frac{f(y)}{f'(y)}=\frac{y}{f'(1)}$

$\therefore \quad \int \frac{f(x)}{f'(x)}dx=\int \frac{x}{f'(1)}dx=\frac{1}{f'(1)}\left(\frac{x^2}{2}+c\right)$

$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{1+h+hg(h)-1}{h}=1$

20.(B) We have $\int \frac{dx}{x^2\left(x^n+1\right)^{(n-1)/n}}=\int \frac{dx}{x^2x^{n-1}\left(1+\frac{1}{x^n}\right)^{(n-1)/n}}=\int \frac{dx}{x^{n+1}\left(1+x^{-n}\right)^{(n-1)/n}}$

Put $1+x^{-n}=t$

$\therefore \quad -nx^{-n-1}dx=dt$ or $\frac{dx}{x^{n+1}}=-\frac{dt}{n}$ $=-\frac{1}{n}\int t^{1/n-1}dt=-\frac{1}{n}.\frac{t^{1/n-1+1}}{1/n-1+1}+c$

21.(C) $I_{4,3} = \int \cos^4 x \sin 3x dx$

Integrating by parts, we have $I_{4,3} = -\frac{\cos 3x \cos^4 x}{3} - \frac{4}{3}\int \cos^3 x \sin x \cos 3x dx$

But $\sin x \cos 3x = -\sin 2x + \sin 3x \cos x$ so, $I_{4,3} = -\frac{\cos 3x \cos^4 x}{3} + \frac{4}{3}\int \cos^3 x \sin 2x dx - \frac{4}{3}\int \cos^4 x \sin 3x dx + C$

Therefore, $\frac{7}{3} I_{4,3} - \frac{4}{3} I_{3,2} = -\frac{\cos 3x \cos^4 x}{3} + C$

22.(AB) Put $t = \sin^2 x$

The integral reduces to $I = \frac{1}{2}\int e^t (2-t) dt = \frac{3}{2}e^t - \frac{te^t}{2} + c = \frac{1}{2}e^{\sin^2 x}\left(3 - \sin^2 x\right) + c = e^{\sin^2 x}\left(1 + \frac{1}{2}\cos^2 x\right) + c$

23.(AB) $I = \int \frac{1}{\sin^2 x - 4\cos^2 x} dx = \int \frac{\sec^2 x}{\tan^2 x - 4} dx$

Let $\tan x = t, \sec^2 x dx = dt$ $= \int \frac{dt}{t^2 - 4} = \int \frac{dt}{(t+2)(t-2)} = \int \frac{1}{4}\left[\frac{1}{t-2} - \frac{1}{t+2}\right] dt$

24.(BD) $I = \int \sqrt{\operatorname{cosec} x + 1}\, dx = \int \frac{\cot x}{\sqrt{\operatorname{cosec} x - 1}} dx$

Put $\operatorname{cosec} x = t$

$$I = -\int \frac{dt}{t\sqrt{t-1}}$$

Put $t - 1 = u^2$

So that $I = -\int \frac{2u du}{u\left(u^2 + 1\right)} = -2\tan^{-1} u + c$

25.(AC) $f(x) = \int \left(x + \sqrt{x^2 + 1}\right) dx = \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2 + 1} + \log\left|x + \sqrt{x^2 + 1}\right| + c$

Putting x= 0 in above equation, we get $f(0) = c \Rightarrow c = \frac{-\left(1 + \sqrt{2}\right)}{2}$

26.(ABD) $F(x) = \int \frac{1}{4 - 3\cos^2 x + 5\sin^2 x} dx = \int \frac{1}{9 - 8\cos^2 x} dx$

$= \int \frac{\sec^2 x}{9\sec^2 x - 8} dx = \int \frac{\sec^2 x}{1 + 9\tan^2 x} dx = \frac{1}{3}\tan^{-1}(3\tan x) + c \Rightarrow g(x) = 3\tan x$

$g\left(\frac{\pi}{4}\right) = 3$

27.(AC) We have $I = \int \frac{\sin x - \cos x}{(\sin x + \cos x)\sqrt{\sin x \cos x + \sin^2 x \cos^2 x}} dx = \operatorname{cosec}^{-1}(g(x)) + c$

$$I = \int \frac{-\cos 2x}{(1+\sin 2x)\sqrt{\sin x \cos x + \sin^2 x \cos^2 x}} dx$$

Let $1 + \sin 2x = y \quad \Rightarrow \quad I = \int \frac{dy}{y\sqrt{y^2 - 1}} \Rightarrow I = \operatorname{cosec}^{-1}(y) + c$

28.(AB) We have been given that $I = \int \frac{x\sqrt{2\sin(x^2+1) - \sin 2(x^2+1)}}{2\sin(x^2+1) + \sin 2(x^2+1)} dx \quad \Rightarrow I = \int x \tan \frac{(x^2+1)}{2} dx$

29.(ABC) $f'(x) = b^2 + (a-1)b + 2 - \sin^2 x - \cos^4 x$

$b^2 + (a-1)b + 2 - 1$ for minimum value $(a-1)^2 - 4 < 0 \Rightarrow a \in (-1, 3)$

30.(BCD) Let $I = \int \operatorname{cosec}^2 x \log(\cos x + \cos 2x)\, dx$

$$= -\cot x \log\left(\cos x + \sqrt{\cos 2x}\right) + \int \frac{\cot x\left(-\sin x + \frac{-2\sin 2x}{2\sqrt{\cos 2x}}\right)}{\cos x + \sqrt{\cos 2x}} dx$$

$$= -\cot x \log\left(\cos x + \sqrt{\cos 2x}\right) - \int \frac{\cot x \sin x\sqrt{\cos 2x} + \sin 2x}{\sqrt{\cos 2x}\left(\cos x + \sqrt{\cos 2x}\right)} dx$$

Multiplying by $(\cos 2x)^{1/2}\left\{\cos x + (\cos 2x)^{1/2}\right\}$ in numerator and denominator, we get

$$I = -\cot x \log\left(\cos x + \sqrt{\cos 2x}\right) - \int \frac{\cos x}{\sin^2 x\sqrt{\cos 2x}} dx + \int \cot^2 x dx$$

Now put $t = \frac{1}{u}$

$\Rightarrow \quad \int \frac{\cos x}{\sin^2 x\sqrt{\cos 2x}} dx = -\sqrt{\operatorname{cosec}^2 x - 2} + c_1$ (where c_1 is integration constant)

31.(ABC) $\int \frac{\sin 2x}{8\sin^2 x + 17\cos^2 x} dx = \int \frac{2\sin x.\cos x}{8\sin^2 x + 17 - 17\sin^2 x} dx = \int \frac{\sin 2x}{17 - 9\sin^2 x} dx$

Let $\sin^2 x = t \Rightarrow \sin 2x dx = dt$

$$\int \frac{dt}{17 - 9t} = -\frac{1}{9}\log\left(17 - 9\sin^2 x\right) + c$$

32.(BC) $I = \int \sec^3 2\theta d\theta = \int \sec^2 2\theta \sec 2\theta d\theta = \frac{\tan 2\theta}{2}\sec 2\theta - \int \frac{2\tan^2 2\theta \sec 2\theta}{2} d\theta$

$$\Rightarrow 2I = \frac{\tan 2\theta}{2}\sec 2\theta + \frac{1}{2}\ln[\sec 2\theta + \tan 2\theta] + c$$

33.(CD) $I_1=\int 2^x dx=\frac{2^x}{\log 2}=p(x)+c_1$; $I_2=\int 2^{-x}dx=-\frac{2^{-x}}{\log 2}=m(x)+c_1$

34.(ABD) According to given problem $\int\frac{dx}{p^2\sin^2 x+r^2\cos^2 x}=\int\frac{\sec^2 xdx}{r^2+p^2\tan^2 x}$

Let *tanx* = t

$$\Rightarrow \quad \int\frac{dt}{r^2+p^2t^2}=\frac{1}{pr}\tan\left(\frac{pt}{r}\right)+c=\frac{1}{pr}=12$$

$$\frac{p}{r}=3\Rightarrow p^2=36\Rightarrow p=\pm 6\Rightarrow r=\pm 2$$

We know that $-\sqrt{p^2+r^2}\le p\sin x+r\cos x\le\sqrt{p^2+r^2}$

$$-40\le 6\sin x\pm 2\cos x\le 40\ \forall\ x$$

35.(ABC) $I=\int\frac{dx}{\left(1+\sqrt{x}\right)^8}$ Let $1+\sqrt{x}=t\Rightarrow\frac{1}{2}x^{-1/2}dx=dt\Rightarrow\frac{dx}{\sqrt{x}}=2dt$

36.(BC) $I=\int\frac{\sin\left(\overline{x-\alpha}+\alpha\right)}{\sin(x-\alpha)}dx\Rightarrow I=\int\{\cos\alpha+\sin\alpha\cot(x-\alpha)\}dx \quad =x\cos\alpha+\sin\alpha\log\left(\sin(x-\alpha)\right)+c$

37.(BC) $I=\int\left(4\sin\theta-4\sin^3\theta\right)e^{\sin\theta}\cos\theta d\theta=\left(A\sin^3\theta-B\sin^2\theta+C\sin\theta+D\cos\theta+B+E\right)e^{\sin\theta}+F$

Taking $\sin\theta=t$

$$I=\int\left(4t-4t^3\right)e^t dt=\underbrace{\left(At^3-Bt^2+Ct+D\cos\theta+E\right)e^t}_{f(t)}+F$$

D = 0 as $f(t)$ is a polynomial in t.

$\left(4t-4t^3\right)e^t=\left(At^3+Bt^2+Ct+E\right)e^t+\left(3t^2-2Bt+C\right)e^t\Rightarrow A-4$; 3A = B $\Rightarrow B=-12$

38.(CD) $\int\frac{(x-1)dx}{(x+1)\sqrt{x^3+x^2+x}}$

Multiplying numerator and denominator by $(x+1)$, we get: $\int\frac{\left(x^2-1\right)}{\left(x^2+2x+1\right)\sqrt{x^3+x^2+x}}dx=\int\frac{1-\frac{1}{x^2}}{\left(x+2+\frac{1}{x}\right)\sqrt{x+1+\frac{1}{x}}}$

Either substitute $x+1+\frac{1}{x}=t^2$ to get (A) or substituting $x+2+\frac{1}{x}=t^2$ to get (C)

39.(AD) Let $3x=\cos\theta\ \Rightarrow-\frac{1}{3}\int\frac{\frac{\cos\theta}{3}+\theta^2}{\sin\theta}\sin\theta d\theta=-\frac{1}{3}\int\left(\frac{1}{3}\cos\theta+\theta^2\right)d\theta=-\frac{1}{9}\sin\theta-\frac{\theta^3}{9}+c$

40.(AD) Let $I=\int\frac{\sec x(2+\sec x)}{(1+2\sec x)^2}dx=\int\frac{2\cos x+1}{(2+\cos x)^2}dx\Rightarrow I=\int\frac{\left(\cos x(\cos x+2)+\sin^2 x\right)}{(2+\cos x)^2}dx$

41.(BC) $f(x)+(\sin x)f(x+\pi)=\sin^2 x.$ ...(1)

$x \to x+\pi$

$f(x+\pi)+(\sin(x+\pi))f(x+2\pi)=\sin^2(x+\pi)$

$f(x+\pi)-(\sin x)f(x)=\sin^2 x$ ... (2)

from (1) & (2) $f(x)=\dfrac{\sin^2 x(1-\sin x)}{1+\sin^2 x}$

Now $\displaystyle\int \frac{\sin^2 x(1-\sin x)}{(1+\sin^2 x)}dx=\int\frac{(1+\sin^2 x-1)(1-\sin x)dx}{(1+\sin^2 x)}=x-\int\frac{(1-\sin x)+\sin x+\sin^3 x}{(1+\sin^2 x)}dx$

$\displaystyle = x-\int\frac{dx}{1+\sin^2 x}+\int\frac{\sin x dx}{(2-\cos^2 x)}-\int \sin x\ dx$

$\displaystyle = x-\frac{1}{2}\int\frac{\sec^2 xdx}{\tan^2 x+\frac{1}{2}}+\int\frac{\sin xdx}{(2-\cos^2 x)}-\int\sin x\ dx = x-\frac{1}{2}.\sqrt{2}\tan^{-1}(\sqrt{2}\tan x)-\frac{1}{2\sqrt{2}}\ln\frac{(\sqrt{2}+\cos x)}{\sqrt{2}-\cos x}+\cos x+C$

42.(AC) $\displaystyle\int\frac{dx}{x(x+1)(x+2).........(x+n)}$

Consider $\displaystyle\frac{1}{x(x+1)(x+2)........(x+n)}=\frac{1}{n!}\left[\frac{1}{x}-\frac{n}{1.(x+1)}+\frac{n(n-1)}{1.2(x+2)}-\frac{n(n-1)(n-2)}{1.2.3(x+3)}.......+\frac{(-1)^n.n(n-1)(n-2)....1}{(1.2.3.....n)(x+n)}\right]$

$\displaystyle\frac{1}{x(x+1)(x+2)........(x+n)}=\frac{1}{n!}\left[\frac{{}^nC_0}{x}-\frac{{}^nC_1}{(x+1)}+\frac{{}^nC_2}{(x+2)}.....+\frac{(-1)^n.{}^nC_n}{(x+n)}\right]$

So, $\displaystyle I=\int\frac{dx}{x(x+1)(x+2)......(x+n)}=\frac{1}{n!}\int\left(\frac{{}^nC_0}{x}-\frac{{}^nC_1}{(x+1)}+\frac{{}^nC_2}{(x+2)}.....+\frac{(-1)^n.{}^nC_n}{(x+n)}\right)dx$

$\displaystyle I=\frac{1}{n!}\left[{}^nC_0(\ln x)-{}^nC_1\ln(x+1)+{}^nC_2\ln(x+2)+..................+(-1)^n.{}^nC_n.\ln(x+n)\right]+c$

$\displaystyle\frac{1}{n!}\left[\sum_{r=0}^{n}(-1)^r.{}^nC_r\ln(x+r)\right]+c$ or $\displaystyle I=\frac{1}{n!}\left[\ln\left(\prod_{r=0}^{n}(x+r)^{(-1)^r.{}^nC_r}\right)\right]$

43.(ABC) Let $h(x)=f(x).g(x)$

Differentiate both sides w.r.t x

$h'(x)=f'(x).g(x)+f(x)g'(x)$

But due to mistake $h'(x)=f'(x).g'(x)$

So, $f'(x).g'(x)=f'(x).g(x)+f(x)g'(x)$

$3x^2.g'(x)=3x^2.g(x)+x^3.g'(x)$

$\displaystyle\int\frac{g'(x)}{g(x)}=\int\frac{-3}{(x-3)}\Rightarrow g(x)=\frac{9}{(x-3)^3}$

44.(ACD)

$$I=\begin{bmatrix}1&0\\0&1\end{bmatrix}, A=\begin{bmatrix}x&x\\x&x\end{bmatrix},\quad A^2=\begin{bmatrix}2x^2&2x^2\\2x^2&2x^2\end{bmatrix}$$

$$A^3=\begin{bmatrix}4x^3&4x^3\\4x^3&4x^3\end{bmatrix}\ldots\ldots\ldots\ldots\ldots\ldots$$

So $I+A+\dfrac{A^2}{2!}+\dfrac{A^3}{3!}+\ldots\ldots\ldots\ldots=\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}x&x\\x&x\end{bmatrix}+\dfrac{1}{2!}\begin{bmatrix}2x^2&2x^2\\2x^2&2x^2\end{bmatrix}$

$$\begin{bmatrix}\dfrac{f(x)}{2}&\dfrac{g(x)}{2}\\\dfrac{g(x)}{2}&\dfrac{f(x)}{2}\end{bmatrix}=\begin{bmatrix}1+x+\dfrac{2x^2}{2!}+\dfrac{4x^2}{3!}\ldots\ldots\ldots & x+\dfrac{2x^2}{2!}+\dfrac{4x^2}{3!}+\ldots\ldots\ldots\ldots\\ x+\dfrac{2x^2}{2!}+\dfrac{4x^3}{3!}+\ldots\ldots\ldots\ldots & 1+x+\dfrac{2x^2}{2!}+\dfrac{4x^3}{3!}+\ldots\ldots\end{bmatrix}$$

So $\qquad f(x)=2+(2x)+\dfrac{(2x)^2}{2!}+\dfrac{(2x)^3}{3!}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f(x)=1+e^{2x}\Rightarrow \qquad f(x)\in(1,\infty)\in R^+$

Similarly $\qquad g(x)=(2x)+\dfrac{(2x)^2}{2!}+\dfrac{(2x^3)}{3!}+\ldots\ldots\ldots$

$\qquad g(x)=e^{2x}-1\Rightarrow g(x)\in(-1,\infty)\in R$

How $\displaystyle\int\frac{g(x)}{f(x)}dx=\int\frac{(e^{2x}-1)}{(e^{2x}+1)}dx=\int\frac{(e^x-e^{-x})}{(e^x+e^{-x})}dx=\ln\left(e^x+e^{-x}\right)+c$

& $\displaystyle\int(g(x)+1)\sin x dx=\int e^{2x}.\sin x dx=\frac{e^{2x}}{5}(2\sin x-\cos x)+c$

45.(AC) $\displaystyle I=\int\frac{(x^2-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^2(x+1)^2}dx$

$$I=\int\frac{\left(1-\dfrac{1}{x^2}\right)\sqrt{\left(x^2+\dfrac{1}{x^2}\right)+2\left(x+\dfrac{1}{x}\right)-1}}{\left(x+\dfrac{1}{x}+2\right)}dx$$

Let $x+\dfrac{1}{x}=t\Rightarrow\left(1-\dfrac{1}{x^2}\right)dx=dt$

$$I=\int\frac{\sqrt{(t^2-2)+2t-1}}{(t+2)}dt=\int\frac{\sqrt{t^2+2t-3}}{(t+2)}dt$$

$$I=\int\frac{(t^2+2t-3)dt}{(t+2)\sqrt{t^2+2t-3}}=\int\frac{t(t+2)}{(t+2)\sqrt{t^2+2t-3}}dt-3\int\frac{dt}{(t+2)\sqrt{t^2+2t}}$$

Let $I_1 = \int \frac{tdt}{\sqrt{t^2+2t-3}}, I_2 = \int \frac{dt}{(t+2)\sqrt{t^2+2t-3}}$

$$I_1 = \int \frac{tdt}{\sqrt{(t+1)^2-4}} = \int \frac{(t+1)dt}{(t+1)^2-4} - \int \frac{dt}{\sqrt{(t+1)^2-4}}$$

$I_1 = \sqrt{t^2+2t-3} - \ln\left|(t+1)+\sqrt{t^2+2t-3}\right|$ & $I_2 = \int \frac{dy}{\sqrt{1-2y-3y^2}}$ put $t+2 = \frac{1}{y}$

$I_2 = \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{y+1/3}{2/3}\right) = \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{5+t}{2+t}\right)$; Now $I = \sqrt{t^2+2t-3} - \ln\left|(t+1)+\sqrt{t^2+2t-3}\right| - \sqrt{3}\sin^{-1}\left(\frac{5+t}{2+t}\right)+c$

46.(ABCD)

$$f(x) = (x-1)(x+2)(x-3)(x-6)-100$$

$$f(x) = \left(x^2-4x+3\right)\left(x^2-4x-12\right)-100$$

$$f(x) = \left(x^2-4x\right)^2 - 9\left(x^2-4x\right)-136$$

$$f(x) = \left(x^2-4x+8\right)\left(x^2-4x-17\right)$$

So equation $f(x)=0$ has two real & two imaginary roots

Now $f(x) = \left(x^2-4x-17\right)\left(x^2-4x+8\right)$

$$f(x) = \left((x-2)^2-21\right)\left((x-2)^2+4\right)$$

$(f(x))_{\min} = (-21)(4) = -84$ at $x=2$

$$\frac{g(x)}{f(x)} = \frac{g(x)}{\left(x^2-4x-17\right)\left(x^2-4x+8\right)} = \frac{(Ax+B)}{\left(x^2-4x-17\right)} + \frac{(cx+D)}{\left(x^2-4x+8\right)}$$

clearly A, B and C must be zero.

$$\therefore \frac{g(x)}{\left(x^2-4x-17\right)\left(x^2-4x+8\right)} = \frac{D}{\left(x^2-4x+8\right)}$$

$$g(x) = D\left(x^2-4x-17\right)$$

$$g(-2) = D(4+8-17)$$

So $\frac{g(x)}{f(x)} = \frac{2\left(x^2-4x-17\right)}{\left(x^2-4x-17\right)\left(x^2-4x+8\right)}$

$\frac{g(x)}{f(x)} = \frac{2}{x^2-4x+8}$; $\quad \int \frac{g(x)}{f(x)}dx = \int \frac{2}{x^2-4x+8}dx = 2\int \frac{dx}{(x-2)^2+(2)^2}$

$$\int \frac{g(x)}{f(x)}dx = 2.\frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right)+c = \tan^{-1}\left(\frac{x-2}{2}\right)+c$$

47.(BC) $\ln(1-x) = -\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\right\}$

Replace x by x^2

$$\ln\left(1-x^2\right) = -\left\{x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \frac{x^8}{4} + ...\right\}$$

$$\ln\left(1-x^2\right) = -\left\{x^4 + \frac{x^6}{2} + \frac{x^8}{3} + \frac{x^8}{4} + ...\right\}$$

$$x^2\ln\left(1-x^2\right) = -\left\{x^4 + \frac{x^6}{2} + \frac{x^8}{3} + \frac{x^{10}}{4} + ...\right\}$$

$$\int x^2\ln\left(1-x^2\right)dx = -\left\{\frac{x^5}{1.5} + \frac{x^7}{2.7} + \frac{x^9}{2.7} + \frac{x^9}{3.9} + \frac{x^{11}}{4.11}........\right\}$$

Applying I.B.P with limits 0 to 1 for LHS

$$\left(\frac{x^3}{3}\ln\left(1-x^2\right)\right)_0^1 - \int_0^1 \frac{x^3}{3}.\frac{(-2x)}{\left(1-x^2\right)}dx$$

$$\left[\frac{x^3}{3}\ln\left(1-x^2\right)\right]_0^1 + \frac{2}{3}\left(\frac{-x^3}{3} - x + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right)_0^1$$

$$\frac{1}{3}\ln 2 + \frac{1}{3}\ln 2 - \frac{2}{3} - \frac{2}{9} + \lim_{x\to 1}\left(\frac{x^3}{3} - \frac{1}{3}\right)\ln(1-x)$$

$$\therefore\ \lim_{x\to 1}\left(x^3-1\right)\ln(1-x) = 0$$

So $\frac{2}{3}\ln 2 - \frac{8}{9} = RHS$

$$\therefore\ \frac{1}{1.5} + \frac{1}{2.7} + \frac{1}{3.9} + ... = \frac{2}{3}\ln 2 - \frac{8}{9}$$

48.(AC) $f(x+2y) = f(x)e^{2y} + f(2y)e^x + x^2\left(1-e^{2y}\right) + 4y^2\left(1-e^x\right) + 4xy$ ……(1)

Putting $x = y = 0$ in equation (1), we get $f(0) = 2f(0) \Rightarrow f(0) = 0$

Now, $f'(x) = \lim_{y\to 0}\frac{f(x+2y) - f(x)}{2y}$

$$= \lim_{y\to 0}\frac{f(x)e^{2y} + e^x f(2y) + x^2\left(1-e^{2y}\right) + 4y^2\left(1-e^x\right) + 4xy - f(x)}{2y}$$

$$= \lim_{y\to 0}\frac{f(x)\left(e^{2y}-1\right) + e^x\left(f(2y) - f(0)\right) - \left(e^{2y}-1\right)x^2 + 4y^2\left(1-e^x\right) + 4xy}{2y}$$

$$= f(x)\lim_{y\to 0}\frac{\left(e^{2y}-1\right)}{2y}+e^x\lim_{y\to 0}\frac{f(2y)-f(0)}{2y}-x^2\lim_{y\to 0}\frac{\left(e^{2y}-1\right)}{2y}+2x+\lim_{y\to 0}2y\left(1-e^x\right)$$

$$f'(x)=f(x)+e^x-x^2+2x$$

$$e^{-x}f'(x)-e^{-x}f(x)=1-x^2e^{-x}+2xe^{-x}$$

$$\frac{d}{dx}\left(e^{-x}f(x)\right)=\frac{d}{dx}\left(x+x^2e^{-x}\right)\Rightarrow e^{-x}f(x)=x+x^2e^{-x}+c$$

put $x=0$, we get $c=0$ Hence $f(x)=xe^x+x^2$.

49.(AB) $\int\left(\frac{1}{1-x^8}\right)\left[\cos^{-1}\left(\frac{2x}{1+x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]dx$

$$=\int\left(\frac{1}{1-x^8}\right)\left[\frac{\pi}{2}-\sin^{-1}\frac{2x}{1+x^2}+\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]dx$$

$$=\frac{\pi}{8}\int\left(\frac{1}{1+x^2}+\frac{1}{1-x^2}\right)dx+\frac{\pi}{4}\int\frac{dx}{x^4+1}\;=\frac{\pi}{8}\left(\tan^{-1}x+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right)+\frac{\pi}{8}\int\left[\frac{x^2+1}{x^4+1}-\frac{x^2-1}{x^4+1}\right]dx$$

$$=\frac{\pi}{8}\left[\tan^{-1}x+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+\frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{x\sqrt{2}}-\frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right|\right]+c.$$

50.(ABC) $A+Bi=\int a^{ax}.e^{ibx}.dx=\int e^{(a+ib)x}dx=\frac{e^{(a+ib)x}}{a+ib}$

$\Rightarrow \quad (A+iB)(a+ib)=e^{(a+ib)x}=e^{ax}.e^{ibx}$ . . . (1)

Comparing the modulus of both sides, $\sqrt{A^2+B^2}\sqrt{a^2+b^2}=e^{ax}$

Squaring, we get the result $\left(A^2+B^2\right)\left(a^2+b^2\right)=e^{2ax}$

Comparing the arguments of both sides in equation (1), $\tan^{-1}\frac{B}{A}+\tan^{-1}\frac{b}{a}=bx.$

51.(CD) Let $I=\int\sqrt{\frac{\cos x-\cos^3 x}{\left(1-\cos^3 x\right)}}dx=\int\frac{\sqrt{\cos x}\left(\sqrt{1-\cos^2 x}\right)}{\sqrt{1-\left(\cos^{\frac{3}{2}}x\right)}}dx$

If $\cos^{\frac{3}{2}}x=p$, then $\left(-\frac{3}{2}\cos^{\frac{1}{2}}x\sin x\right)dx=dp$

$$I=-\frac{2}{3}\int\frac{dp}{\sqrt{1-p^2}},=-\frac{2}{3}\sin^{-1}\left(\cos^{\frac{3}{2}}x\right)+C=\frac{2}{3}\cos^{-1}\left(\cos^{\frac{3}{2}}x\right)+C_1$$

52.(CB) $I=\int \frac{dx}{x^{29}\left(1-\frac{6}{x^7}\right)}$

Let $\left(1-\frac{6}{x^7}\right)=p \Rightarrow \frac{42}{x^8}dx=dp$ and $x^7=\left(\frac{6}{1-p}\right)$

$I=\frac{1}{42}\int \frac{(1-p)^3}{(6)^3\,p}dp=\frac{1}{(42).(216)}\int \frac{1-p^3-3p+3p^2}{p}dp=\frac{1}{54432}\left[\ln p^6+9p^2-2p^3-18p\right]+c$

53.(BC) $\int e^{\tan^{-1}x}\left(1+x+x^2\right)d\left(\cot^{-1}x\right)$

$\int e^{\tan^{-1}x}\left(1+x+x^2\right)\left(\frac{-1}{1+x^2}\right)dx$

$\int e^{t}\left(1+\tan t+\tan^2 t\right)(-dt)$

Let $\tan^{-1}x=t$

$\frac{dx}{1+x^2}=dt$

$-\int e^t\left(\tan t+\sec^2 t\right)dt \Rightarrow -e^t\tan t+c. \quad \Rightarrow \quad -xe^{\tan^{-1}x}.+c$

54.(BD) $\int \frac{x^3}{\sqrt{1+2x^2}}dx=\frac{1}{m}\left(1+2x^2\right)^{1/2}\left(x^2-1\right)+2$

$\int \frac{x^2.x}{\sqrt{1+2x^2}}dx \qquad$ Put $1+2x^2=t^2 \Rightarrow 4xdx=2tdt$

$\frac{1}{2}\int \frac{\left(t^2-1\right)tdt}{2\quad t} \qquad 2xdx=tdt$

$\frac{1}{4}\int\left(t^2-1\right)dt \Rightarrow \frac{1}{4}\left(\frac{t^3}{3}-t\right)+c$

$\Rightarrow \qquad \frac{1}{4}t\left(\frac{t^2}{3}-1\right)+c\ ,\quad \frac{1}{4}t\frac{\left(t^2-3\right)}{3}+c$

$\Rightarrow \qquad \frac{t}{12}\left(t^2-3\right)+c$

$\Rightarrow \qquad \frac{\left(\sqrt{1+2x^2}\right)\left(2x^2+1-3\right)}{12}+c$

$\Rightarrow \qquad \frac{\left(\sqrt{1+2x^2}\right)\left(x^2-1\right)}{6}+c$

$m=6$

55.(AC) $I=\int \frac{\sin x+\sin^3 x}{\cos 2x}dx$

$$I=\int \frac{\left(1+\sin^2 x\right)\sin x}{\left(2\cos^2 x-1\right)}dx$$

$$I=\int \frac{\left(2-\cos^2 x\right)\left(-\sin x\right)}{\left(1-2\cos^2 x\right)}dx$$

Let $\cos x=t$

$-\sin x dx=dt$

$$I=\int \frac{\left(2-t^2\right)}{\left(1-2t^2\right)}dt$$

$$I=\int \frac{2}{\left(1-2t^2\right)}dt+\frac{1}{2}\int\left(\frac{\left(-2t^2\right)-1}{\left(1-2t^2\right)}\right)dt$$

$$I=\int \frac{2dt}{\left(1-2t^2\right)}+\frac{1}{2}\int \frac{\left(1-2t^2\right)-1}{\left(1-2t^2\right)}dt$$

$$I=\frac{3}{2}\int \frac{dt}{\left(1-2t^2\right)}+\frac{1}{2}\int dt$$

$$I=\frac{1}{2}t\frac{-3}{4}\int \frac{dt}{\left(t^2-\frac{1}{2}\right)}$$

$$I=\frac{t}{2}-\frac{3}{4\sqrt{2}}\ln\left|\frac{t-\frac{1}{\sqrt{2}}}{t+\frac{1}{\sqrt{2}}}\right|+c$$

$$I=\frac{\cos x}{2}-\frac{3}{4\sqrt{2}}\ln\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right|+c$$

56.(ABC)

Here, $f(x)+g(x)+h(x)$

$=\sqrt{f(x).g(x)}+\sqrt{g(x).h(x)}+\sqrt{h(x).f(x)}$

$$\Rightarrow \quad \frac{1}{2}\left\{\left(\sqrt{f(x)}-\sqrt{g(x)}\right)^2+\left(\sqrt{g(x)}-\sqrt{h(x)}\right)^2+\left(\sqrt{h(x)}-\sqrt{f(x)}\right)^2\right\}=0$$

$$\Rightarrow \quad f(x)=g(x)=h(x)$$

$$\therefore\ f(x)+g(x)-2h(x)=0$$

57.(ABD) We have $f(x)=x^2+\int_0^x e^{-t}f(x-t)dt$

$\Rightarrow \quad f(x)=x^2+\int_0^x e^{-(x-t)}f(x-(x-t))dt$

$\left[u\sin g\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

$\Rightarrow \quad f(x)=x^2+e^{-x}\int_0^x e^t f(t)dt$ ....(i)

On differentiating both the sides, get

$f'(x)=2x+e^{-x}\left[e^x f(x)\right]-e^{-x}\int_0^x e^t f(t)dt$ ....(ii)

[using Leibnitz rule]

$\Rightarrow \quad f(x)+f'(x)=x^2+2x+f(x) \quad \Rightarrow \quad f'(x)=x^2+2x$ ....(iii)

On integrating both the sides of equation (iii), we get $f(x)=\frac{x^3}{3}+x^2+C$

But $f(0)=0\Rightarrow C=0$ Hence, $f(x)=\frac{x^3}{3}+x^2$

58.(BC) It is given that $f'(x)=f(x)+\int_0^2 f(x)dx$

$\Rightarrow f'(x)=f(x)+c \qquad \left[\therefore c=\int_0^2 f(x)dx\right]$

$\Rightarrow \quad \frac{f'(x)}{f(x)+c}=1$

On integrating both the sides of above expression, $\log_e\{f(x)+c\}=k\Rightarrow f(x)+c=ke^x$

[k being constant of integration]

Since, $f(0)=\frac{4-e^2}{3}$ and $f(0)+c=k \qquad \therefore k=\frac{4-e^2}{3}+c\Rightarrow c=k-\left(\frac{4-e^2}{3}\right)$

From Eqs. (i) and (ii), We get $f(x)+k-\left(\frac{4-e^2}{3}\right)=k\left(e^x\right) \Rightarrow f(x)=k\left(e^k-1\right)+\left(\frac{4-e^2}{3}\right)$

Now, to find constant of integration k.

On integrating both the sides form 0 to 2, we get $\int_0^2 f(x)dx=k\int_0^2\left(e^x-1\right)dx+\frac{4-e^2}{3}\int_0^2 dx$

$\Rightarrow \quad c=k\left(e^2-2-1\right)+\frac{4-e^2}{3}(2) \Rightarrow c=\frac{2}{3}\left(4-e^2\right)+k\left(e^2-3\right)$

From Eqs. (ii) and (iv), we get $k-\left(\frac{4-e^2}{3}\right)=\frac{2}{3}\left(4-e^2\right)+k\left(e^2-3\right)\Rightarrow k=1$

On putting $k=1$ in Eq. (iii), we get $f(x)=\left(e^x-1\right)+\frac{4-e^2}{3}$ Hence, $f(x)=e^x-\frac{\left(e^2-1\right)}{3}$

59.(AD) We have, $f(x).f(y)+2=f(x)+f(y)+f(xy)$ ….(i)

On replacing $x, y \to 1$, we get $(f(1))^2+2=3f(1)$

$\Rightarrow \quad f^2(1)-3f(1)+2=0 \quad \Rightarrow \quad f(1)=2$ or 1

But $f(1)$ cannot be equal to one as $f(0)=1 \quad \Rightarrow \quad f(1)=2$ …(ii)

On replacing y by $1/x$ in Eq. (i) we get $f(x).f(1/x)+2=f(x)+f(1/x)+f(1)$

$\Rightarrow \quad f(x).f(1/x)+2=f(x)+f(1/x)+2$ [using Eq. (ii)

$\Rightarrow \quad f(x)=\dfrac{f(1/x)}{f(1/x)-1}$ and $f(1/x)=\dfrac{f(x)}{f(x)-1}$ ...(iii)

Now, $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{f(x+h)+\dfrac{f(1/x)}{1-f(1/x)}}{h}=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x+h).f(1/x)+f(1/x)}{h\{1-f(1/x)\}}$

$=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x+h)-f(1/x)-f\left(\dfrac{x+h}{x}\right)+2+f(1/x)}{h\{1-f(1/x)\}}$ [using Eq. (i)]

$=\lim\limits_{h\to 0}\dfrac{f\left(1+\dfrac{h}{x}\right)-2}{h\{f(1/x)-1\}}=\lim\limits_{h\to 0}\dfrac{f(1+h/x)-f(1)}{\dfrac{h}{x}.x\{f(1/x)-1\}}$ [using Eq. (ii)

$=\dfrac{f'(1)}{x\{f(1/x)-1\}}$ $\left[\because \text{ from eq. (iii)}, f(x).f\left(\dfrac{1}{x}\right)=\dfrac{f(x)f\left(\dfrac{1}{x}\right)}{\left\{f\left(\dfrac{1}{x}\right)-1\right\}\{f(x)-1\}}\right]$

$f'(x)=\dfrac{2\{f(x)-1\}}{x}\Rightarrow xf'(x)=2\{f(x)-1\}$

On integrating both the sides of the above expression, we get $xf(x)-\int f(x)dx=2\int f(x)dx-2x+\lambda$,

Where λ is the constant of integral

$\Rightarrow \quad 3\int f(x)dx=x\{2+f(x)\}-\lambda$ Hence, $3\int f(x)dx-x\{2+f(x)\}=\lambda$ (constant)

60. **[A-p, q]** **[B-r, s]** **[C-p]** **[D-p, q]**

(A) Let $I=\int\dfrac{2^x}{\sqrt{1-4^x}}dx=\dfrac{1}{\log 2}\int\dfrac{1}{\sqrt{1-t^2}}dt$

Putting $2^x=t, 2^x\log 2dx=dt$

$I=\dfrac{1}{\log 2}\sin^{-1}\left(\dfrac{t}{1}\right)+C=\dfrac{1}{\log 2}\sin^{-1}\left(2^x\right)+C$

(B) $$\int \frac{dx}{\left(\sqrt{x}\right)^2+\left(\sqrt{x}\right)^7}=\int \frac{dx}{\left(\sqrt{x}\right)^7\left(1+\frac{1}{\left(\sqrt{x}\right)^5}\right)}$$

Put $\frac{1}{\left(\sqrt{x}\right)^5}=y, \frac{dy}{dx}=-\frac{5}{2\left(\sqrt{x}\right)^7}$

$$\therefore\ I=\int -\frac{2dy}{5(1+y)}=-\frac{2}{5}\ln|1+y|+C=\frac{2}{5}\ln\left(\frac{1}{1+\frac{1}{\left(\sqrt{x}\right)^5}}\right)$$

(C) Add and subtract $2x^2$ in the numerator, then k = 1 and m = 1

(D) $$I=\int \frac{dx}{5+4\cos x}=\int \frac{dx}{5\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+4\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

Let $t=\tan\frac{x}{2}\Rightarrow 2dt=\sec^2\frac{x}{2}dt \Rightarrow I=\int \frac{2dt}{9+t^2}=\frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+C$

61. **[A-r]** **[B-s]** **[C-q]** **[D-p]**

(A) $$\int \frac{e^{2x}-1}{e^{2x}+1}dx=\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx=\log\left(e^x+e^{-x}\right)$$

(B) $$I=\int \frac{1}{\left(e^x+e^{-x}\right)^2}dx=\int \frac{e^{2x}}{\left(e^{2x}+1\right)^2}dx$$

Put $e^{2x}+1=t\Rightarrow 2e^{2x}dx=dt,\ we\,get \Rightarrow I=\frac{1}{2}\int \frac{1}{t^2}dt=-\frac{1}{2}\frac{1}{t}+c$

(C) $$I=\int \frac{e^{-x}}{1+e^x}dx=\int \frac{e^{-x}e^{-x}}{e^{-x}+1}dx$$

Put $e^{-x}+1=t\Rightarrow -e^{-x}dx=dt$

$$\Rightarrow\quad I=-\int \frac{(t-1)}{t}dt=\int\left(\frac{1}{t}-1\right)dt=\log t-t+C=\log\left(e^{-x}+1\right)-\left(e^{-x}+1\right)+C$$

(D) $$I=\int \frac{1}{\sqrt{1-e^{2x}}}dx=\int \frac{e^{-x}}{\sqrt{e^{-2x}-1}}dx$$

Put $e^{-x}=t\Rightarrow -e^{-x}dx=dt$

$$\Rightarrow\quad I=-\int \frac{1}{\sqrt{t^2-1}}dt=-\log\left[t+\sqrt{t^2-1}\right]+C=-\log\left[e^{-x}+\sqrt{e^{-2x}-1}+C\right]$$

62. **[A-s] [B-t] [C-r] [D-q]**

(A) $\int \frac{dx}{\sqrt{x}(x+9)}, x=t^2 = 2\int \frac{dt}{t^2+9} = \frac{2}{3} tan^{-1}\left(\frac{\sqrt{x}}{3}\right)+c$

(B) $\int e^x \left(1-cot\,x+cot^2\,x\right)dx = \int e^x\left(-cot\,x+cosec^2 x\right)dx = -e^x\,cot\,x+c$

(C) $\int \frac{sin^3 x+cos^3 x}{cos^2 x\,sin^2 x}dx = \int \left(tan\,x\,sec\,x+cot\,x\,cosec\,x\right)dx = sec\,x-cosecx+c$

(D) $\int \frac{dx}{1-cos\,x-sin\,x} = \int \frac{dx}{2sin^2\frac{x}{2}-2sin\frac{x}{2}cos\frac{x}{2}} = \frac{1}{2}\int \frac{cosec^2\frac{x}{2}}{1-cot\frac{x}{2}}dx = \ell n\left|1-cot\frac{x}{2}\right|+c$

63.(1) $f(x) = \int x^{\sin x}(1+x\cos x.\ln x+\sin x)dx$

If $F(x) = x^{\sin x} = e^{\sin x \ln x}$

$\therefore \quad f(x) = \int\left(F(x)+xF'(x)\right) = xF(x)+C$

$f(x) = x.x^{\sin x}+C$

64.(4) $g(x) = \int \frac{\cos x(\cos x+2)+\sin^2 x}{(\cos x+2)^2}dx = \int \underset{II}{\cos x}\underbrace{\frac{1}{(\cos x+2)}}_{I}dx + \int \frac{\sin^2 x}{\cos x+2}dx$

$= \frac{1}{\cos x+2}\sin x - \int \frac{\sin^2 x}{(\cos x+2)^2}dx + \int \frac{\sin^2 x}{(\cos x+2)^2}dx$

$g(x) = \frac{\sin x}{\cos x+2}+C$

$g(0) = 0 \Rightarrow C = 0$

65.(2) $k(x) = \int \frac{\left(x^2+1\right)dx}{\left(x^3+3x+6\right)^{1/3}}$

Put $\quad x^3+3x+6 = t^3 \Rightarrow 3\left(x^2+1\right)dx = 3t^2 dt$

$k(x) = \int \frac{t^2 dt}{t} = \frac{t^2}{2}+C$

$k(x) = \frac{1}{2}\left(x^3+3x+6\right)^{2/3}+C$

$k(-1) = \frac{1}{2}(2)^{2/3}+C \Rightarrow C = 0$

66.(3) $\frac{d}{dx}\left(A\ln|\cos x+\sin x-2|+Bx+C\right) = A\frac{\cos x-\sin x}{\cos x+\sin x-2}+B$

$\therefore 2 = A+B, -1 = -A+B, \lambda = -2B \qquad \therefore A = 3/2, B = 1/2, \lambda = -1$

67.(0) $\int\left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ln x dx$

Put $\left(\frac{x}{e}\right)^x=t$ Or $x\ln\left(\frac{x}{e}\right)=\ln t$

$\therefore$ $\left(x.\frac{1}{x/e}\frac{1}{e}+\ln\left(\frac{x}{e}\right)\right)dx=\frac{1}{t}dt$ $\therefore$ $(1+\ln x-\ln e)dx=\frac{1}{t}dt$

$\therefore$ $(\ln e+\ln x-\ln e)dx=\frac{1}{t}dt$

$\therefore$ $(\ln x)dx=\frac{1}{t}dt$ Or $I=\int\left(t+\frac{1}{t}\right)\frac{1}{t}dt=\int 1.dt+\int\frac{1}{t^2}dt=t-\frac{1}{t}+C$

68.(2) $I=\int\left(x^9+x^6+x^3\right)\left(2x^6+3x^3+6\right)^{1/3}dxI=\int x\left(x^8+x^5+x^2\right)\left(2x^6+3x^3+6\right)^{1/3}dx$

Let $t=2x^9+3x^6+6x^3$

$dt=18\left(x^8+x^5+x^2\right)dx=\frac{1}{18}\int t^{1/3}dt=\frac{1}{24}t^{4/3}+c$

69.(1) $I=\int\frac{dx}{(1+\sin x)}=\int\frac{dx}{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}=\int\frac{\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}+1}$

Let $t=\tan\frac{x}{2}\Rightarrow 2dt=\sec^2\frac{x}{2}dx=2\int\frac{dt}{(t+1)^2}=-\frac{2}{(1+t)}+c=-\frac{2}{\left(1+\tan\frac{x}{2}\right)}+c$

$=1-\frac{2}{\left(1+\tan\frac{x}{2}\right)}+c-1=\left(\frac{\tan\frac{x}{2}-1}{\tan\frac{x}{2}+1}\right)+c-1$

70.(10) $x+7=t^4$

$\int\frac{4t^3dt}{t^2-t}=4\int\frac{t^2-1+1}{t-1}dt=4\int(t+1)+\frac{1}{t-1}dt$

$=4\left(\frac{t^2}{2}+t+\ell n|t-1|\right)+c=2\sqrt{x+7}+4(x+7)^{1/4}+4\ell n\left|\sqrt[4]{x+7}-1\right|+c$

$P=2, Q=4, R=4$

71.(6) $\frac{x^2+2}{\left(x^2+1\right)\left(x^2+4\right)}=\frac{1}{3}\frac{1}{x^2+1}+\frac{2}{3}\frac{1}{x^2+4}$

$$\int \frac{x^2+2}{(x^2+1)(x^2+4)}dx = \frac{1}{3}tan^{-1}(x) + \frac{1}{3}tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= \frac{1}{3}tan^{-1}\left(\frac{3x}{2-x^2}\right) \Rightarrow K = \frac{1}{3}, m = 3, c = 2$$

72.(4) $$\int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}} = \int \left(\frac{x+2}{x-1}\right)^{3/4} \frac{dx}{(x+2)^2}$$

$$\frac{x-1}{x+2} = t \quad = \frac{1}{3}\int \frac{dt}{t^{3/4}} = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$$

73.(3015) $$I = \int \frac{x^{2009}}{(1+x^2)^{1006}}dx = \frac{1}{n}\left(\frac{x^2}{1+x^2}\right)^m + c$$

Put $1+x^2 = t$

$2xdx = dt$

So $I = \frac{1}{2}\int \frac{(t-1)^{1004}}{t^{1006}}dt = \frac{1}{2}\int \left(1-\frac{1}{t}\right)^{1004} . \left(\frac{1}{t^2}\right)dt$

Put $1-\frac{1}{t} = y \Rightarrow \frac{1}{t^2}dt = dy$

$I = \frac{1}{2}\int y^{1004}dy = \frac{y^{1005}}{2\times 1005} + c$

$I = \frac{1}{2}\int y^{1004}dy = \frac{y^{1005}}{2\times 1005} + c$

$\Rightarrow \quad m = 1005, \qquad n = 2010 \qquad$ So $\qquad m+n = 3015$

74.(521) Let $g(x) = \int \frac{f(x)dx}{x^2(x+1)^2}$

$g(x) = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2}\right)dx$; $\quad g(x) = A\ln|x| - \frac{B}{x} + C\ln|x+1| - \frac{D}{x+1} + E$

$\therefore \; g(x)$ is rational function

$\Rightarrow \quad A = C = 0$

Hence $g(x) = \int \left(\frac{B}{x^2} + \frac{D}{(x+1)^2}\right)dx$

Hence $f(x)$ must be of the form of $f(x) = B(x+1)^2 + Dx^2$

$f(0) = 1 \Rightarrow B = 1; \quad f(-1) = 4 \Rightarrow D = 4$

So $f(x) = 5x^2 + 2x + 1$

$f(10) = 521$

75.(8) $I = \int \frac{f'(x)g(x) - g'(x)f(x)dx}{\left(f(x)+g(x)\right)\sqrt{f(x)g(x)-g^2(x)}}$

$$I = \int \frac{\frac{\{f'(x)g(x)-g'(x)f(x)\}dx}{(g(x))^2}}{\left(\frac{f(x)}{g(x)}+1\right)\sqrt{\frac{f(x)}{g(x)}-1}}dx$$

Put $\frac{f(x)}{g(x)} - 1 = t^2$

$$\frac{\left(f'(x)g(x)-g'(x)f(x)\right)}{(g(x))^2}dx = 2tdt$$

$$I = \int \frac{2tdt}{\left(t^2+2\right)t} = \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+c$$

Hence $m = 2, \quad n = 2$ So $m^2 + n^2 = 8$

76.(3) $Det(A) = \begin{vmatrix} 5^x & 0 & 0 \\ 0 & 5^{5^x} & 0 \\ 0 & 0 & 5^{5^{5^x}} \end{vmatrix}$

$$Det\,(A) = 5^x.5^{5^x}.5^{5^{5^x}}$$

$$I = \int 5^x.5^{5^x}.5^{5^{5^x}}\,dx$$

Put $5^{5^{5^x}} = t$

$$5^{5^{5^x}}.5^{5^x}.5^x.dx = \frac{dt}{(\ln 5)^3}\;;\quad I = \int \frac{dt}{(\ln 5)^3} \Rightarrow I = \frac{t}{(\ln 5)^3}+c\;;\qquad I = \frac{5^{5^{5^x}}}{(\ln 5)^3}+c$$

77.(2.5) $I = \int \left\{2\cos^2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \ln\frac{(\cos\theta+\sin\theta)}{(\cos\theta-\sin\theta)}\right\}d\theta$

$$I = \int \left\{\left(2\cos^2\theta-1\right)\ln\frac{(\cos\theta+\sin\theta)}{(\cos\theta-\sin\theta)}\right\}d\theta$$

use $I.B.P$

$$I = \left(\ln\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right|\right).\frac{\sin 2\theta}{2} - \int \frac{2}{\cos 2\theta}.\frac{\sin 2\theta}{2}\,d\theta$$

$$I = \frac{\sin 2\theta}{2}\ln\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right| + \frac{1}{2}\ln\left|\cos 2\theta\right| + c$$

Hence $a = 2, b = \frac{1}{2}$

78.(4) $f'(x)+g'(x)=0 \Rightarrow \quad f(x)+g(x)=k$

Put $x=0$

$f(0)+g(0)=k \Rightarrow k=6$

$f(x)+g(x)=6 \quad \ldots(i)$

again

$f'(x)-g'(x)=2\left(f(x)-g(x)\right)$

$\dfrac{f'(x)-g'(x)}{f(x)-g(x)}=2$

By integrating both sides w.r.t. x

$\ln\left|f(x)-g(x)\right|=2x+c$

Put $x=0$

$\ln|5-1|=c \Rightarrow c=\ln 4$

so $\quad \ln\left|\dfrac{f(x)-g(x)}{4}\right|=2x.$

$f(x)-g(x)=4e^{2x} \quad \ldots(ii)$

From (i) & (ii)

$f(x)=3+2e^{2x} \quad \& \quad g(x)=3-2e^{2x}$

$f(\ln 2)=3+2e^{2\ln 2}=3+2e^{\ln 4}=11$

$g(\ln 3)=3-2e^{2\ln 3}=3-2e^{\ln 9}=3-18=-15$

$\left|f(\ln 2)+g(\ln 3)\right|=4$

79.(0.5) $I=\displaystyle\int \frac{\left(x^3+x+1\right)dx}{\left(x^2+x+1\right)\left(x^2-x+1\right)}=\int\left(\frac{x\left(x^2-x+1\right)+\left(x^2+x+1\right)-x}{\left(x^2+x+1\right)\left(x^2-x+1\right)}\right)dx$

$$I=\underbrace{\int \frac{(xdx)}{\left(x^2+x+1\right)}}_{I_1}+\underbrace{\int \frac{dx}{\left(x^2-x+1\right)}}_{I_2}-\underbrace{\int \frac{xdx}{\left(x^4+x^2+1\right)}}_{I_3}$$

For I_3, put $x^2=t \Rightarrow 2xdx=dt$

$$I_3=\frac{1}{2}\int \frac{dt}{t^2+t+1}=\frac{1}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$I_3=\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x^2+1}{\sqrt{3}}\right) \quad \ldots(i)$

Similarly $I_2 = \int \frac{dx}{x^2 - x + 1} = \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$I_2 = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$ ...(ii)

& $I_1 = \int \frac{xdx}{x^2 + x + 1} = \frac{1}{2}\int \frac{(2x+1)dx}{(x^2 + x + 1)} - \frac{1}{2}\int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$I_1 = \frac{1}{2}\ln\left(x^2 + x + 1\right) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$

Hence $I = I_1 + I_2 - I_3$

$I = \frac{1}{2}\ln\left(x^2 + x + 1\right) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$

so $A_1 = \frac{1}{2}, A_2 = \frac{-1}{\sqrt{3}}, A_3 = \frac{2}{\sqrt{3}}, A_4 = \frac{-1}{\sqrt{3}}$

$A_1 + A_2 + A_3 + A_4 = \frac{1}{2}$

$\left[A_1 + A_2 + A_3 + A_4\right] = 0$

80.(3.6) $\int \frac{\ln\left(x + \sqrt{1+x^2}\right)}{\sqrt{1+x^2}} dx = \frac{\left(\ln\left(x + \sqrt{1+x^2}\right)\right)^2}{2} + c^1$

So $f(x) = \frac{x^2}{2}$ & $g(x) = \ln\left(x + \sqrt{1+x^2}\right)$

Now $\int \frac{x^2}{2}.\ln\left(x + \sqrt{1+x^2}\right)dx = \frac{x^3}{6}.\ln\left(x + \sqrt{1+x^2}\right) - \int \frac{1}{\sqrt{1+x^2}}.\frac{x^3}{6}dx$

$\int f(x).g(x) = \frac{1}{6}x^3 g(x) - \frac{1}{6}\int \frac{x^2.x}{\sqrt{1+x^2}}dx$

Put $1 + x^2 = t^2$

$1 + x^2 = t^2$

$\int f(x).g(x) = \frac{1}{6}x^3 g(x) - \frac{1}{6}\int \frac{\left(t^2 - 1\right)tdt}{t}$

$= \frac{x^3}{6} g(x) - \frac{1}{6}\left(\frac{t^3}{3} - t\right) + d = \frac{x^3}{6} g(x) - \frac{1}{18}\left(1 + x^2\right)^{3/2} + \frac{1}{6}\left(1 + x^2\right)^{1/2} + d$

So $a = \frac{1}{6}, b = \frac{-1}{18}, c = \frac{1}{6} \Rightarrow \frac{1}{a+b+c} = \frac{18}{5}$

81.(2.05) $\int \sqrt{x}\tan\left\{2\tan^{-1}\left(\frac{\sqrt{\sqrt{1+\sqrt{x}}+1}-\sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1}+\sqrt{\sqrt{1+\sqrt{x}}-1}}\right)\right\}dx$

Put $\sqrt{x}=\tan^2\theta$

$\frac{1}{2\sqrt{x}}dx = 2\tan\theta\sec^2\theta d\theta$

$dx = 4\tan^3\theta\sec^2\theta d\theta$

$\int \tan^2\theta\tan\left\{2\tan^{-1}\left(\frac{\sqrt{\sqrt{1+\tan^2\theta}+1}-\sqrt{\sqrt{1+\tan^2\theta}-1}}{\sqrt{\sqrt{1+\tan^2\theta}+1}+\sqrt{\sqrt{1+\tan^2\theta}-1}}\right)\right\}4\tan^3\sec^2\theta d\theta$

$I = 4\int \tan^5\theta\sec^2\theta.\tan\theta(2\tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)d\theta$

$I = 4\int \tan^5\theta\sec^2\theta.\tan\theta(2\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)d\theta$

$I = 4\int \tan^5\theta\sec^2\theta.\tan\left(\frac{\pi}{2}-\theta\right)d\theta$

$I = 4\int \tan^4\theta\sec^2\theta d\theta$

$I = \frac{4}{5}\tan^5\theta + c$

$I = \frac{4}{5}x^{5/4} + c$

82.(1) $I = \int \frac{\left(e^x+\cos x+1\right)-\left(e^x+\sin x+x\right)}{\left(e^x+\sin x+x\right)}dx$

$I = \ln\left|e^x+\sin x+x\right| - x + c$; $f(x) = e^x+\sin x+x$

$g(x) = -x$ so $f(x)+g(x) = e^x+\sin x \Rightarrow \frac{f(x)+g(x)}{e^x+\sin x} = 1$

83.(403) $I = \int x.\left(x^{2009}+x^{803}+x^{401}\right)\left(2x^{1608}+5x^{402}+10\right)^{1/402}dx$

$I = \int \left(x^{2009}+x^{508}+x^{401}\right)\left(2x^{2010}+5x^{804}+10x^{10}\right)^{1/402}dx$

Put $2x^{2010}+5x^{804}+10x^{402} = t$

$4020\left(x^{2009}+x^{803}+x^{401}\right)dx = dt$

$I = \int \frac{1}{4020}(t)^{1/402}dt$; $I = \frac{1}{4020}.\frac{t^{403/402}}{\left(403/402\right)}+c$; $I = \frac{\left(2x^{2010}+5x^{804}+10x^{402}\right)^{403/402}}{4030}+c$

$a = 403 \Rightarrow a-400 = 3$

84.(1) $u(x)=7v(x)\Rightarrow u'(x)=7v'(x) \quad\Rightarrow\quad p=7$

$$\frac{u(x)}{v(x)}=7\Rightarrow\left(\frac{u(x)}{v(x)}\right)'=0 \quad\Rightarrow\quad q=0 \qquad \text{Now } \frac{p+q}{p-q}=\frac{7+0}{7-0}=1$$

85.(0) $I=\int\frac{dx}{\sqrt{2-x-x^2}}+2\int\frac{dx}{x^2\sqrt{2-x-x^2}}-\int\frac{dx}{x\sqrt{2-x-x^2}}$

$$I=-\int\frac{dx}{\sqrt{\frac{9}{4}-\left(x+\frac{1}{2}\right)}}+2\int\frac{-dt}{\sqrt{2-\frac{1}{t}-\frac{1}{t^2}}}+\int\frac{\frac{1}{t}\,dt}{2-\frac{1}{t}-\frac{1}{t^2}}$$

Put $x=\frac{1}{t} \quad\Rightarrow\quad dx=\frac{-1}{t^2}dt$

$$I=-\sin^{-1}\left(\frac{2x+1}{3}\right)-\frac{1}{2}\int\frac{(4t-1)dt}{\sqrt{2t^2-t-1}}+\frac{1}{2}\int\frac{dt}{\sqrt{2t^2-t-1}}$$

$$=-\sin^{-1}\left(\frac{2x+1}{3}\right)-\frac{1}{2}\frac{\sqrt{2t^2-t-1}}{\left(\frac{1}{2}\right)}+\frac{1}{2\sqrt{2}}\int\frac{dt}{\sqrt{t^2-\frac{t}{2}-\frac{1}{2}+\frac{1}{16}-\frac{1}{16}}}$$

$$I=-\sin^{-1}\left(\frac{2x+1}{3}\right)-\frac{\sqrt{2-x-x^2}}{x}+\frac{1}{2\sqrt{2}}\ln\left|\frac{(4-x)+4\sqrt{2-x-x^2}}{4x}\right|+k$$

$$I=\frac{-\sqrt{2-x-x^2}}{x}+\frac{1}{2\sqrt{2}}\ln\left|\frac{(4-x)+\sqrt{2-x-x^2}}{x}\right|-\sin^{-1}\left(\frac{2x+1}{3}\right)+c \quad\Rightarrow\quad A=-1\ \&\ B=1 \quad\Rightarrow A+B=0$$

86.(12) $\int\left(\frac{1-\left(4x^2\right)^3}{x^6.\left(4x^2+2x+1\right)}.\frac{x^6}{\left(4x^2-4x+1\right)}\right)-\frac{4x^2(2x+1)}{(1-2x)}dx$

$$I=\int\left\{\frac{(1-2x)(1+2x)\left(4x^2-2x+1\right)\left(4x^2+2x+1\right)}{\left(4x^2+2x+1\right)(2x-1)^2}-\frac{4x^2(2x+1)}{(1-2x)}\right\}dx$$

$$I=\int\left\{\frac{(1+2x)\left(4x^2-2x+1\right)}{(2x-1)}-\frac{4x^2(2x+1)}{(1-2x)}\right\}dx; \quad I=\int(2x+1)\,dx=x^2+x+c$$

87.(1.59) $f(x)=y=x+\sin x$

$$\frac{dy}{dx}=1+\cos x;\quad g'(y)=\frac{dx}{dy}=\frac{1}{1+\cos x}$$

$$y=\frac{\pi}{4}+\frac{1}{\sqrt{2}}=x+\sin x\Rightarrow x=\frac{\pi}{4}$$

So $g'\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)=\frac{1}{1+\cos\frac{\pi}{4}}=\frac{1}{1+\frac{1}{\sqrt{2}}}; \quad g'\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)=\frac{\sqrt{2}\left(\sqrt{2}-1\right)}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}=2-\sqrt{2}$

88.(2019) $I=\int \sin(2020x).\sin^{2018}x\ dx.$

$I=\int\{\sin x(2019x+x)\}.\sin^{2018}x\ dx.$

$I=\int \sin^{2018}x\left((\sin(2019x))\cos x+(\cos 2019x).\sin x\right)dx$

$$I=\underset{I}{\int(\sin 2019x)\left(\sin^{2018}x\right)\cos x.dx}+\underset{II}{\int(\cos 2019x)\left(\sin^{2019}x\right)dx}$$

$$I=\frac{(\sin 2019x)\left(\sin^{2019}x\right)}{(2019)}-\int\frac{(2019)(\cos(2019x))\left(\sin^{2019}x\right)}{2019}dx+\int(\cos 2019x)\left(\sin^{2019}x\right)dx$$

$$I=\frac{(\sin 2019x)\left(\sin^{2019}x\right)}{(2019)}+c$$

89.(1.8) Let $\sqrt{7x-10-x^2}=(x-2)t$

$(5-x)(x-2)=(x-2)^2t^2$

$(5-x)=(x-2)t^2$

$$x-2=\frac{3}{t^2+1}$$

$$1+\sqrt{7x-10-x^2}=\frac{t^2+3t+1}{t^2+1}$$

$$x=\frac{2t^2+5}{t^2+1}\Rightarrow\ dx=\frac{-6t}{\left(t^2+1\right)}dt$$

$$I=\int\frac{\left(t^2+1\right)}{3}.\frac{\left(t^2+1\right)}{\left(t^2+3t+1\right)}.\left(\frac{-6t}{\left(t^2+1\right)^2}\right)dt$$

$$I=-\int\frac{(2t+3-3)}{\left(t^2+3t+1\right)}dt=-\int\frac{(2t+3)dt}{\left(t^2+3t+1\right)}+3\int\frac{dt}{\left(t+\frac{3}{2}\right)^2-\frac{5}{4}}$$

$$I=-\ln\left|t^2+3t+1\right|+\frac{3}{\sqrt{5}}\int\left[\frac{1}{\left(t+\frac{3}{2}\right)-\frac{\sqrt{5}}{2}}-\frac{1}{\left(t+\frac{3}{2}\right)+\frac{\sqrt{5}}{2}}\right]dt$$

$$I=-\ln\left|t^2+3t+1\right|+\frac{3}{\sqrt{5}}\ln\left|\frac{2t+3-\sqrt{5}}{2t+3+\sqrt{5}}\right|+c$$

90.(2.14) since $\int e^{f(x)}.\left(x\ f'(x)+\frac{f''(x)}{\left(f'(x)\right)^2}\right)dx=e^{f(x)}.\left(x-\frac{1}{f'(x)}\right)+c$

Hence $I=\int e^{x\sin x+\cos x}.\left(\frac{x^4\cos^3 x-x\sin x+\cos x}{x^2\cos^2 x}\right)dx$

$$I=\int e^{x\sin x+\cos x}\left(x.(x\cos x)+\frac{(\cos x-x\sin x)}{(x\cos x)^2}\right)dx$$

Let $f(x)=x\sin x+\cos x$

$f'(x)=x\cos x$

$f''(x)=\cos x-x\sin x$

So $I=e^{x\sin x+\cos x}\left(x-\frac{1}{x\cos x}\right)+c$

91.(2) $\left(f'(x)f'(x)\right)-f(x).f''(x)=0$

$$\frac{\left(f'(x)\right)^2-f(x)f''(x)}{\left(f'(x)\right)^2}=0$$

$$\frac{d}{dx}\left(\frac{f(x)}{f'(x)}\right)=0\Rightarrow\frac{f(x)}{f'(x)}=c$$

at $x=0,\quad \frac{f(0)}{f'(0)}=c\Rightarrow c=\frac{1}{2}$

$\Rightarrow\qquad \frac{f'(x)}{f(x)}=2\Rightarrow\ln\left|f(x)\right|=2x+k.$

$\therefore\qquad f(0)=1\Rightarrow k=0$

$f(x)=e^{2x}\qquad ;\quad \lambda=2,\mu=0$

92. (11) Let $I=\int\sqrt{\frac{1+x^{2n}}{x^{2n}}}\,\frac{\left\{\ln\left(1+x^{2n}\right)\right\}-2n\ln x}{x^{2n+1}}dx$

$$I=\int\sqrt{\frac{1+x^{2n}}{x^{2n}}}\left\{\frac{\ln\left(\frac{1+x^{2n}}{x^{2n}}\right)}{x^{2n+1}}\right\}dx=\frac{-1}{2n}\int p\ln p^2(2p)dp$$

For $\left(\frac{1}{x^{2n}}+1\right)=p^2,\frac{-2n}{x^{(2n+1)}}dx=2p\ dp$

$$I=\frac{4}{-2n}\int p^2\ln p dp=-\frac{2}{n}\left[\frac{p^3\ln p}{3}-\int\frac{p^3}{3}\left(\frac{1}{p}\right)dp\right]=-\frac{2}{n}\left[\frac{p^3\log_e p}{3}-\frac{p^3}{9}\right]=\frac{2p^3}{9n}\left[1-3\ln p\right]+c.$$

INTEGRAL CALCULUS-2

1.(A) Let q = p + d, r = p + 2d, s = p + 3d

$$\therefore \qquad f(x)=\begin{vmatrix} p+\sin x & p+d+\sin x & -2d+\sin x \\ p+d+\sin x & p+2d+\sin x & -1+\sin x \\ p+2d+\sin x & p+3d+\sin x & 2d+\sin x \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3 - 2R_2$, we get :

$$f(x)=\begin{vmatrix} 0 & 0 & 2 \\ p+d+\sin x & p+2d+\sin x & -1+\sin x \\ p+2d+\sin x & p+3d+\sin x & 2d+\sin x \end{vmatrix}$$

Given, $\int_0^2 f(x)\,dx = -4 \;\Rightarrow\; \int_0^2 \left(-2d^2\right)dx = -4 \;\Rightarrow d^2 = 1 \;\Rightarrow = \pm 1$

2.(C) $A=\int_1^{\sin\theta}\frac{t\,dt}{1+t^2}$ Let $t=\frac{1}{x}, dt=-\frac{1}{x^2}dx \;=\int_1^{\operatorname{cosec}\theta}\frac{1}{x}\cdot\frac{-1}{x^2}\cdot\frac{x^2+1}{x^2}dx=-\int_1^{\operatorname{cosec}\theta}\frac{dx}{x\left(1+x^2\right)}=-B$

3.(C) If $-8<x<8,$ then y = 2

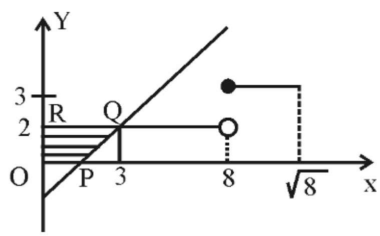
If $x\in\left(-8\sqrt{2},-8\right]\cup[8,\sqrt{2}),$, then y = 3, etc

Intersection of y = x – 1 and y = 2

We get $x=3\in(-8,8)$

Intersection of y = x – 1 and y = 3

We get $x=4\notin(-8\sqrt{2},-8]\cup[8,8\sqrt{2})$



4.(B) Let $I(a)=\int_0^1\frac{x^a-1}{\log x}dx$

Differentiating w.r.t. a keeping x as constant

$$\therefore \qquad \frac{dI(a)}{da}=\int_0^1\frac{d}{da}\left(\frac{x^a-1}{\log x}\right)dx$$

$$=\int_0^1\frac{x^a\log x}{\log x}dx=\int_0^1 x^a dx=\left.\frac{x^{a+1}}{a+1}\right|_0^1$$

$$=\frac{1}{(a+1)}$$

Integrating both sides w.r.t. a, we get

$I(a)=\log(a+1)+c$

For $a=0, I(0)=\log 1+c$

$0=0+c$

$\therefore \quad I=\log(a+1)$

5.(C) Let $F(k)=\int\limits_0^{\pi/2}\ln(\sin^2\theta+k^2\cos^2\theta)d\theta$

$$F'(k)=\int\limits_0^{\pi/2}\frac{1}{\sin^2\theta+k^2\cos^2\theta}2k\cos^2\theta d\theta$$

$$= 2k\int_0^{\pi/2}\frac{\cos^2\theta}{\sin^2\theta + k^2\cos^2\theta}d\theta = 2k\int_0^{\pi/2}\frac{d\theta}{\tan^2\theta + k^2}$$

$$= 2k\int_0^{\pi/2}\frac{\sec^2\theta - \tan^2\theta}{\tan^2\theta + k^2}d\theta = 2k\int_0^{\infty}\frac{dt}{t^2+k^2} - 2k\int_0^{\pi/2}d\theta + 2k^3\int_0^{\pi/2}\frac{d\theta}{\tan^2\theta + k^2} \text{ (Putting } t = \tan\theta)$$

$$= 2k\frac{1}{k}\tan^{-1}\frac{1}{k}\Big|_0^{\infty} - 2k\frac{\pi}{2} + k^2F'(k)$$

$\Rightarrow (1-k^2)F'(k) = \pi - k\pi = \pi(1-k) \Rightarrow F'(k) = \dfrac{\pi}{1+k} \quad \Rightarrow F(k) = \pi\log(1+k) + c$

For $k = 1$, $F(1) = 0 \Rightarrow c = -\pi\log 2 \Rightarrow F(k) = \pi\log(1+k) - \pi\log 2$

6.(A) Let $I(a) = \displaystyle\int_0^{\pi/2}\log\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{dx}{\sin x}$

$$\frac{dI}{da} = \int_0^{\pi/2}\frac{2\sin x}{1-a^2\sin^2 x}\frac{dx}{\sin x} = \int_0^{\pi/2}\frac{2\sec^2 x dx}{1+\tan^2 x - a^2\tan^2 x} = \int_0^{\pi/2}\frac{2\sec^2 x dx}{1+(1-a^2)\tan^2 x} = \int_0^{\infty}\frac{2dt}{1+(1-a^2)t^2} \text{ (Put } \tan x = t)$$

$$= \frac{2}{\sqrt{1-a^2}}\left[\tan^{-1}\left(t\sqrt{1-a^2}\right)\right]_0^{\infty} = \frac{\pi}{\sqrt{1-a^2}} \qquad \Rightarrow I = \pi\sin^{-1}a[\text{as } I(0) = 0]$$

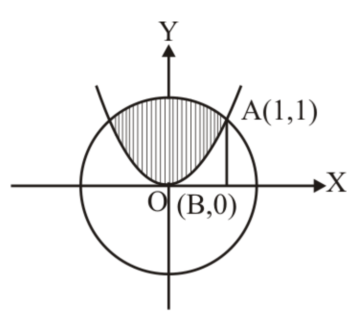
7.(C) $\displaystyle\int_0^{\pi}\frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{a^2-1}}$

Differentiating both sides with respect to a, we get : $\displaystyle -\int_0^{\pi}\frac{dx}{(a-\cos x)^2} = \frac{\pi}{(a^2-1)^{3/2}}$

Again differentiating with respect to a, we get : $\displaystyle 2\int_0^{\pi}\frac{dx}{(a-\cos x)^3} = \frac{\pi(1+2a^2)}{(a^2-1)^{5/2}}$

Put $a = \sqrt{10}$, we get $\displaystyle\int_0^{\pi}\frac{dx}{(\sqrt{10}-\cos x)^3} = \frac{7\pi}{81}$

8.(D) Since, $\displaystyle\lim_{x\to 0}\left[1+\frac{f(x)}{x^3}\right]^{1/x}$ exists, so, $\displaystyle\lim_{x\to 0}\frac{f(x)}{x^3} = 0$



$\therefore \quad f(x) = a_4x^4 + a_5x^5 + ... + a_nx^n, a_n \neq 0, n \geq 4$

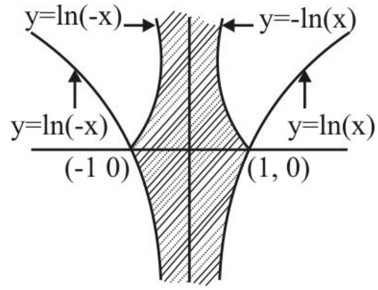
Since, f(x) is of least degree $\Rightarrow f(x) = a_4x^4$

Again $\displaystyle\lim_{x\to 0}\left[1+\frac{f(x)}{x^3}\right]^{\frac{1}{x}} = e \quad \Rightarrow \quad a_4 = 1 \quad \Rightarrow \quad f(x) = x^4$

The graph of $y = x^4$ and $x^2 + y^2 = 2$ are shown in figure

$\therefore$ The required area = $\displaystyle 2\int_0^1\left(\sqrt{2-x^2} - x^4\right)dx = \frac{\pi}{2} - \frac{3}{5}$

9.(C) y = ln x, y = ln | x |, y = |ln x |, y = |ln | x||

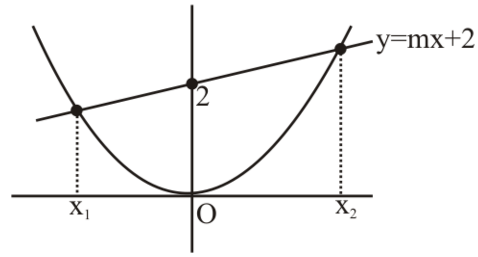


Required area = 4 $\left|\int_0^1 \ln x dx\right| = 4. \lim\limits_{\varepsilon\to 0}\left|\int_\varepsilon^1 \ln x dx\right|$

$= 4\left|\lim\limits_{\varepsilon\to 0}\left[x(\ln x - 1)\right]_\varepsilon^1\right|$

= 4 sq. units

10.(D) Eliminating y from two equations,



We get $2mx + 4 = x^2 \Rightarrow x^2 - 2mx - 4 = 0$

x_1 and x_2 are roots of this quadratic equation

$\therefore \quad x_1 + x_2 = 2m$ and $x_1 x_2 = -4$

Now,

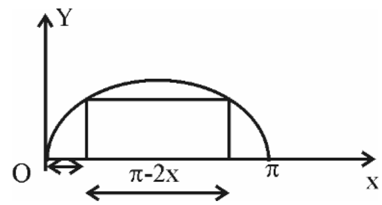
$$\int_{x_1}^{x_2}\left(mx + 2 - \frac{x^2}{2}\right)dx = \frac{m}{2}\left(x_2^2 - x_1^2\right) + 2\left(x_2 - x_1\right) - \frac{1}{6}\left(x_2^3 - x_1^3\right)$$

$$= \left(x_2 - x_1\right)\left[\frac{m}{2}\left(x_2 + x_1\right) + 2 - \frac{1}{6}\left(x_1^2 + x_1x_2 + x_2^2\right)\right]$$

$$= \sqrt{\left(x_2 - x_1\right)^2 - 4x_2x_1}\left[\frac{m}{2}\left(x_2 + x_1\right) + 2 - \frac{1}{6}\left\{\left(x_2 + x_1\right)^2 - x_2x_1\right\}\right] = \sqrt{4m^2 + 16}\left[\frac{m^2 + 4}{3}\right]$$

Clearly above is minimum if m = 0.

11.(B) A = Area = $\sin x(\pi - 2x)$



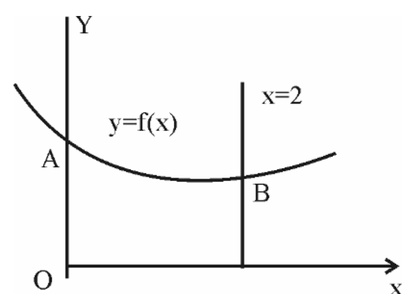
$$\frac{dA}{dx} = (\pi - 2x)\cos x - 2\sin x = 0 \Rightarrow \tan x = \frac{\pi}{2} - x$$

Let $\quad f(x) = \tan x + x - \frac{\pi}{2}$

$f\left(\frac{\pi}{6}\right)$ is negative; $f\left(\frac{\pi}{4}\right)$ is positive

So one root lies between $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

12.(D) Area of OABL = $\int\limits_0^2 y dx = \int\limits_0^2 \left(a + bx + cx^2\right)dx = \left(2a + 2b + \frac{8}{3}c\right) = \frac{1}{3}[6a + 6b + 8c]$



But $\quad f(x) = a + bx + cx^2$

$f(0) = a, f(1) = a + b + c$

$f(2) = a + 2b + 4c$

$$\frac{1}{3}\{f(0) + 4f(1) + f(2)\} = \frac{1}{3}\{a + 4(a + b + c) + a(a + 2b + 4c)\}$$

$$= \frac{1}{3}\{6a + 6b + 8c\}$$

13-15. **13.(C)** **14.(D)** **15.(C)**

$$f(x)-\lambda\int_0^{\pi/2}\sin x\cos t\, f(t)dt=\sin x \Rightarrow f(x)-\lambda\sin x\int_0^{\pi/2}\cos t\, f(t)dt=\sin x$$

$\Rightarrow$ $f(x)-A\sin x=\sin x$ or $f(x)=(A+1)\sin x,$ where $A=\lambda\int_0^{\pi/2}\cos t\, f(t)dt$

$\Rightarrow$ $A=\lambda\int_0^{\pi/2}(\cos t)(A+1)\sin t\, dt$

$$=\frac{\lambda(A+1)}{2}\int_0^{\pi/2}\sin 2t\, dt=\frac{\lambda(A+1)}{2}\left[\frac{-\cos 2t}{2}\right]_0^{\pi/2}=\frac{\lambda(A+1)}{2}$$

$\Rightarrow$ $A=\dfrac{\lambda}{2-\lambda}\Rightarrow f(x)=\left(\dfrac{\lambda}{2-\lambda}+1\right)\sin x\Rightarrow f(x)=\left(\dfrac{2}{2-\lambda}\right)\sin x$

$$\left(\frac{2}{2-\lambda}\right)\sin x=2$$

$\Rightarrow$ $\sin x=(2-\lambda)\Rightarrow |2-\lambda|\le 1\Rightarrow -1\le\lambda-2\le 1\Rightarrow 1\le\lambda\le 3$

$$\int_0^{\pi/2}f(x)dx=3\Rightarrow\int_0^{\pi/2}\frac{2}{2-\lambda}\sin x dx=3\Rightarrow -\left[\frac{2}{2-\lambda}\cos x\right]_0^{\pi/2}=3\quad\Rightarrow\quad\frac{2}{2-\lambda}=3\Rightarrow\lambda=4/3$$

16.(D) The L.H.S. of given inequality is equal to

$$\left[a^2\left(\frac{\sin 3x}{12}+\frac{3}{4}\sin x\right)-a\cos x-20\sin x\right]_0^{\pi/2}=a^2\left(-\frac{1}{12}+\frac{3}{4}\right)-a(0-1)-20=\frac{2a^2}{3}+a-20$$

Thus the given inequality is $\dfrac{2a^2}{3}+a-20\le-\dfrac{a^2}{3}$

$$a^2+a-20\le 0\Leftrightarrow -5\le a\le 4$$

Since a is a positive integer so a = 1,2,3,4

17.(D) For $0<x<1$, we have $\dfrac{1}{2}x^2<x^2<x, i.e., -x^2>-x$, so that $e^{-x^2}>e^{-x}$

Hence $\displaystyle\int_0^1 e^{-x^2}\cos^2 xdx>\int_0^1 e^{-x}\cos^2 xdx$. Also $\cos^2 x\le 1$, therefore $\displaystyle\int_0^1 e^{-x^2}\cos^2 xdx\le\int_0^1 e^{-x^2}dx<\int_0^1 e^{-x^2/2}=I_4$

18.(A) We have $2\sin\dfrac{x}{2}\left(\dfrac{1}{2}+\cos x+\cos 2x+........+\cos nx\right)$

$$=\sin\frac{x}{2}+2\sin\frac{x}{2}\cos x+2\sin\frac{x}{2}\cos 2x+.......+2\sin\frac{x}{2}\cos nx$$

$$=\sin\frac{x}{2}+\sin\frac{3x}{2}-\sin\frac{x}{2}+\sin\frac{5x}{2}-\sin\frac{3x}{2}+.......+\sin\left(n+\frac{1}{2}\right)x-\sin\left(n-\frac{1}{2}\right)x=\sin\left(n+\frac{1}{2}\right)x$$

$$\frac{1}{2}+\cos x+\cos 2x+...+\cos nx=\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

$\Rightarrow$ $\displaystyle\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin x/2}dx=2\left(\int_0^{\pi}\frac{1}{2}dx+\int_0^{\pi}\cos xdx+...+\int_0^{\pi}\cos nxdx\right)=2\left(\frac{\pi}{2}+\sin x\Big|_0^{\pi}+...+\frac{\sin nx}{n}\Big|_0^{\pi}\right)=\pi$

19.(D) $\int_0^1 \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^{k+2}2^k}{k!}dx=\int_0^1 x^2\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(2x)^k}{k!}dx=\int_0^1 x^2e^{2x}dx$

20.(A) Since the required function is a polynomial, the absciassae of the points of inflection can only be among the roots of the second derivative. Consequently $p''(x)=ax(x-1)(x+1)=a\left(x^3-x\right)$

Since at the point x = 0, $p'(0)=\tan 60^\circ=\sqrt{3}$

$$p'(x)=\int_0^x p''(x)dx+\sqrt{3}=a\left(\frac{x^4}{4}-\frac{x^2}{2}\right)+\sqrt{3}$$

Then, since P(1) = 1, we get : $p(x)=\int_1^x p'(x)dx+1=a\left(\frac{x^5}{20}-\frac{x^3}{6}+\frac{7}{60}\right)+\sqrt{3}(x-1)+1$

Since $p(-1)=-1, so\, a=\frac{60\left(\sqrt{3}-1\right)}{7}$

21.(B) We have $\int_0^x \sin^2\frac{t}{2}dt=\frac{1}{2}\int_0^x(1-\cos t)dt=a^2x^2-\frac{3x}{2}+\frac{1}{2}+\frac{1}{a^2}$

$\Rightarrow \quad \frac{x}{2}-\frac{(\sin x)}{2}=\left(ax-\frac{1}{a}\right)^2+\frac{x}{2}+\frac{1}{2} \quad \Rightarrow \quad 2\left(ax-\frac{1}{a}\right)^2+1=-\sin x$

Since $\left(ax-\frac{1}{a}\right)^2\geq 0$ for all x, so the only possibility is $ax=\frac{1}{a}$ and sin x = -1

i.e., $x=2n\pi-\frac{\pi}{2}$. Thus $a=\frac{1}{\sqrt{x}}=\pm\frac{1}{\sqrt{2n\pi-\frac{\pi}{2}}}\, n\in N$

22.(D) $\int_0^a f(x)g(x)h(x)dx=\int_0^a f(a-x)g(a-x)h(a-x)dx$

$$=-\int_0^a f(x)g(x)\left(\frac{3h(x)-5}{4}\right)dx=-\frac{3}{4}\int_0^a f(x)g(x)h(x)dx+\frac{5}{4}\int_0^a f(x)g(x)dx$$

Therefore, $\frac{7}{4}\int_0^a f(x)g(x)h(x)dx=\frac{5}{4}\int_0^a f(x)g(x)dx=0$

$\therefore \quad \int_0^a f(x)g(x)dx=\int_0^a f(a-x)g(a-x)dx=\int_0^a f(x)\{-g(x)\}dx$

$\Rightarrow \quad \int_0^a f(x)g(x)dx=0 \qquad$ So, $\int_0^a f(x)g(x)h(x)=0$

23-26.

23.(B) $f(x)$ is an odd function $\Rightarrow f(x)=-f(-x)$

$$\phi(-x)=\int_a^{-x} f(t)\,dt,\ \text{put } t=-y$$

$$\Rightarrow \quad \phi(-x)=\int_{-a}^{x} f(-t)(-dt)=\int_{-a}^{x} f(t)dt=\int_{-a}^{x} f(t)dt+\int_a^x f(t)dt=0+\int_a^x f(t)dt=\phi(x).$$

24.(D) If $f(x)$ is an even function, then $\phi(-x) = -\int_{-a}^{x} f(t)dt = -\int_{-a}^{a} f(t)dt - \int_{a}^{x} f(t)dt$

$$= -2\int_{0}^{a} f(t)dt - \int_{a}^{x} f(t)dt \qquad \text{(As } f(x) \text{ is an even function)}$$

Now, $\int_{0}^{a} f(t)dt = \int_{0}^{a} f(a-t)dt = \int_{0}^{a} -f(t)dt$ [Using $f(a-x) = -f(x)$)]

$\Rightarrow \int_{0}^{a} f(t)dt = 0 \Rightarrow \phi(-x) = -\int_{a}^{x} f(t)dt = -f(x) \Rightarrow \phi(x)$ is an odd function.

25.(D) $g(x+\alpha) + g(x) = 0$

$\Rightarrow g(x+2\alpha) + g(x+\alpha) = 0 \Rightarrow g(x+2\alpha) = g(x) \Rightarrow g(x)$ is periodic with period 2α

$\Rightarrow \int_{b}^{2k} g(t)dt = \int_{a}^{b+c} g(x)dx$ ($\because$ b, k, c are in A.P.)

This is independent of *b*, then *c* has least value 2α.

26.(D) $\int_{p+m\alpha}^{q+m\alpha} g(t)dt = \int_{p+m\alpha}^{p} g(x)dx + \int_{p}^{q} g(x)dx + \int_{q}^{q+m\alpha} g(x)dx$

$$= -m\int_{0}^{\alpha} g(x)dx + \int_{p}^{q} g(x)dx + n\int_{0}^{\alpha} g(x)dx = \int_{p}^{q} g(x)dx + (n-m)\int_{0}^{\alpha} g(x)dx$$

27.(C) $G(0,t) = 0$ for $t \geq 0$ so $g(0) = \int_0^1 f(t).\, 0\text{ dt} = 0$

$G(1,t) = t.(1-1) = 0$ for $t < 1$. Hence $g(1) = \int_0^1 f(t).\, 0\text{ dt} = 0,$

Also $g(x) = \int_0^x f(t)t(x-1)dt + \int_x^1 f(t)x(t-1)dt = (x-1)\int_0^x tf(t)dt + x\int_x^1 f(t)(t-1)dt$

Hence, $g'(x) = (x-1)xf(x) + \int_0^x tf(t)dt + \int_x^1 f(t)(t-1)dt - xf(x)(x-1)$

$= \int$ if $(t)dt - \int_0^x f(t)dt + .$ Thus $g''(x) = f(x)$

28.(B) Let $I = \int_{4\pi-2}^{4\pi} \frac{\sin(t/2)}{4\pi+2-t}dt$

$$= (4\pi-(4\pi-2))\int_0^1 \frac{\sin\left(\frac{(4\pi-(4\pi-2))t+4\pi-2}{2}\right)}{4\pi+2-((4\pi-(4\pi-2))t+4\pi-2)}dt \quad \left[\therefore \int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)x+a)dx\right]$$

$$= 2\int_0^1 \frac{\sin(t-1)dt}{(4-2t)}$$

29.(D) $\int_{-\pi}^{\pi} \left|x\sin\left[x^2-\pi\right]\right| dx = 2\int_{0}^{\pi} \left|x\sin\left[x^2-\pi\right]\right| dx = 2\int_{0}^{\pi} x\left|\sin\left[x^2-\pi\right]\right| dx$

Since $-\pi \leq x^2 - \pi \leq \pi^2 \leq \pi^2 - \pi \leq 7$ so the last integral is equal to

$$2\left[\int_0^{\sqrt{\pi-3}} x\left|\sin\left[x^2-\pi\right]\right|dx+\int_{\sqrt{\pi-3}}^{\sqrt{\pi-2}} x\left|\sin\left[x^2-\pi\right]\right|dx+\int_{\sqrt{\pi-2}}^{\sqrt{\pi-1}} x\left|\sin\left[x^2-\pi\right]\right|dx\right.$$

$$+\int_{\sqrt{\pi-1}}^{\sqrt{\pi}} x\left|\sin\left[x^2-\pi\right]\right|dx+\int_{\sqrt{\pi}}^{\sqrt{1+\pi}} x\left|\sin\left[x^2-\pi\right]\right|dx+\int_{\sqrt{\pi+1}}^{\sqrt{\pi+2}} x\left|\sin\left[x^2-\pi\right]\right|dx$$

$$+\int_{\sqrt{2+\pi}}^{\sqrt{\pi+3}} x\left|\sin\left[x^2-\pi\right]\right|dx+\int_{\sqrt{\pi+3}}^{\sqrt{x+4}} x\left|\sin\left[x^2-\pi\right]\right|dx+\int_{\sqrt{4+\pi}}^{\sqrt{\pi+5}} x\left|\sin\left[x^2-\pi\right]\right|dx$$

$$\left.+\int_{\sqrt{\pi+5}}^{\sqrt{\pi+6}} x\left|\sin\left[x^2-\pi\right]\right|dx+\int_{\sqrt{\pi+6}}^{\pi} x\left|\sin\left[x^2-\pi\right]\right|dx\right]$$

30.(B) Since $4-x^2 \ge 4-x^2-x^3 \ge 4-2x^2 \; \forall x \in [0,1]$

$\Rightarrow \quad \sqrt{4-x^2} \ge \sqrt{\left(4-x^2-x^3\right)} \ge \sqrt{4-2x^2} \, \forall x \in [0,1] \Rightarrow \quad \dfrac{1}{\sqrt{4-x^2}} \le \dfrac{1}{\sqrt{\left(4-x^2-x^3\right)}} \le \dfrac{1}{\sqrt{4-2x^2}} \, \forall x \in [0,1]$

$\therefore \quad \displaystyle\int_0^1 \frac{dx}{\sqrt{2^2-x^2}} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{2}}\int_0^1 \frac{dx}{\sqrt{\left(\left(\sqrt{2}\right)^2-x^2\right)}}$

31.(A) Let $f(x)=\dfrac{\sin x}{x} \quad \therefore f'(x)=\dfrac{x\cos x-\sin x}{x^2}=\dfrac{(x-\tan x)\cos x}{x^2} \; \forall x \in \left[\dfrac{\pi}{4},\dfrac{\pi}{3}\right]$

$f(x)=\dfrac{\sin x}{x}$ is decreased on the interval $\left[\dfrac{\pi}{4},\dfrac{\pi}{3}\right]$

$\Rightarrow \quad$ the least value of the function $m=f\left(\dfrac{\pi}{3}\right)=\dfrac{\sin(\pi/3)}{(\pi/3)}=\dfrac{3\sqrt{3}}{2\pi}$

And the greatest value of the function M = $f\left(\dfrac{\pi}{4}\right)=\dfrac{\sin(\pi/4)}{(\pi/4)}=\dfrac{2\sqrt{2}}{\pi}$

32.(A) Let $P=\lim\limits_{n\to\infty}\left\{\dfrac{\sqrt{n}}{\left(3+4\sqrt{n}\right)^2}+\dfrac{\sqrt{n}}{\sqrt{2}\left(3\sqrt{2}+4\sqrt{n}\right)^2}+\dfrac{\sqrt{n}}{\sqrt{3}\left(3\sqrt{3}+4\sqrt{n}\right)^2}+\ldots+\dfrac{1}{49n}\right\}$

$$P=\lim_{n\to\infty}\left\{\frac{\sqrt{n}}{\sqrt{1}\left(3\sqrt{1}+4\sqrt{n}\right)^2}+\frac{\sqrt{n}}{\sqrt{2}\left(3\sqrt{2}+4\sqrt{n}\right)^2}+\frac{\sqrt{n}}{\sqrt{3}\left(3\sqrt{3}+4\sqrt{n}\right)^2}+\ldots+\frac{\sqrt{n}}{\sqrt{n}\left(3\sqrt{n}+4\sqrt{n}\right)^2}\right\}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{1}{\sqrt{\frac{r}{n}}\left\{3\sqrt{\frac{r}{n}}+4\right\}^2}=\int_0^1\frac{dx}{\sqrt{x}\left(3\sqrt{x}+4\right)^2}$$

Put $\quad 3\sqrt{x}+4=t\,; \qquad \dfrac{3}{2\sqrt{x}}dx=dt$

When x = 0, $\Rightarrow t=4$ and when x = 1 $\Rightarrow t=7 \qquad \therefore \qquad P=\dfrac{2}{3}\displaystyle\int_4^7\frac{dt}{t^2}=\frac{2}{3}\left(-\frac{1}{t}\right)\Bigg|_4^7$

33.(AB) $f(x)=e^x+\int_0^1 e^x f(t)dt = e^x + k e^x$ where $k=\int_0^1 f(t)dt$

$\therefore \quad k=\int_0^1 \left(e^t+ke^t\right)dt = e+ke-1-k \qquad \therefore\ k=\frac{e-1}{2-e}$, thus $f(x)=e^x\left(1+\frac{e-1}{2-e}\right)=\frac{e^x}{2-e}$

Obviously, $f(0)=\frac{1}{2-e}<0$

Also, $f'(x)=\frac{e^x}{2-e}<0$ for $\forall x \in R$.

Also, $\int_0^1 f(x)dx = \int_0^1 \frac{e^x}{2-e}dx = \left[\frac{e^x}{2-e}\right]_0^1 = \frac{e-1}{2-e}<0$

34.(A) $f'(x)=\frac{3^x}{1+x^2}>0 \forall x>0 \Rightarrow f'(x)=\frac{3^x}{1+x^2}>\frac{1}{1+x^2}, \forall x \ge 1$

$\Rightarrow \quad \int_1^x f'(x)dx > \int_1^x \frac{1}{1+x^2}dx \quad \Rightarrow \quad f(x)>\tan^{-1}x-\tan^{-1}1 \Rightarrow f(x)+\pi/4>\tan^{-1}x$

35.(ABC) For $a \le 0$,

Given equation becomes $\int_0^2 (x-a)dx \ge 1 \Rightarrow a \le \frac{1}{2} \Rightarrow a \le 0$

For $0<a<2$,

$\int_0^2 |x-a|dx \ge 1 \Rightarrow \int_0^a (a-x)dx + \int_a^2 (x-a)dx \ge 1$

$\Rightarrow \quad \frac{a^2}{2}+2-2a+\frac{a^2}{2} \ge 1 \Rightarrow a^2-2a+1 \ge 0 \Rightarrow (a-1)^2 \ge 0$

For $a \ge 2$,

$\int_0^2 |x-a|dx \ge 1 \Rightarrow \int_0^2 (a-x)dx \ge 1 \Rightarrow 2a-2 \ge 1 \Rightarrow a \ge \frac{3}{2}$

$= a \ge 2$

36.(ABD) We know $\int_a^b |\sin x|dx$ represents the area under the curve from $x=a$ to $x=b$. We also know that area from $x=a$ to $x=a+\pi$ is 2.

$\therefore \quad \int_a^b |\sin x|dx = 8 \Rightarrow b-a=\frac{8\pi}{2} \qquad (1)$

Similarly, $\int_0^{a+b} |\cos x|dx = 9 \Rightarrow a+b-0=\frac{9\pi}{2} \qquad (2)$

From (1) and (2), $a=\frac{\pi}{4}$ and $b=\frac{17\pi}{4}$

$\Rightarrow \quad |a+b|=\frac{9\pi}{2}, |a-b|=4\pi, \frac{a}{b}=17$ and $\int_a^b \sec^2 x dx = [\tan x]_{\pi/4}^{17\pi/4}=0$.

37.(ABC) $g(x)=\int_0^x 2|t|\,dt=\begin{cases}\int_0^x -2t\,dt, & x<0\\ \int_0^x 2t\,dt, & x\ge 0\end{cases}=\begin{cases}\left[-t^2\right]_0^x, & x<0\\ \left[t^2\right]_0^x, & x\ge 0\end{cases}=\begin{cases}-x^2, & x<0\\ x^2, & x\ge 0\end{cases}=x|x|$

Clearly, continuous and differentiable at $x=0$

Also, $g'(x)=\begin{cases}-2x, & x<0\\ 2x, & x>0\end{cases}$ which is non-differentiable at $x=0$.

38.(AB) $f(x)=x\int_1^x \frac{e^t}{t}dt-e^x \quad\Rightarrow\quad f'(x)=x\frac{e^x}{x}+\int_1^x \frac{e^t}{t}dt-e^x$

$f'(x)=\int_1^x \frac{e^t}{t}dt>0\left[\because x\in[1,\infty)\right]$ $\quad f(x)$ is an increasing function.

39.(ACD) $\int_0^1 \frac{2x^2+3x+3}{(x+1)\left(x^2+2x+2\right)}=\int_0^1 \frac{2(x^2+2x+2)-(x+1)}{(x+1)\left(x^2+2x+2\right)}dx=\int_0^1\left(\frac{2}{x+1}-\frac{1}{x^2+2x+2}\right)dx$

$\left[2\log(x+1)-\tan^{-1}(x+1)\right]_0^1$

$2\log 2-\tan^{-1}2+\tan^{-1}1$

$2\log 2-\tan^{-1}2+\frac{\pi}{4}$

$\log 4-\left(\frac{\pi}{2}-\cot^{-1}2\right)+\frac{\pi}{4}$

$-\frac{\pi}{4}+\log 4+\cot^{-1}2$

From equation (1), $I=2\log 2-\tan^{-1}\left(\frac{2-1}{1+2\times 1}\right)=2\log 2-\tan^{-1}\frac{1}{3}=2\log 2-\cot^{-1}3$

40.(AD) $A_{n+1}-A_n=\int_0^{\pi/2}\frac{\sin(2n+1)x-\sin(2n-1)x}{\sin x}=\int_0^{\pi/2}2\cos 2nx\,d\,x=0$

$\Rightarrow \quad A_{n+1}=A_n$

$B_{n+1}-B_n=\int_0^{\pi/2}\frac{\sin^2(n+1)x-\sin^2 nx}{\sin^2 x}dx=\int_0^{\pi/2}\frac{\sin(2n+1)x}{\sin x}dx=A_{n+1}$

41.(ABC) $f(x)=\int_0^x \frac{1}{f(x)}dx\Rightarrow f'(x)=\frac{1}{f(x)}\cdot 1-0\Rightarrow f(x)f'(x)=1$

$\Rightarrow \quad \int f(x)f'(x)dx=\int 1dx \quad\Rightarrow\quad \frac{1}{2}[f(x)]^2=x+c \qquad (1)$

Now given that $\int_0^1[f(x)]^{-1}dx=\sqrt{2}\Rightarrow f(1)=\sqrt{2}$

$\Rightarrow$ From (1), $\frac{1}{2}[f(1)]^2=1+c\Rightarrow c=0 \quad\Rightarrow\quad f(x)=\pm\sqrt{2x}$

But $f(1)=\sqrt{2}\Rightarrow f(x)=\sqrt{2x}\Rightarrow f(2)=2$

Also, $f'(x)=\frac{1}{\sqrt{2x}} \Rightarrow f'(2)=1/2$

$$\int_0^1 f(x)dx=\int_0^1 \sqrt{2x}dx=\left[\frac{(2x)^{3/2}}{3}\right]_0^1=\frac{(2)^{3/2}}{3}$$

Also, $f^{-1}(x)=\frac{x^2}{2} \Rightarrow f^{-1}(2)=2$

42.(BC) $I=\int_0^\infty \frac{dx}{1+x^4}$ (1)

$$=\int_0^\infty \frac{x^2+1-x^2}{1+x^4}dx=\int_0^\infty \frac{x^2}{1+x^4}dx+\int_0^\infty \frac{1-x^2}{1+x^4}dx=I_1+I_2$$

$$I_2=\int_0^\infty \frac{\frac{1}{x^2}-1}{\frac{1}{x^2}+x^2}dx.$$

Put $x+\frac{1}{x}=y$

$\Rightarrow$ $I_2=\int_\infty^\infty \frac{-1}{y^2-2}dy=0$ $\Rightarrow$ $I=\int_0^\infty \frac{dx}{1+x^4}=\int_0^\infty \frac{x^2dx}{1+x^4}$ (2)

Adding equations (1) and (2), we get :

$\Rightarrow$ $2I=\int_0^\infty \frac{1+x^2dx}{1+x^4}=\int_0^\infty \frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+x^2}dx,$ put $x-\frac{1}{x}=y$

$\Rightarrow$ $2I=\int_{-\infty}^\infty \frac{dy}{y^2+2}=\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}}\right]_{-\infty}^\infty=\frac{\pi}{\sqrt{2}}$ $\Rightarrow$ $I=\frac{\pi}{2\sqrt{2}}$

43.(ABD) Given than $f(x)=\int_0^x |t-1|\,dt$ $\Rightarrow$ $f(x)=\int_0^x (1-t)dt, 0\le x\le 1=x-\frac{x^2}{2}$

Also $f(x)=\int_0^1 (1-t)dt+\int_1^x (t-1)dt,$ where $1\le x\le 2=\frac{1}{2}+\frac{x^2}{2}-x+\frac{1}{2}=\frac{x^2}{2}-x+1$

Thus, $f(x)=\begin{cases} x-\frac{x^2}{2}, & 0\le x\le 1 \\ \frac{x^2}{2}-x+1, & 1<x\le 2\end{cases} \Rightarrow f'(x)=\begin{cases}1-x, & 0\le x<1\\ x-1, & 1<x<2\end{cases}$

Thus, $f(x)$ is continuous as well as differentiable at $x=1$.

Also, $f(x)=\cos^{-1}x$ has one real root, draw the graph and verify.

44.(ABC) Let $I=\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx$ (1)

$=\int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}dx$ (2)

Adding equations (1) and (2), we get

$\Rightarrow \quad 2I=\int_a^b 1dx=b-a \Rightarrow I=\left(\frac{b-a}{2}\right)=10$ (Given) $\quad \therefore \quad b-a=20$

45.(AB) $I_n=\int_0^1 \frac{dx}{\left(1+x^2\right)^n}=\int_0^1\left(1+x^2\right)^{-n}dx=\left.\frac{x}{\left(1+x^2\right)^n}\right|_0^1-\int_0^1(-n)\left(1+x^2\right)^{-n-1}2x\times xdx=\frac{1}{2^n}+2n\int_0^1\frac{x^2dx}{\left(1+x^2\right)^{n+1}}$

$=\frac{1}{2^n}+2n\int_0^1\frac{1+x^2-1}{\left(1+x^2\right)^{n+1}}dx=\frac{1}{2^n}+2nI_n-2nI_{n+1}$

$\Rightarrow \quad 2nI_{n+1}=2^{-n}+(2n-1)I_n \Rightarrow 2I_2=\frac{1}{2}+I_1=\frac{1}{2}+\tan^{-1}x\Big|_0^1 \Rightarrow I_2=\frac{1}{4}+\frac{\pi}{8}$

Also $\quad 4I_3=2^{-2}+3I_2=\frac{1}{4}+3\left(\frac{1}{4}+\frac{\pi}{8}\right)=\frac{1}{4}+\frac{3\pi}{32}$

46.(BC) $\lim_{n\to\infty}\frac{1}{n}\sum_{r=n+1}^{2n}f\left(\frac{r}{n}\right)=\int_1^2 f(x)dx$

$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}f\left(\frac{r+n}{n}\right)=\int_0^1 f(1+x)dx=\int_1^2 f(t)dt=\int_1^2 f(x)dx$

$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}f\left(\frac{r}{n}\right)=\int_0^1 f(x)dx$

$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}f\left(\frac{r}{n}\right)=\int_0^2 f(x)dx$

47.(ABD) $f(2-x)=f(2+x),\ f(4-x)=f(4+x)$

$\Rightarrow \quad f(4+x)=f(4-x)=f(2+2-x)=f(2-(2-x))=f(x) \Rightarrow 4$ is a period of $f(x)$

$\int_0^{50} f(x)dx=\int_0^{48} f(x)dx+\int_{48}^{50} f(x)dx=12\int_0^4 f(x)dx+\int_0^2 f(x)dx$

(In second integral replacing x by $x+48$ and then using $f(x)=f(x+48)$)

$=12\left(\int_0^2 f(x)dx+\int_0^2 f(4-x)dx\right)+5=12\left(\int_0^2 f(x)dx+\int_0^2 f(4+x)dx\right)+5=24\int_0^2 f(x)dx+5=125$

$\int_{-4}^{46} f(x)dx=\int_{-4}^{-2} f(x)dx+\int_{-2}^{-2+48} f(x)dx$

$=\int_0^2 f(x+4)dx+12\int_0^4 f(x)dx=\int_0^2 f(x)dx+24\int_0^4 f(x)dx$

Also $\int_{2}^{52} f(x)dx = \int_{2}^{4} f(x)dx + \int_{4}^{4+48} f(x)dx$

$= \int_{0}^{2} f(4-x)dx + 12\int_{0}^{4} f(x)dx = \int_{0}^{2} f(4+x)dx + 24\int_{0}^{2} f(x)dx = \int_{0}^{2} f(x)dx + 24\int_{0}^{2} f(x)dx$

$\int_{1}^{51} f(x)dx = \int_{1}^{3} f(x)dx + \int_{3}^{3+48} f(x)dx$

$\int_{1}^{3} f(x)dx + 12\int_{0}^{4} f(x)dx$

$\int_{0}^{2} f(x+1)dx + 24\int_{0}^{2} f(x)dx \neq 125$

48.(AB) L.H.S. $= \int_0^x \left\{ \int_0^u f(t)dt \right\} du$

Integrating by parts choose '1' as the second function

$= \left\{ u\int_0^u f(t)dt \right\}_0^x - \int_0^x f(u)u\,du = x\int_0^x f(t)dt - \int_0^x f(u)u\,du$

$= x\int_0^x f(u)du - \int_0^x f(u)u\,du = \int_0^x f(u)(x-u)du$ = R.H.S.

49.(ACD) The expression $f(x)f(c) \forall x \in (c-h, c+h)$ where $h \to 0^+$ is equivalent to $\lim_{x\to 0} f(x)f(c)$ which equals to $(f(c))^2$ because $f(x)$ is continuous.

(A) We have $I = \lim_{n\to\infty} \frac{1}{n} \ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right]$

$= \lim_{n\to\infty} \frac{1}{n} \ln \prod_{k=1}^{n} \left(1+\frac{k}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(1+\frac{k}{n}\right) = \int_{1}^{2} \ln x dx = [x(\ln x - 1)]_1^2 = -1 + 2\ln 2$

(C) Given $f(x) \ge 0 \Rightarrow \int_{a}^{b} f(x)dx \ge 0.$

But given $\int_{a}^{b} f(x)dx = 0,$ so this can be true only when $f(x) = 0.$

(D) $\int_{a}^{b} f(x)dx = 0 \Rightarrow y = f(x)$ cuts x axis at least once.

So, there exists at least one $c \in (a,b)$ for which $f(x) = 0.$

50.(AC) $\int_{0}^{1} e^{x^2-x} dx$

For $x \in (0,1), x^2 - x \in (-1/4, 0)$ $\quad \Rightarrow e^{-1/4} < e^{x^2-x} < e^0$ $\quad \Rightarrow e^{\frac{-1}{4}} < \int_{0}^{1} e^{x^2-x} dx < 1$

51.(AD) $f(x+\pi)=\int_0^{x+\pi}(\cos(\sin t)+\cos(\cos t))dt=\int_0^{\pi}(\cos(\sin t)+\cos(\cos t))dt+\int_x^{x+\pi}(\cos(\sin t)+\cos(\cos t))dt$

$=f(x)+\int_0^{\pi}(\cos(\sin t)+\cos(\cot t))dt$ ($\because$ for $g(x)=\cos(\sin x)+\cos(\cos x), f(x+\pi)=f(x))=f(\pi)+f(x)$

$=f(x)+2f\left(\frac{\pi}{2}\right)$ ($\because g(x)$ has period $\pi/2$)

52.(AB) Let $f(x)=\int_0^{x^2}\left(\frac{t^2-5t+4}{2+e^t}\right)dt$; $f^1(x)=\left(\frac{x^4-5x^2+4}{2+e^{x^2}}\right)\times 2x$ For extremum $f^1(x)=0$

53.(BD) For x = 0

$G(0,t)=0, t\ge 0$

$g(1)=\int_0^1 f(t)G(0,t)dt=\int_0^1 0.dt=0$

And for x = 1

$G(1,t)=0, t<1$

$g(1)=\int_0^1 f(t)G(1,t)dt=0$

Also, $g(x)=\int_0^x f(t)G(x,t)dt+\int_x^1 f(t)G(x,t)dt=(x-1)\int_0^x tf(t)dt=x\int_x^1(t-1)f(t)dt$

$\therefore$ $g^{11}(x)=xf(x)+\{0-(x-1)f(x)\}=f(x)$

54.(ACD) Let $I=\int_{-1/2}^{1/2}\sqrt{\left(\frac{x+1}{x-1}-\frac{x-1}{x+1}\right)}dx=\int_{-1/2}^{1/2}\left|\frac{4x}{x^2-1}\right|dx=-2\int_0^{1/2}\frac{4x}{(x^2-1)}dx=-4\left\{ln\left|x^2-1\right|\right\}_0^{1/2}=-4ln\left(\frac{3}{4}\right)$

$=4ln\left(\frac{4}{3}\right)=ln\left(\frac{256}{81}\right)=-ln\left(\frac{81}{256}\right)$

55.(AC) $\int_a^{a+nT}f(x)dx=n\int_0^T f(x)dx$ **56.(ABCD)** $f'(x)=\sin x\times\frac{2x}{1+\cos^2 x}+\cos x\int_{\pi^2/4}^{x^2}\frac{1}{1+\cos^2\sqrt{t}}dt$

57.(ABC) $I_{n+2}-I_n=\int_{-\pi}^{\pi}\frac{\sin(n+2)x-\sin nx}{\sin x}dx=\int_{-\pi}^{\pi}2\cos(n+1)dx=0\Rightarrow I_1=I_3=I_5=....$

$I_{2m}=2\int_0^{\pi}\frac{sin 2mx}{sin x}dx=0$ $(\because f(\pi-x)=-f(x))$

58.(BD) Differentiating both sides w.r.t. x

$\int_0^x y(t)\,dt+xy(x)=\int_0^x t\,y(t)\,dt+(x+1)x\,y(x)$ or $\int_0^x y(t)\,dt=\int_0^x t\,y(t)\,dt+x^2y(x)$

again differentiating $y(x)=xy(x)+x^2y'(x)+2xy(x)$ or $x^2\frac{dy}{dx}+(3x-1)y=0$

separating variable and integrating $y=\frac{e}{x^3}e^{-1/x}$]

59.(ABCD)

Consider $Q = \int_0^\infty \frac{x\,dx}{1+x^4}$ $\quad x^2 = t;\ 2xdx = dt \quad = \frac{1}{2}\int_0^\infty \frac{dt}{1+t^2} \quad = \frac{1}{2}\tan^{-1} t\Big]_0^\infty \quad = \frac{\pi}{4}$

Consider $P = \int_0^\infty \frac{x^2}{1+x^4}dx$ $\qquad x = 1/t;\ dx = -1/t^2\ dt$

$$= \int_0^\infty \frac{1}{t^2}\cdot\frac{t^4}{1+t^4}\frac{1}{t^2}dt = \int_0^\infty \frac{dt}{1+t^4}$$

$$\therefore \qquad P = \int_0^\infty \frac{dt}{1+t^4} = R$$

Now $P+R = \int_0^\infty \frac{1+x^2}{1+x^4}dx = \int_0^\infty \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int_0^\infty \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx$ $\quad$ put $x - \frac{1}{x} = t \Rightarrow 1+\frac{1}{x^2}dx = dt$

$$= \int_{-\infty}^\infty \frac{dt}{t^2+2} = 2\int_0^\infty \frac{dt}{t^2+2} = 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\Big]_0^\infty \quad = \sqrt{2}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{2}}$$

60.(AB) $v = \ = \int_0^{\pi/4} \frac{1+\sin 2x}{\cos^2 x}dx = \int_0^{\pi/4} (\sec^2 x + 2\tan x)\,dx = \tan x + 2\ln(\sec x)]_0^{\pi/4} = (1+\ln 2) - 0 = 1+\ln 2$

again $u = \int_0^{\pi/4} \frac{\cos^2 x}{1+\sin 2x}dx$ $\qquad$ putting $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$

$$u = \int_0^{\pi/4} \frac{dx}{1+\tan^2 x + 2\tan x} = \int_0^{\pi/4} \frac{dx}{(1+\tan x)^2} = \int_0^{\pi/4} \frac{dx}{(1+\tan(\pi/4)-x)^2}$$

$$= \int_0^{\pi/4} \frac{dx}{\left(1+\frac{1-\tan x}{1+\tan x}\right)^2} = \int_0^{\pi/4} \frac{(1+\tan x)^2}{4}dx \quad u = \frac{1}{4}\left[\int_0^{\pi/4} (1+2\tan x+\tan^2 x)dx\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{4} + 2\ln\sec x\Big|_0^{\pi/4} + \tan x - x\Big|_0^{\pi/4}\right] = \frac{1}{4}\left[\frac{\pi}{4}+\frac{2}{2}(\ln 2)+1-\frac{\pi}{4}\right] = \frac{1}{4}[(\ln 2)+1]$$

Hence $\quad u = \frac{1}{4}[(\ln 2)+1]$ and $v = (1+\ln 2)$; $\qquad \frac{v}{u} = 4$

61.(ABC) $\quad$ Let $I = \int_0^2 \frac{\ln(1+2x)}{1+x^2}dx$

put $x = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta d\theta$; when $x = 2,\ \theta = \tan^{-1}(2)$

$$I=\int_0^{\tan^{-1}2}\ln(1+2\tan\theta)\,d\theta \qquad ....(1)$$

applying king

$$I=\int_0^{\tan^{-1}2}\ln\left(1+2\tan(\tan^{-1}2-\theta)\right)d\theta \;=\int_0^{\tan^{-1}2}\ln\left(1+2\left(\frac{2-\tan\theta}{1+2\tan\theta}\right)\right)d\theta$$

$$I=\int_0^{\tan^{-1}2}\ln 5\,d\theta \;-\underbrace{\int_0^{\tan^{-1}2}\ln(1+2\tan\theta)\,d\theta}_{I}$$

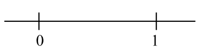
$$2I=\tan^{-1}1.\ln 5$$

$$I=\frac{1}{2}\tan^{-1}2.\ln 5=\tan^{-1}2.\ln\sqrt{5}$$

Hence $a=2$ and $b=5$ $\quad\therefore\; a^2+b^2=29$ $\quad$ Note that: $\displaystyle\int_0^a\frac{\ln(1+ax)}{1+x^2}dx\;=\tan^{-1}a.\ln\sqrt{(1+a^2)}$

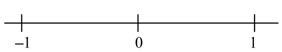
62.(ABD)

$$D=4+4\left(k+\int_0^1|k+t|\,dt\right)=4+4k+4\int_0^1|k+t|\,dt$$



let $\quad k\ge 0$

$$I=\int_0^1|k+t|\,dt \quad =\int_0^1(k+t)dt\;=k+\frac{1}{2}$$



Hence D becomes $\qquad 4+4k+4\left(k+\frac{1}{2}\right)$

$$4+8k+2=8k+6>0=4\left(2k+\frac{3}{2}\right)$$

let $\qquad k\le -1, I=-\displaystyle\int_0^1(k+t)\,dt\;=-\left[kt+\frac{t^2}{2}\right]_0^1=-\;\left[k+\frac{1}{2}\right]$

$\therefore \qquad D=4+4k-4\left(k+\frac{1}{2}\right)$

$$D=2>0$$

$-1<k<0 \qquad I=\displaystyle\int_0^1|k+t|dt$

Let $\quad k=-y \quad\Rightarrow\quad 0<y<1$

$$I=\int_0^1 |t-y|\,dt \;=\int_0^y (y-t)\,dt+\int_y^1 (t-y)\,dt=\left[ty-\frac{t^2}{2}\right]_0^y+\left[\frac{t^2}{2}-yt\right]_y^1$$

$$=\left(y^2-\frac{y^2}{2}\right)+\left(\frac{1}{2}-y\right)-\left(\frac{y^2}{2}-y^2\right)=y^2-y+\frac{1}{2}=k^2+k+\frac{1}{2}$$

$$D=4+4k+4\left(k^2+k+\frac{1}{2}\right)=4\left[1+k+k^2+k+\frac{1}{2}\right]=4\left[(k+1)^2+\frac{1}{2}\right]>0$$

Hence $D>0\ \forall\ k\in R$ $\Rightarrow$ roots are real and distinct

63.(AB) $x=k\cos^2\theta+(k+1)\sin^2\theta$ [note $\alpha=k$ and $\beta=k+1$]

$x=k+\sin^2\theta$

$dx=\sin 2\theta\, d\theta$

$$I=\int_0^{\pi/2}\sin\theta\cos\theta\, 2\sin\theta\cos\theta\, d\theta=\frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\, d\theta \quad \text{put } 2\theta=t$$

$$=\frac{1}{4}\int_0^{\pi}\sin^2 t\,dt=\frac{1}{2}\int_0^{\pi/2}\sin^2 t\,dt=\frac{\pi}{8}$$

$$\therefore \quad L=\frac{1}{n^2}\lim_{n\to\infty}\sum_{k=0}^{n-1}k\left(\frac{\pi}{8}\right)=\lim_{n\to\infty}\frac{\pi}{8}\frac{1}{n^2}[1+2+....+n-1]=\lim_{n\to\infty}\frac{\pi}{8}\frac{(n-1)n}{2}\cdot\frac{1}{n^2}=\frac{\pi}{16}$$

64.(AB) **(A)** $g'(x)=cx^{c-1}.e^{2x}+x^c.e^{2x}.2$

$f'(x)=e^{2x}\left(3x^2+1\right)^{1/2}$

$$\underset{x\to\infty}{\text{Limit}}\frac{f'(x)}{g'(x)}=\underset{x\to\infty}{\text{Limit}}\frac{e^{2x}\left(3x^2+1\right)^{1/2}}{c\,x^{c-1}\,.\,e^{2x}+2x^c\,.\,e^{2x}}=\underset{x\to\infty}{\text{Limit}}\frac{x\left(3+\frac{1}{x^2}\right)^{1/2}}{x^c\left(\frac{c}{x}+2\right)}$$

If $x\to\infty$ it will be finite if $c=1$ and $\underset{x\to\infty}{\text{Limit}}$ will be $\frac{\sqrt{3}}{2}$

65.(ABC) For $x>e$, we know that $1<\ln x<\frac{x}{e}$

$$\left(\frac{e}{x}\right)^{1/3}<\frac{1}{\sqrt[3]{\log x}}<1$$

$$\Rightarrow \int_3^4 e^{1/3}x^{-1/3}dx<I<\int_3^4 dx \quad\Rightarrow\quad \frac{3}{2}e^{1/3}\left(4^{2/3}-3^{2/3}\right)<I<1 \quad\Rightarrow\quad 0.92<I<1.$$

66.(ABC) For $0<x<1$, $x^2<x$

$$\Rightarrow -x^2>-x\Rightarrow e^{-x^2}>e^{-x} \quad\Rightarrow\quad \int_0^1 e^{-x^2}\cos^2 x dx>\int_0^1 e^{-x}\cos^2 x dx \text{ and } \cos^2 x\le 1$$

$$\Rightarrow \int_0^1 e^{-x^2}\cos^2 x dx<\int_0^1 e^{-x^2}dx\ ;\ <\int_0^1 e^{-\frac{1}{2}x^2}dx=1;\ <\int_0^1 e^{-\frac{1}{2}x^2}dx=1$$

Hence, I_4 is the greatest integral.

67.(A,B,C,D)

We have $1+x^2+2x^5>1+x^2$ and $1+x^2+2x^5<1+x^2+2x^2=1+3x^2$ $\left[\because\ x^5<x^2\ on(0,1)\right]$

Hence, we have $\dfrac{1}{1+3x^2}<\dfrac{1}{1+x^2+2x^5}<\dfrac{1}{1+x^2}$

$\Rightarrow \quad \int_0^1\dfrac{dx}{1+3x^2}<\int_0^1\dfrac{dx}{1+x^2+2x^5}<\int_0^1\dfrac{dx}{1+x^2} \Rightarrow \left[\dfrac{\tan^{-1}\sqrt{3}x}{\sqrt{3}}\right]_0^1<\int_0^1\dfrac{dx}{1+x^2+2x^5}<\left[\tan^{-1}x\right]_0^1$

$\Rightarrow \quad \dfrac{\pi}{3\sqrt{3}}\le\int_0^1\dfrac{dx}{1+x^2+2x^5}\le\dfrac{\pi}{4}.$

Which is the desired result.

$\sin^4 x\le\sin^3 x\le\sin^2 x$

$\Rightarrow \quad -\sin^2 x\le-\sin^3 x\le-\sin^4 x \Rightarrow \quad 1-\sin^2 x\le 1-\sin^3 x\le 1-\sin^4 x$

So, we get $\sqrt{1-\sin^2 x}\le\sqrt{1-\sin^3 x}\le\sqrt{1-\sin^4 x}$

$\int_0^{\pi/2}\sqrt{1-\sin^2 x}dx=\int_0^{\pi/2}\cos x dx=[\sin x]_0^{\pi/2}=1,$ and $\int_0^{\pi/2}\sqrt{1-\sin^4 x}dx=\int_0^{\pi/2}\sqrt{\left(1+\sin^2 x\right)\cos^2 x}dx$

$=\int_0^{\pi/2}\cos x\sqrt{1+\sin^2 x}dx=\int_0^1\sqrt{1+t^2}dt=\left[\dfrac{1}{2}\left(t\sqrt{t^2+1}+\ln\left(t+\sqrt{t^2+1}\right)\right)\right]=\dfrac{1}{2}\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)$

$\therefore \quad 1\le\int_0^{\pi/2}\sqrt{1-\sin^3 x}\le\dfrac{1}{2}\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right).$

68.(BC) Put $x=0$ in the given equation $0=1-c.\int_0^1 f(t)e^{-t}dt$

Let $c=\dfrac{1}{k}$ where $k=\int_0^1 f(t)e^{-t}dt.$

Differentiating the given equation, we get $f(x)=e^x-2e^{2x}$

Now, $k=\int_0^1\left(e^t-2e^{2t}\right)e^{-t}dt=\int_0^1\left(1-2e^t\right)dt=3-2e$ $\quad\therefore\quad c=\dfrac{1}{k}=\dfrac{1}{3-2e}.$

69.(BD) $f(x)=x+x\int_0^1 y^2 f(y)dy+x^2\int_0^1 yf(y)dy=x\left(1+\int_0^1 y^2 f(y)dy\right)+x^2\left(\int_0^1 yf(y)dy\right)$

$\Rightarrow \quad f(x)$ is a quadratic expression.

Let $f(x)=ax+bx^2$, then $f(y)=ay+by^2,$ Where $a=1+\int_0^1 y^2 f(y)dy$

$=1+\int_0^1 y^2\left(ay+by^2\right)dy=1+\left(\dfrac{ay^4}{4}+\dfrac{by^5}{5}\right)_0^1$

$a=1+\dfrac{a}{4}+\dfrac{b}{5}$ & $b=\int_0^1 y\ f(y)dy$

$\dfrac{3a}{4}+\dfrac{b}{5}=1$

70.(ABC)

$$I=\int_{1/2}^{2}\frac{\ln t}{1+t^n}dt \qquad ....(1)$$

put $\quad t=\frac{1}{u} \quad \Rightarrow \quad dt=-\frac{1}{u^2}du$

$$I=\int_{2}^{1/2}\frac{\ln(1/u)}{1+(1/u^n)}\left(-\frac{1}{u^2}\right)du \quad =-\int_{1/2}^{2}\frac{u^n\ln u}{(1+u)^n u^2}du$$

$$I=-\int_{1/2}^{2}\frac{t^{n-2}\ln t}{1+t^n}dt \qquad ....(2)$$

$$2I=\int_{1/2}^{2}\frac{(1-t^{n-2})\ln t}{(1+t^n)}dt$$

for $n=1$, the integrand is $+ve$ in $(1/2,1)\cup(1,2)$ and zero only for $t=1 \Rightarrow 1>0$

for $n=2$, integrand is zero $\quad \Rightarrow \quad I=0$

for $n\ge 3$, integrand is always $-ve$ except for $t=1$ hence $I<0$

Note that for $n=0$ $I=\int_{1/2}^{2}\ln t\,dt=\left(t\ln t-t\right)_{1/2}^{2}=\left(2\ln 2-2\right)-\left(\frac{1}{2}\ln\frac{1}{2}-\frac{1}{2}\right)=\frac{5}{2}\ln 2-\frac{3}{2}>0]$

71.(ABC) $\displaystyle \lim_{x\to 0}\frac{\int_0^x \frac{t^2dt}{\left(a+t^r\right)^{1/p}}}{bx-\sin x}=\lim_{x\to 0}\frac{\frac{x^2}{\left(a+x^r\right)^{1/p}}}{b-\cos x}$ using L' Hosp0ital's rule

For existence of limit, $\displaystyle\lim_{x\to 0}$ denominator $=0$

$\therefore \quad b-1=0\Rightarrow b=1$

$$l=\lim_{x\to 0}\frac{x^2}{\left(a+x^r\right)^{1/p}}\cdot\frac{x^2}{(1-\cos x)}\cdot\frac{1}{x^2}=2\lim_{x\to 0}\frac{1}{\left(a+x^r\right)^{1/p}}=\frac{2}{a^{1/p}}$$

If $p=3$ and $l=1$, then $1=\frac{2}{a^{1/3}}\Rightarrow a=8$ $\qquad$ If, $p=2$ and $a=9$, then $l=\frac{1}{9^{1/2}}=\frac{2}{3}$

72.(ABC)

Here, $f(x)=e^x\underbrace{\int_0^1 e^t.f(t)dt}_{A(say)}$ $\qquad$ ….(i)

$f(x)=Ae^x \Rightarrow \quad f(t)=Ae^t$

Where, $A=\int_0^1 e^t.f(t)dt \;\Rightarrow\; A=\int_0^1 e^t.Ae^t dt;\; A=A\int_0^1 e^{2t}dt\ne 0$

Hence, $f(x)=0\Rightarrow f(1)=0$

Again, $g(x) = e^x \int_0^1 e^t g(t)dt + x$

$\Rightarrow \quad g(x) = Be^x + x \quad$ ….(ii)

$\Rightarrow \quad g(t) = Be^t + t$

Where, $B = \int_0^1 e^t g(t)\,dt \Rightarrow B = \int_0^t e^t\left(Be^t + t\right)dt \;\Rightarrow\; B = B\int_0^1 e^{2t}dt + \int_0^1 e^t.t.dt$

But $\quad \int_0^1 e^{2t}dt = \frac{1}{2}\left(e^2 - 1\right)$ and $\int_0^1 te^t dt = 1$

$\therefore \quad B = \frac{B}{2}\left(e^2 - 1\right) + 1 \;\Rightarrow\; 2B = B\left(e^2 - 1\right) + 2 \;\Rightarrow\; 3B = Be^2 + 2 \;\Rightarrow\; B = \frac{2}{3 - e^2}$

From Equation (ii), $g(x) = \left(\frac{2}{3 - e^2}\right)e^x + x \Rightarrow g(0) = \frac{2}{3 - e^2}$

Also, $f(0) = 0$

$\therefore \quad g(0) - f(0) = \frac{2}{3 - e^2} - 0 = \frac{2}{3 - e^2}$; $g(2) = \frac{2e^2}{3 - e^2} + 2 = \frac{6}{3 - e^2}$ $\therefore$ $\frac{g(0)}{g(2)} = \frac{2}{3 - e^2}.\frac{3 - e^2}{6} = \frac{1}{3}$

73.(CD) We have the equations of the tangents to the curve $y = \int_{-\infty}^{x} f(t)dt$ and $y = f(x)$ at arbitrary pints on them are

$Y - \int_{-\infty}^{x} f(t)dt = f(x)(X - x) \quad$ …(i)

And $Y - f(x) = f'(x)(X - x) \quad$ …(ii)

As Eqs. (i) and (ii) intersect at the same point on the X-axis Putting $Y = 0$ and equating x-coordinates, we have

$$x - \frac{f(x)}{f'(x)} = x - \frac{\int_{-\infty}^{x} f(t)\,dt}{f(x)} \;\Rightarrow\; \frac{f(x)}{\int_{-\infty}^{x} f(t)\,dt} = \frac{f'(x)}{f(x)} \;\Rightarrow\; \int_{-\infty}^{x} f(t)dt = cf(x) \quad \text{…..(iii)}$$

As $f(0) = 1 \Rightarrow \int_{-\infty}^{0} f(t)dt = c \times 1 \Rightarrow c = \frac{1}{2}$

$\Rightarrow \quad \int_{-\infty}^{x} f(t)dt = \frac{1}{2} f(x)$; differentiating both the sides and integrating and using boundary conditions, we get $f(x) = e^{2x}$; $y = 2ex$ is tangent to $y = e^{2x}$.

$\therefore \quad$ Number of solutons $= 1$.

Clearly, $f(x)$ is increasing for all x. $\quad \therefore \quad \lim_{x \to \infty}\left(e^{2x}\right)^{e^{-2x}} = 1 \quad$ (∞° form)

74.(ABC) We have, $g(x) = g(0) + xg'(0) + \frac{x^2}{2}g''(0) = -bx^2 + cx - 6$, $h(x) = g(x) = 4x^4 - ax^3 + bx^2 - cx + 6 = 0$ has 4 distinct real roots.

Given biquadratic equation has 4 distinct positive roots.

Let the roots be x_1, x_2, x_3 and x_4 .

Now, $\dfrac{\dfrac{1}{x_1}+\dfrac{2}{x_2}+\dfrac{3}{x_3}+\dfrac{4}{x_4}}{4} \ge \sqrt[4]{\dfrac{24}{x_1x_2x_3x_4}}$

$\Rightarrow \quad 2 \ge 2 \Rightarrow \dfrac{1}{x_1}=\dfrac{2}{x_2}=\dfrac{3}{x_3}=\dfrac{4}{x_4}=k \;\Rightarrow\; \dfrac{1}{x_1}.\dfrac{2}{x_2}.\dfrac{3}{x_3}.\dfrac{4}{x_4}=k^4 \;\Rightarrow\; \dfrac{24}{3/2}=k^4 \Rightarrow k=2$

$\therefore$ Roots are $\dfrac{1}{2},1.\dfrac{3}{2}$ and 2. Also, $a=20$ and $c=25$

75(CD) Given, $\left(f'(x)\right)^2+\left(g(x)\right)^2=1$

$f(x)+\int_0^x g(t)dt=\sin x(\cos x-\sin x)$

Differentiating both the sides, we get

$f'(x)+g(x)=\cos 2x-\sin 2x$ ….(i)

Squaring both the sides of Eq. (i), we get

$\left(f'(x)\right)^2+\left(g(x)\right)^2+2f'(x).g(x)=1-\sin 4x$

$\Rightarrow \quad 1+2f'(x).g(x)=1-\sin 4x$

$\therefore \quad 2f'(x)g(x)=-\sin 4x$

Now, substituting $g(x)=-\dfrac{\sin 4x}{2f'(x)}$ in Eq. (i), we get

$f'(x)-\dfrac{\sin 4x}{2f'(x)}=\cos 2x-\sin 2x$

Put $f'(x)=t$

$\Rightarrow \quad 2t^2-2(\cos 2x-\sin 2x)t-\sin 4x=0 \Rightarrow \quad t=\dfrac{2(\cos 2x-\sin 2x)\pm\sqrt{4(1-\sin 4x)+8\sin 4x}}{4}$

$\therefore \quad 4t=2(\cos 2x-\sin 2x)\pm\sqrt{4(1-\sin 4x)+8\sin 4x} \Rightarrow 2t=(\cos 2x-\sin 2x)\pm\sqrt{1+\sin 4x}$

Taking + ve sign, $2t=\cos 2x-\sin 2x+\cos 2x+\sin 2x$

$f(x)=\dfrac{1}{2}\sin 2x+C_1$ or $f(x)=\dfrac{\cos 2x}{2}+C_2$

$f(0)=0$

$\Rightarrow \quad C_1=0$ and $C_2=-1/2$

$\therefore \quad f(x)=\dfrac{1}{2}\sin 2x$ or $f(x)=\dfrac{\cos 2x-1}{2}$

If $f'(x)=\cos 2x$, then $g(x)=-\sin 2x$

If $f'(x)=-\sin 2x$, then $g(x)=\cos 2x$

i.e. $f(x)=\dfrac{1}{2}\sin 2x$ and $g(x)=\cos 2x$

$\Rightarrow \quad f(x)=\dfrac{\cos 2x-1}{2}$ and $g(x)=\cos 2x$

76.(ABD)

Given, $f\left(f\left(x\right)\right)=-x+1$

Replacing x by $f\left(x\right)$, we get

$f\left(f\left(f\left(x\right)\right)\right)=-f\left(x\right)+1$

$f\left(1-x\right)=-f\left(x\right)+1$

$f\left(x\right)+f\left(1-x\right)=1$

Now, $\alpha=\int_0^1 f\left(x\right)dx=\int_0^1 f\left(1-x\right)dx$ (using King's property)

$\Rightarrow \quad 2\alpha=\int_0^1\left(f\left(x\right)+f\left(1-x\right)\right)dx \Rightarrow \quad 2\alpha=\int_0^1 dx=1\Rightarrow\alpha=\frac{1}{2}$

Put $x=\frac{1}{4}$ in Eq. (i), $f\left(\frac{1}{4}\right)+f\left(1-\frac{1}{4}\right)=1\Rightarrow f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)=1$

Now $I=\int_0^{\pi/2}\frac{\sin x}{(\sin x+\cos x)^3}dx$

$$I=\int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-x\right)}{\left(\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)\right)^3}dx \Rightarrow I=\int_0^{\pi/2}\frac{\cos x}{(\cos x+\sin x)^3}dx$$

$$\therefore\ 2I=\int_0^{\pi/2}\frac{1}{(\sin x+\cos x)}dx=\frac{1}{2}\int_0^{\pi/2}\frac{dx}{\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)^2}$$

$$=\frac{1}{2}\int_0^{\pi/2}\frac{dx}{\sin^2\left(\frac{\pi}{4}+x\right)}=\frac{1}{2}\int_0^{\pi/2}\operatorname{cosec}^2\left(\frac{\pi}{4}+x\right)dx=-\frac{1}{2}\left[\cot\left(\frac{\pi}{4}+x\right)\right]_0^{\pi/2}=-\frac{1}{2}[-1-1]=1 \Rightarrow I=\frac{1}{2}$$

77. [A-s] [B-s] [C-r] [D-q]

(A) $\int\limits_{-1}^{1}[x+[x+[x]]]dx$ (use property $[x+n]=[x]+n$ if n is integer)

$$=\int\limits_{-1}^{1}3[x]dx=3\int\limits_{-1}^{1}[x]dx=-3$$

(B) $\int_2^5([x]+[-x])dx=\int_2^5 -1dx=-3$

(C) $\operatorname{sgn}(n-[x])=\begin{cases}1,\text{if } x \text{ is not an integer}\\ 0,\text{if } x \text{ is an integer}\end{cases}$ Hence, $\int_{-1}^{3}\operatorname{sgn}(x-[x])dx=4(1-0)=4.$

(D) Let $I=25\int_0^{\pi/4}\left(\tan^6(x-[x])+\tan^4\left(x-[x]\right)\right)dx$ $\left\{\because\ 0<x\le\frac{\pi}{4}\Rightarrow[x]=0\right\}$

$\therefore I=25\int_0^{\pi/4}\left(\tan^6 x+\tan^4 x\right)dx$

$$=25\int_0^{\pi/4}\tan^4 x\left(\tan^2 x+1\right)dx=25\int_0^{\pi/4}\tan^4 x\sec^2 x\,dx=25\left(\frac{\tan^5 x}{5}\right)_0^{\pi/4}=25\times\frac{1}{5}=5$$

78. **[A-r] [B-p] [C-s] [D-q]**

(A) $\lim_{n\to\infty}\left[\dfrac{\int_0^2\left(1+\frac{t}{n+1}\right)^n dt}{n+1}\right] \Rightarrow \lim_{n\to\infty}\left[\left(1+\frac{t}{n+1}\right)^{n+1}\right]_0^2 = \lim_{n\to\infty}\left(1+\frac{2}{n+1}\right)^{n+1}-1$

(B) $f'(x)=f(x)\Rightarrow f(x)=Ce^x$ and since $f(0)=1$

$\therefore \quad 1=f(0)=C \qquad \therefore \quad f(x)=e^x$ and hence $g(x)=x^2-e^x$.

Thus, $\int_0^1 f(x)g(x)dx = \int_0^1 (x^2e^x-e^{2x})dx = x^2e^x\Big| - 2\int_0^1 xe^x dx - \left.\frac{e^{2x}}{2}\right|_0^1 = (e-0)-2xe^x\Big|_0^1 + 2e^x\Big|_0^1 - \frac{1}{2}(e^2-1)$

$= (e-0)-2e+2e-2-\frac{1}{2}(e^2-1) = e-\frac{1}{2}e^2-\frac{3}{2}$

(C) $I=\int_0^1 e^{e^x}(1+xe^x)dx \qquad$ Let $e^x=t \Rightarrow \int_1^e e^t(1+t\log t)\frac{dt}{t} = \int_1^e e^t\left(\frac{1}{t}+\log t\right)dt = \left[e^t\log t\right]_1^e = e^e$

(D) $L=\lim_{k\to 0}\dfrac{\int_0^k (1+\sin 2x)^{1/x}dx}{k}\left(\text{form}\frac{0}{0}\right) \qquad \Rightarrow L=\lim_{k\to 0}(1+\sin 2k)^{\frac{1}{k}} = e^{\lim_{k\to 0}\frac{1}{k}(\sin 2k)} = e^2$

79. **[A-q] [B-r, s] [C-p] [D-p]**

(A) $I_1=\int_{\pi/6}^{\pi/3}\sec^2\theta\, f(2\sin 2\theta)d\theta$

Applying property $\int_a^b f(a+b-x)dx=\int_a^b f(x)dx \Rightarrow I_1=\int_{\pi/6}^{\pi/3}\sec^2\left(\frac{\pi}{2}-\theta\right)f\left(2\sin 2\left(\frac{\pi}{2}-\theta\right)\right)d\theta$

$I_1=\int_{\pi/6}^{\pi/3}\text{cosec}^2\theta\, f\left(2\sin 2\theta\right)d\theta = I_2$.

(B) $f(x+1)=f(x+3)\Rightarrow f(x)=f(x+2) \Rightarrow f(x)$ is periodic with period 2.

Then $\int_a^{a+b} f(x)dx$ is independent of *a*, for which *b* is multiple of 2.

$\Rightarrow b=2,4,6...$

(C) Let $I=\int_1^4 \dfrac{\tan^{-1}[x^2]}{\tan^{-1}[x^2]+\tan^{-1}[25+x^2-10x]}$ ……(1)

Applying $\int_a^b f(x)dx=\int_a^b \cdot f(a+b-x)dx$, we get : $I=\int_1^4\dfrac{\tan[(5-x)^2]}{\tan^{-1}[(5-x)^2]+\tan^{-1}[x^2]}dx$ …….(2)

Adding equations (1) and (2), we get : $2I=\int_1^4 dx \Rightarrow 2I=3\Rightarrow I=3/2$

(D) Let $y=\sqrt{x+\sqrt{x+\sqrt{x+...}}}=\sqrt{x+y}$

$\Rightarrow \quad y^2 - y - x = 0 \qquad \Rightarrow \quad y = \dfrac{1 \pm \sqrt{1+4x}}{2.1}$

$\Rightarrow \quad y = \dfrac{1+\sqrt{1+4x}}{2} \qquad (\because y > 1)$

$\Rightarrow \quad I = \int_0^2 \dfrac{1+\sqrt{1+4x}}{2} dx = \left[\dfrac{x}{2} + \dfrac{(1+4x)^{3/2}}{\frac{3}{2}.2.4}\right]_0^2 = \left[\left(1+\dfrac{27}{12}\right) - \left(0+\dfrac{1}{12}\right)\right] = 1 + \dfrac{26}{12} = \dfrac{19}{6}$

80. **[A-p, q] [B-p, q, r] [C-q, s] [D- s]**

(A) $I = \int_{-2}^{2} (\alpha x^3 + \beta x + \gamma) dx$

$\alpha x^3 + \beta x$ is an odd function

$I = 0 + 2\int_0^2 \gamma dx = 2.2\gamma = 4\gamma$

(B) $I = \dfrac{1}{2}\int_0^1 2\sin\alpha x \sin\beta x \, dx = \dfrac{1}{2}\int_0^1 (\cos(\alpha-\beta)x - \cos(\alpha+\beta)x) dx = \dfrac{1}{2}\left[\dfrac{\sin(\alpha-\beta)x}{\alpha-\beta} - \dfrac{\sin(\alpha+\beta)x}{\alpha+\beta}\right]_0^1$

$= \dfrac{1}{2}\left[\dfrac{\sin(\alpha-\beta)}{\alpha-\beta} - \dfrac{\sin(\alpha+\beta)}{\alpha+\beta}\right]$

Also, $2\alpha = \tan\alpha$ and $2\beta = \tan\beta \Rightarrow 2(\alpha-\beta) = \tan\alpha - \tan\beta$ and $2(\alpha+\beta) = \tan\alpha + \tan\beta$

$2(\alpha-\beta) = \dfrac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$ and $2(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$

Substituting these values, we get : $I = (\cos\alpha\cos\beta) - (\cos\alpha\cos\beta) = 0.$

(C) $f(x+\alpha) + f(x) = 0 \Rightarrow f(x+2\alpha) + f(x+\alpha) = 0 \Rightarrow f(x+2\alpha) = f(x)$

$\Rightarrow f(x)$ is periodic with period $2\alpha \Rightarrow \int_\beta^{\beta+2\gamma\alpha} f(x)dx = \gamma\int_0^{2\alpha} f(x)dx.$

(D) Let $I = \int_0^\alpha [\sin x]dx, \alpha \in [(2\beta+1)\pi, (2\beta+2)\pi], \beta \in N$, [where $[\cdot]$ denotes the greatest integer function.]

$I = \int_0^{2\beta\pi} [\sin x]dx + \int_{2\beta\pi}^{(2\beta+1)\pi} [\sin x]dx + \int_{(2\beta+1)\pi}^{\alpha} [\sin x]dx$

$= \beta\int_0^{2\pi} [\sin x]dx + 0 + \int_{(2\beta+1)\pi}^{\alpha} (-1)dx = -\beta\pi + (2\beta+1)\pi - \alpha = (\beta+1)\pi - \alpha$

$\Rightarrow \gamma\int_0^\alpha [\sin x]dx$ depends on α, β and α, β

81. **[A-r] [B-p] [C-q] [D-s]**

To solve I(n) elegantly, take $f(x) = \dfrac{1}{4}x^4(\log x)^n, n \geq 1$

$f'(x) = x^3(\log x)^n + \dfrac{n}{4}x^4(\log x)^{n-1}\left(\dfrac{1}{x}\right) = x^3(\log x)^n + \dfrac{n}{4}x^3(\log x)^{n-1}$

Integrating in the interval [1, e], we get $\dfrac{1}{4}e^4 = I(n) + \dfrac{n}{4}I(n-1) \Rightarrow \dfrac{4I(n) + nI(n-1)}{e^4} = 1$

Now $\quad I(0) = \dfrac{e^4 - 1}{4} \Rightarrow I(1) = \dfrac{3e^4 + 1}{16} \Rightarrow I(2) = \dfrac{5e^4 - 1}{32}$

Hence $\dfrac{64}{5e^4-1}\ I(2)=2$. $\displaystyle\lim_{n\to\infty} I(n)=\int_1^e x^3\left(\lim_{n\to\infty}(\log x)^n\right)dx$ $\therefore \displaystyle\lim_{n\to\infty} I(n)=0$

Finally, observe that $I(2)=\dfrac{5e^4-1}{32}>\dfrac{4}{3}$ (verify)

Now $I(3)=\dfrac{e^4-3I(2)}{4}<\dfrac{e^4-4}{4}$ $\therefore\ n=3$

82.(3) We have $f(x)=\sin x+\displaystyle\int_{-\pi/2}^{\pi/2}(\sin x+t\,f(t))dt=\sin x+\pi\sin x+\int_{-\pi/2}^{\pi/2}t\,f(t)dt$

$\therefore\ f(x)=(\pi+1)\sin x+A$

Now, $A=\displaystyle\int_{-\pi/2}^{\pi/2}t((\pi+1)\sin t+A)dt=2(\pi+1)\left(\int_0^{\pi/2}\underset{(I)}{t}\,\underset{(II)}{\sin t}\,dt\right)$ (By part)

$\Rightarrow\ A=2(\pi+1)$

Hence, $f(x)=(\pi+1)\sin x+2(\pi+1)$.

Therefore, $f_{\max.}=3(\pi+1)=M$

and $f_{\min.}=(\pi+1)=m.$ $\Rightarrow$ $\dfrac{M}{m}=3$

83.(4) Given $f(x)=x^3-\dfrac{3x^2}{2}+x+\dfrac{1}{4}=\dfrac{1}{4}(4x^3-6x^2+4x+1)$

$$=\frac{1}{4}(4x^3-6x^2+4x-1+2)$$

$$f(x)=\frac{1}{4}[x^4-(1-x)^4]+\frac{2}{4}$$

$$\therefore\quad f(1-x)=\frac{1}{4}[(1-x)^4-x^4]+\frac{2}{4}$$

$$\therefore\quad f(x)+f(1-x)=\frac{2}{4}+\frac{2}{4}=1 \qquad (1)$$

Replacing x by $f(x)$ we have

$$f[f(x)]+f[1-f(x)]=1 \qquad (2)$$

Now $I=\displaystyle\int_{1/4}^{3/4}f(f(x))dx$ (3)

Also, $I=\displaystyle\int_{1/4}^{3/4}f(f(1-x))dx=\int_{1/4}^{3/4}f(1-f(x))dx$ (4) [using (1)]

Adding (3) and (4), $2I=\displaystyle\int_{1/4}^{3/4}\left[f(f(x))+f(1-f(x))\right]dx=\int_{1/4}^{3/4}dx$

$\Rightarrow 2I=\dfrac{1}{2}\ \Rightarrow I=\dfrac{1}{4}$

$\therefore I=\dfrac{1}{4}\ \ \therefore I^{-1}=4$

84.(6) Given $f^3(x)=\int_0^x t.f^2(t)dt$

Differentiating, $3f^2(x)f'(x)=xf^2(x)$

$$f(x)\neq 0 \therefore f'(x)=\frac{x}{3};\ \therefore f(x)=\frac{x^2}{6}+C$$

But $f(0)=0 \Rightarrow C=0$

$f(6)=6$

85.(8) Let $I=\int_0^1 {}^{207}C_7.\underset{II}{x^{200}}.\underbrace{(1-x)^7}_{I}dx$

$$I={}^{207}C_7\left[\underbrace{\left.(1-x)^7\cdot\frac{x^{201}}{201}\right|_0^1}_{zero}+\frac{7}{201}\int_0^1(1-x)^6\cdot x^{201}dx\right]={}^{207}C_7\cdot\frac{7}{201}\int_0^1(1-x)^6\cdot x^{201}dx$$

Integrating by parts again 6 more times $={}^{207}C_7\cdot\frac{7!}{201.202.203.204.205.206.207}\int_0^1 x^{207}dx$

$$\frac{(207)!}{7!(200)!}\cdot\frac{7!}{201.202\cdots 207}\cdot\frac{1}{208}=\frac{(207)!}{(207)!7!}\cdot\frac{7!}{208}=\frac{1}{208}=\frac{1}{k}\Rightarrow k=208$$

86.(8) $I=\lim\limits_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{6n}}{n\sqrt{n}}=\lim\limits_{n\to\infty}\frac{1}{n}\sum\sqrt{\frac{r}{n}}=\int_0^6\sqrt{x}\,dx=\left[\frac{2}{3}x^{3/2}\right]_0^6=\frac{2}{3}\cdot 6\sqrt{6}=\sqrt{96}$

87.(4) $I_1=\int_0^1 x^{1004}(1-x)^{1004}dx=2\int_0^{1/2}x^{1004}(1-x)^{1004}dx$ (1)

And $I_2=\int_0^1 x^{1004}(1-x^{2010})^{1004}dx$ Put $x^{1005}=t$ $\Rightarrow 1005x^{1004}dx=dt$

$$\Rightarrow I_2=\frac{1}{1005}\int_0^1(1-t^2)^{1004}dt=\frac{1}{1005}\int_0^1\left(t(2-t)\right)^{1004}dt=\frac{1}{1005}\int_0^1 t^{1004}(2-t)^{2004}dt$$

Now put $t=2y \Rightarrow dt=2dy$

$$\Rightarrow I_2=\frac{1}{1005}\int_0^{1/2}(2y)^{1004}(2-2y)^{1004}dt=\frac{1}{1005}2\cdot 2^{1004}\cdot 2^{1004}\int_0^{1/2}y^{1004}(1-y)^{1004}dy$$

$$=\frac{1}{1005}2^{2009}\int_0^{1/2}y^{1004}(1-y)^{1004}dy=\frac{1}{1005}2^{2008}I_1\Rightarrow\frac{I_1}{I_2}=\frac{1005}{2^{2008}}\Rightarrow\frac{2^{2010}}{1005}\frac{I_1}{I_2}=4$$

88.(0) We have $J=\int_{-5}^{-4}(3-x^2)\tan(3-x^2)dx.$

Put $(x+5)=t,$ we get : $J=\int_0^1(3-(t-5)^2)\tan(3-(t-5)^2)dt=\int_0^1(-22+10t-t^2)\tan(-22+10t-t^2)dt.$

Now, $\quad K = \int_{-2}^{-1} (6-6x+x^2)\tan(6x-x^2-6)dx.$

Put $\quad (x+2) = z,$ we get :

$$K = \int_0^1 (6-6(z-2)+(z-2)^2)\tan(6(z-2)-(z-2)^2-6)dz = \int_0^1 (22-10z+z^2)\tan(-22+10z-z^2)dz$$

Hence, $\quad (J+K) = 0.$

89.(1) $\quad I_n = \int_0^1 x^n (1-x^2)^{1/2} dx$

$$= \int_0^1 x^{n-1}\cdot x(1-x^2)^{1/2} dx = \left(-x^{n-1}\frac{(1-x^2)^{3/2}}{3} \right)_0^1 + \int_0^1 (n-1)x^{n-2}\frac{(1-x^2)^{3/2}}{3}\, dx$$

$$= O + \left(\frac{n-1}{3}\right)\int_0^1 x^{n-2}(1-x^2)\sqrt{1-x^2}\,dx$$

$3\, I_n + (n-1)\, I_n = (n-1)\, I_{n-2}$

$$\frac{I_n}{I_{n-2}} = \frac{(n-1)}{(n+2)} \Rightarrow \lim_{n\to\infty} \frac{I_n}{I_{n-2}} = 1$$

90.(1) $\quad I = \int_0^n f'(x)[x]dx - \int_0^n xf'(x)dx + \frac{1}{2}\int_0^n f'(x)dx$

$$\sum_{r=1}^{n}\int_{r-1}^{r} f'(x)[x]dx - \left\{ \left(xf(x)\right)_0^n - \int_0^n f(x)dx \right\} + \frac{1}{2}\left(f(x)\right)_0^n$$

$$= \sum_{r=1}^{n}(r-1)\int_{r-1}^{r} f'(x)dx - nf(n) + \frac{1}{2}f(n) - \frac{1}{2}f(0) + \int_0^n f(x)dx$$

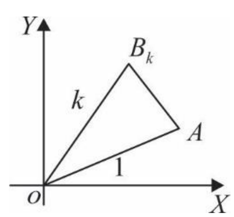
$$= \sum_{r=1}^{n}(r-1)\left\{f(r) - f(r-1)\right\} - nf(n) + \frac{1}{2}f(n) + \int_0^n f(x)dx - \frac{1}{2}f(0)$$

$$= -f(1) - f(2) - .... - f(n-1) - f(n) + \frac{1}{2}f(n) + \frac{1}{2}f(0) + \int_0^n f(x)dx$$

$$= -\sum_{r=1}^{n} f(r) + \frac{1}{2}f(n) + \frac{1}{2}f(0) + \int_0^n f(x)dx$$

91.(2) $\quad OB_k = k, \ \angle AOB_k = \frac{k\pi}{2n}$

$S_k = \frac{1}{2}k\sin\frac{k\pi}{2n} \quad$ (using $\Delta = \frac{1}{2}ab\sin\theta$)



$\therefore \quad L = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} S_k = \frac{k}{2n^2}\sum_{n=1}^{\infty}\sin\frac{k\pi}{2n}$

$$= \frac{1}{2n}\sum_{n=1}^{\infty}\frac{k}{n}\sin\frac{k\pi}{2n} = \frac{1}{2}\int_0^1 x.\sin\frac{\pi x}{2}dx = \frac{1}{2}\left[\frac{-2}{\pi}x\cos\frac{\pi x}{2}\Big| + \frac{2}{\pi}\int_0^1 x.\sin\frac{\pi x}{2}dx\right] = \frac{1}{2}\left[0 + \frac{2}{\pi}.\frac{2}{\pi}\left(\sin\frac{\pi x}{2}\right)_0^1\right] = \frac{2}{\pi^2}.$$

92.(0.72) $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{{}^nC_k}{n^k(k+3)}=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k+3}\,{}^nC_k\cdot\frac{1}{n^k}$

$=\lim_{n\to\infty}\sum_{k=0}^{n}{}^nC_k\cdot\frac{1}{n^k}\int_0^1 x^{k+2}dx\left\{\because \int_0^1 x^{k+2}dx=\frac{1}{k+3}\right\}$

$=\int_0^1\left(x^2\lim_{n\to\infty}\sum_{k=0}^{n}{}^nC_k\cdot\left(\frac{x}{n}\right)^k\right)dx=\int_0^1 x^2\left\{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right\}dx=\int_0^1 x^2.e^x dx\qquad\left\{\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x\right\}$

$=\left(x^2.e^x\right)_0^1-\int_0^1 2xe^x dx=e-2$

93.(3.14) $f'(x)+f^2(x)\geq -1$

$\therefore\quad f^2(x)+1\geq -f'(x)$ in (a,b)

$1\geq -\frac{f'(x)}{1+f^2(x)}$ in (a,b)

$\therefore\quad \int_a^b dx\geq -\int_a^b\frac{f'(x)}{1+f^2(x)}dx$

$b-a\geq -(\tan^{-1}(f(x))_a^b=-\left[\left(-\frac{\pi}{2}\right)-\frac{\pi}{2}\right]$

$\therefore\quad (b-a)\geq \pi.$

94.(101) Given $(f(x))^{101}=1+\int_0^x f(t)dt$

Differentiating $101.(f(x))^{100}.f'(x)=f(x)$

$\therefore\quad 101.(f(x))^{99}.f'(x)=1(as\ f(x)>0)$

Integrating, $\frac{(101)(f(x))^{100}}{100}=x+C$

but $f(0)=1$

$\Rightarrow\quad C=\frac{101}{100}$

$\therefore\quad \frac{101}{100}(f(x))^{100}=x+\frac{101}{100}$

Putting $x=101$,

$\frac{101}{100}(f(101))^{100}+\frac{101}{100}=\frac{(101)(101)}{100}$

$\therefore\quad (f(101))^{100}=101.$

95.(4) We have $\int_x^{xy} f(t)dt.$

Differentiating with respect of x, treating y as constant, we get

$$\frac{d}{dx}\int_x^{xy} f(t)dt = 0 \Rightarrow \quad yf(xy) - f(x) = 0 \qquad \ldots(1)$$

Putting $y = \frac{1}{x}$ in equation (1), we have $\frac{1}{x} f(1) - f(x) = 0$

$$\Rightarrow \quad f(x) = f(1).\frac{1}{x} \quad \Rightarrow \quad \int_1^x f(x)dx = f(1)\int_1^x \frac{1}{x}dx = f(1).\ln x$$

Now, putting $y = \frac{1}{2}$ and $x = 2$ in equation (1),

We have $\frac{1}{2} f(1) - f(2) = 0 \quad \Rightarrow \quad f(1) = 2f(2) = 4$ Hence, we have $\int_1^x f(x)dx = 4\ln x.$

96.(8) Let $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$f'(2) = 4a + b = 1$ ....(1)

now $I = \int\limits_{2-\pi}^{2+\pi} f(x)\cdot\sin\left(\frac{x-2}{2}\right)dx$ put $x - 2 = t \quad \Rightarrow \quad dx = dt$

$I = \int\limits_{-\pi}^{\pi} f(2+t)\cdot\sin\left(\frac{t}{2}\right)dt$ ....(2)

Also, $I = \int\limits_{-\pi}^{\pi} f(2-t)\cdot\sin\left(-\frac{t}{2}\right)dt$ (using $a+b-x$ property)

$I = -\int\limits_{-\pi}^{\pi} f(2-t)\cdot\sin\left(\frac{t}{2}\right)dt$ ....(3)

(2) + (3)

$2I = \int\limits_{-\pi}^{\pi} [f(2+t) - f(2-t)]\sin\frac{t}{2}dt$ ....(4)

now $f(2+t) - f(2-t)$

$\left[a(2+t)^2 + b(2+t) + c\right] - \left[a(2-t)^2 + b(2-t) + c\right]$

$a\left[(2+t)^2 - (2-t)^2\right] + b[2+t-2+t]$

$= 8at + 2bt = 2t(4a+b) = 2t$ (as $4a + b = 1$)

hence $2I = \int\limits_{-\pi}^{\pi} 2t\cdot\sin\frac{t}{2}dt \quad \Rightarrow \quad I = \int\limits_{-\pi}^{\pi} t\cdot\sin\frac{t}{2}dt = 2\int\limits_{0}^{\pi} t\cdot\sin\frac{t}{2}dt$ (put $\frac{t}{2} = y$)

$I = 8\int\limits_{0}^{\pi/2} y\sin y\,dy \quad \Rightarrow \quad I = 8$ as $\int\limits_{0}^{\pi/2} y\sin y\,dy = 1$

97.(10.50) $\int_0^{\pi/2}(\sin x+a\cos x)^3dx-\frac{4a}{\pi-2}\int_0^{\pi/2}x\cos x\,dx=2$

Let $I_1=\int_0^{\pi/2}(\sin x+a\cos x)^3dx=\int_0^{\pi/2}(\sin^3x+a^3\cos^3x+3a\sin^2x\cos x+3a^2\sin x\cos^2x)dx$

$=\int_0^{\pi/2}\sin^3x\,dx\ +a^3\int_0^{\pi/2}\cos^3x\,dx+\ 3a\int_0^{\pi/2}\sin^2x\cos x\,dx\ +3a^2\int_0^{\pi/2}\sin x\cos^2x\,dx$

$=\frac{2}{3}+\ a^3\left(\frac{2}{3}\right)\ +3a\int_0^{\pi/2}(1-\cos^2x)\cos x\,dx\ +3a^2\int_0^{\pi/2}\sin x(1-\sin^2x)\,dx$

$=\frac{2}{3}\left(1+a^3\right)+3a\ \left(1-\frac{2}{3}\right)+3a^2\left(1-\frac{2}{3}\right)=\frac{2}{3}+\frac{2a^2}{3}+a+a^2$

$I_1=\frac{2a^3}{3}+a^2+a+\frac{2}{3}$; $\quad I_2=\int_0^{\pi/2}\underset{I}{x}\cdot\underset{II}{\cos x}\,dx\ =x\sin x\Big|_0^{\pi/2}-\int_0^{\pi/2}\sin x\,dx$

$I_2=x\sin x+\cos x\Big|_0^{\pi/2}$; $\quad I_2=\frac{\pi}{2}-1=\frac{\pi-2}{2}$

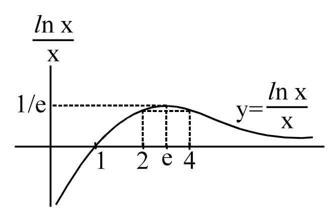
$I=\frac{2a^3}{3}+a^2+a+\frac{2}{3}-\frac{4a}{\pi-2}.\frac{\pi-2}{2}\quad\Rightarrow\quad\frac{2a^3}{3}+a^2-a+\frac{2}{3}=2$

$\Rightarrow\qquad 2a^3+3a^2-3a+2=6\qquad\Rightarrow\qquad 2a^3+3a^2-3a-4=0$

$a_1+a_2+a_3=-\frac{3}{2}\qquad\Rightarrow\sum a_1a_2\ =-\frac{3}{2}\qquad\therefore\qquad\sum a_1^{\ 2}=\frac{9}{4}+\frac{6}{2}+\frac{21}{4}$

98.(2) $I=\int_0^{\pi/2}\frac{\ell n\,(x\tan\theta)\cdot x\sec^2\theta}{x^2(1+\tan^2\theta)}\ d\theta=\frac{1}{x}\int_0^{\pi/2}\ \left(\ln x+\ln\tan\theta\right)d\theta$

$=\frac{\ell n\,x}{x}\int_0^{\pi/2}\ d\theta+\frac{1}{x}\int_0^{\pi/2}\ \ln\tan\theta d\theta=\frac{\pi\ \ell n\ x}{2\ x}+$ zero



Hence $\quad\frac{\pi\ \ell n\ x}{2\ x}=\frac{\pi\ \ell n\ 2}{4}\Rightarrow\frac{\ell n\ x}{x}=\frac{\ell n\ 2}{2}\Rightarrow x=2$ or 4

Note from the graph of $y=\frac{\ell n\ x}{x}$ that for all values of $y=\frac{\ell n\ x}{x}\in\left(0,\frac{1}{e}\right)$, there can be two values of x on either side of $x=e$ for which $\frac{\ell n\ x}{x}$ will have the same value.]

99.(0) $F\left(x\right)=\int_{-1}^{x}f(t)\,dt$ and $\qquad G\left(x\right)=\int_{x}^{1}f(t)\,dt\quad$ where $f\left(t\right)=\sqrt{4+t^2}$

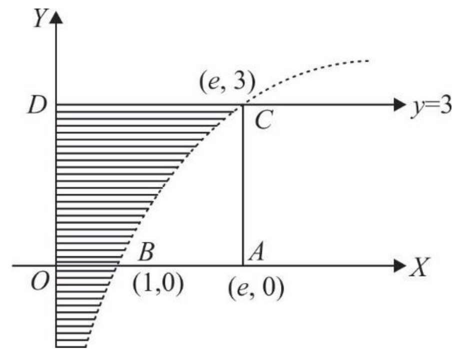
now $\qquad H\left(x\right)=F\left(x\right).G\left(x\right)$; $\qquad H'\left(x\right)=F\left(x\right).G'\left(x\right)+G\left(x\right).F\left(x\right)$

$H'\left(x\right)=\left(\int_{-1}^{x}f(t)\,dt\right)\left(-\sqrt{4+x^2}\right)\ +\left(\int_{x}^{1}f(t)\,dt\right)\left(\sqrt{4+x^2}\right)$

$$H'(x)=\sqrt{4+x^2}\left[\int_x^1\sqrt{4+t^2}\,dt-\int_{-1}^{x}\sqrt{4+t^2}\,dt\right];H'(0)=2\left[\int_0^1\sqrt{4+t^2}\,dt-\int_{-1}^{0}\sqrt{4+t^2}\,dt\right]$$

put $\quad t=-y \quad =\left[\int_0^1\sqrt{4+t^2}dt+\int_1^0\sqrt{4+y^2}dy\right] \quad =\left[\int_0^0\sqrt{4+t^2}dt\right]=$ zero

100.(8.15) Given, $f\left(\frac{x}{y}\right)=f(x)-f(y)$ Putting, $x=y=1, f(1)=0$



Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{h}$ [from Eq. (i)]

$=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}.x}$

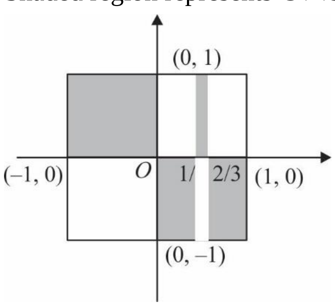
$\Rightarrow \quad f'(x)=\frac{3}{x} \qquad \left[\text{since, } \lim_{x\to 0}\frac{f(1+x)}{x}=3\right]$

$\Rightarrow \quad f(x)=3\log x+c$

Putting $x=1\Rightarrow c=0 \;\Rightarrow\; f(x)=3\log x=y$

$\therefore$ Required area $\int_{-\infty}^{3}x\,dy=\int_{-\infty}^{3}e^{y/3}dy=3\left[e^{y/3}\right]_{-\infty}^{3}=3(e-0)=3e$ sq units

101.(2) Shaded region represents $S\cap S'$ clearly are enclosed is 2 sq units.



102.(1) Differentiating both sides w.r.t. x

$2f(x).f'(x)=(f(x))^2+(f'(x))^2$ or $(f(x)-f'(x))^2=0 \quad\Rightarrow\quad f'(x)=f(x)$

(from the given relation $f(0)^2=e^2\Rightarrow f(0)=e$ or $-e$ (to be rejected))

now $\quad \frac{f'(x)}{f(x)}=1 \quad\Rightarrow\quad \ln(f(x))=x+C;$ but $f(0)=e$

$\therefore \quad \ln(e)=C \quad\Rightarrow\quad C=1 \quad \therefore \quad \ln(f(x))=x+1 \Rightarrow \quad f(x)=e^{x+1}$

103.(8) $I=\int_1^2\frac{(x^2-1)\,dx}{x^5\cdot\sqrt{2-2x^{-2}+x^{-4}}}$ (taking x^2 out from the radical sign)

$=\int_1^2\frac{(x^{-3}-x^{-5})\,dx}{\sqrt{2-2x^{-2}+x^{-4}}}$

put $2-2x^{-2}+x^{-4}=t^2$; when $x=2,\ t\to 5/4$

when $x=1, t\to 1$

$+4\left(x^{-3}-x^{-5}\right)dx=2tdt$

$$\left(x^{-3}-x^{-5}\right)dx=\frac{tdt}{2} = \frac{1}{2}\int_1^{5/4}\frac{t\,dt}{t} = \frac{1}{2}[t]_1^{5/4} = \left(\frac{5}{4}-1\right).\frac{1}{2}=\frac{1}{8}$$

104.(85) $\frac{d}{dx}\left(h(x)\right) = -\frac{\sin x}{\cos^2(\cos x)} \Rightarrow -\int\frac{\sin x}{\cos^2(\cos x)}dx = h(x)$

$\therefore \quad h(x) = -\int\frac{\sin x}{\cos^2(\cos x)}dx$; $\quad h(x)=\tan(\cos x)+C \quad$ ....(1)

comparing (1) with $h(x)=(fog)(x)+K$

we have $f(x)=\tan x$ and $g(x)=\cos x$

hence $j(0)=\int_{f(0)}^{g(0)}\frac{\tan t}{\cos t}dt = \int_0^1(\tan t\ \sec t)dt = \sec t\Big]_0^1 = \sec(1)-1$

105.(101) $I=\int_0^\infty\frac{dx}{x^2+\frac{1}{x^2}+(a^2-2)} = \int_0^\infty\frac{x^2\,dx}{x^4+(a^2-2)x^2+1} \qquad \left(a^2-2=k\geq 0\right)$

$$=\int_0^\infty\frac{x^2\,dx}{x^4+kx^2+1} = \frac{1}{2}\int_0^\infty\frac{(x^2+1)+(x^2-1)}{x^4+kx^2+1}dx = \underbrace{\frac{1}{2}\int_0^\infty\frac{1+(1/x^2)}{x^2+(1/x^2)+k}dx}_{I_1} + \underbrace{\frac{1}{2}\int_0^\infty\frac{1-(1/x^2)}{x^2+(1/x^2)+k}dx}_{I_2}$$

now proceed, $I_1=\frac{\pi}{2a}$ and $I_2=0$

$\therefore \boxed{I=\frac{\pi}{2a}}$; $\frac{\pi}{2a} = \frac{\pi}{5050} \Rightarrow \quad a=2525$

106.(153) $I=\int_0^\pi\sqrt{3+2(\cos x+\cos x+\cos 2x)}\,dx = \int_0^\pi\sqrt{3+2(2\cos x+2\cos^2 x-1)}\,dx$

$$=\int_0^\pi\sqrt{4\cos^2 x+4\cos x+1}\,dx = \int_0^\pi\left|1+2\cos x\right|dx = \int_0^{2\pi/3}(1+2\cos x)\,dx - \int_{2\pi/3}^{\pi}(1+2\cos x)\,dx$$

$$= x+2\sin x\Big|_0^{2\pi/3} - x+2\sin x\Big|_{2\pi/3}^{0} = \left(\frac{2\pi}{3}+\sqrt{3}\right)-\left(\pi-\frac{2\pi}{3}-\sqrt{3}\right) = \frac{\pi}{3}+2\sqrt{3}=\frac{\pi}{3}+\sqrt{12}$$

$\therefore \quad k=3$ and $w=12 \Rightarrow \left(k^2+w^2\right)=9+144=153$

107.(61) Consider $I_1=\int\frac{\cos 2x\cos x}{\sin^7 x}dx = \int\frac{(1-2\sin^2 x)\cos x}{\sin^7 x}dx$

$x=t \Rightarrow \sin x dx = dt$

$$I_1=\int\frac{(1-2t^2)dt}{t^7} = \int t^{-7}\,dt - 2\int t^{-5}\,dt = \frac{2}{4}\frac{1}{t^4} - \frac{1}{6t^6} = \frac{1}{2\sin^4 x} - \frac{1}{6\sin^6 x} = \frac{\text{cosec}^4 x}{2} - \frac{\text{cosec}^6 x}{6}$$

now integrating by parts the integral

$$I=\int_{\pi/4}^{\pi/2}\underset{I}{x}\cdot\underbrace{\frac{\cos 2x\cos x}{\sin^7 x}}_{II}dx = x\left(\frac{\text{cosec}^4 x}{2}-\frac{\text{cosec}^6 x}{6}\right)\Bigg|_{\pi/4}^{\pi/2}-\int_{\pi/4}^{\pi/2}\left(\frac{\text{cosec}^4 x}{2}-\frac{\text{cosec}^6 x}{6}\right)dx$$

$$I=\underbrace{\left[\left\{\frac{\pi}{2}\left(\frac{1}{2}-\frac{1}{6}\right)\right\}-\left\{\frac{\pi}{4}\left(2-\frac{4}{3}\right)\right\}\right]}_{zero}=-J$$

now $\quad -J=\int_{\pi/4}^{\pi/2}\left(\frac{\text{cosec}^2 x}{2}-\frac{\text{cosec}^4 x}{6}\right)\text{cosec}^2 x\,dx=\int_{\pi/4}^{\pi/2}\left(\frac{1+\cot^2 x}{2}-\frac{(1+\cot^2 x)^2}{6}\right)\text{cosec}^2 x\,dx$

$$\cot x = y, -\cos ec^2 x dx = dy$$

$$-J=+\int_0^1\left(\frac{1+y^2}{2}-\frac{(1+y^2)^2}{6}\right)dy$$

$$-J=\frac{16}{45}\quad\Rightarrow\quad J=-\frac{16}{45}\quad\Rightarrow\quad a+b=16+45=61$$

108.(10) $I=\int_0^{\pi}\frac{x\sin^3 x}{4-\cos^2 x}dx$

$I=\int_0^{\pi}\frac{(\pi-x)\sin^3 x}{4-\cos^2 x}dx$ add $2I=\pi\int_0^{\pi}\frac{\sin^3 x}{4-\cos^2 x}dx = 2\pi\int_0^{\pi/2}\frac{\sin^3 x}{4-\cos^2 x}dx$; $I=\pi\int_0^{\pi/2}\frac{\sin^3 x}{4-\cos^2 x}dx$

put $\cos x = t$

$$I=\pi\int_0^1\frac{1-t^2}{4-t^2}dt=\pi\int_0^1\frac{4-t^2-3}{4-t^2}dt=\pi\left[1-\int_0^1\frac{3\,dt}{4-t^2}\right]=\pi\left[1-3\frac{1}{2\cdot 2}\ln\frac{2+t}{2-t}\Bigg]_0^1\right]=\pi\left[1-\frac{3\ln 3}{4}\right]$$

hence $\quad a=3, b=3, c=4 \quad\Rightarrow\quad a+b+c=10$

109.(4.14) We have $f(x)=\sin x+\int_{-\pi/2}^{\pi/2}(\sin x+tf(t))dt.$

$$=\sin x+\pi\sin x+\int_{-\pi/2}^{\pi/2}tf(t)dt$$

$\therefore\quad f(x)=(\pi+1)\sin x+A$

Now, $A=\int_{-\pi/2}^{\pi/2}t((\pi+1)\sin t+A)dt=2(\pi+1)\int_0^{\pi/2}t\sin t\,dt$

$\Rightarrow\quad A=2(\pi+1)$

Hence, $\quad f(x)=(\pi+1)\sin x+2(\pi+1)$

$\therefore\quad f_{\max}=3(\pi+1)$ and $f_{\min}=(\pi+1).$

DIFFERENTIAL EQUATIONS

1.(C) $\frac{xdy}{x^2+y^2}=\left(\frac{y}{x^2+y^2}-1\right) \Rightarrow \frac{xdy-ydx}{x^2+y^2}=-dx$

$\int d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)=-\int dx \Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=-x+c \quad \Rightarrow \frac{y}{x}=\tan(c-x)$

2.(C) The given equation can be written as $y(1+x^{-1})dx+(x+\log x)dx+\sin ydx+x\cos ydy=0$
$\Rightarrow \; d(y(x+\log x))+d(x\sin y)=0 \;\Rightarrow\; y(x+\log x)+x\sin y=C$

3.(B) Rewriting the given equation the form $x^4\cos y\frac{dy}{dx}+4x^3\sin y=xe^x \;\Rightarrow \frac{d}{dx}(x^4\sin y)=xe^x$

$\Rightarrow \quad x^4\sin y=\int xe^x dx+C=(x-1)e^x+C$

Since $\quad y(1)=0,$ so $C=0$ $\qquad$ Thus $\quad \sin y=x^{-4}(x-1)e^x$

4.(A) Taking, $x=r\cos\theta$ and $y=r\sin\theta$, so that $x^2+y^2=r^2$ and $y/x=\tan\theta,$ we have $xdx+ydy=rdr$ and $xdy-ydx=x^2\sec^2\theta d\theta=r^2d\theta.$

The given equation can be transformed into $\frac{rdr}{r^2d\theta}=\sqrt{\frac{a^2-r^2}{r^2}} \Rightarrow \frac{dr}{d\theta}=\sqrt{a^2-r^2}$

$\Rightarrow \quad C+\sin^{-1}r/a=\theta=\tan^{-1}y/x$

$\Rightarrow \quad y=x\tan\left(C+\sin^{-1}\frac{1}{a}\sqrt{x^2+y^2}\right)$ or $\sqrt{x^2+y^2}=a\sin\left(\text{const.}+\tan^{-1}\frac{y}{x}\right)$

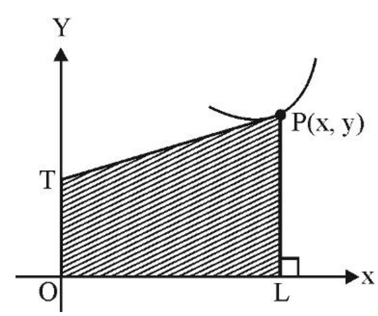
5.(A) Let $P(x,y)$ be any point on the curve. Length of intercept on y-axis by any tangent at

$$P(x,y)=OT=y-x\frac{dy}{dx}$$

$\therefore$ Area of trapezium OLPTO $=\frac{1}{2}(PL+OT)OL=\frac{1}{2}\left(y+y-x\frac{dy}{dx}\right)x=\frac{1}{2}\left(2y-x\frac{dy}{dx}\right)x$

According to question,

Area of trapezium $\quad$ OLPTO $=\frac{1}{2}x^2$



i.e., $\quad \frac{1}{2}\left(2y-x\frac{dy}{dx}\right)x=\pm\frac{1}{2}x^2 \Rightarrow 2y-x\frac{dy}{dx}=\pm x$

$\frac{xdy}{dx}-2y=\pm x$

$\frac{xdy}{dx}-\frac{2y}{x}=\pm 1$

I.F., $=e^{-\int\frac{2}{x}}=\frac{1}{x^2}$

Now solution is: $\frac{y}{x^2}=\int\pm\frac{1}{x^2}dx \quad\Rightarrow\quad \frac{y}{x^2}=\pm\frac{1}{x}+c \quad\Rightarrow\quad y=cx^2\pm x$

6.(C) The substitution $y = u^m \Rightarrow \frac{dy}{dx} = mu^{m-1}\frac{du}{dx}$ changes the equation to, $2.x^4.u^m.mu^{m-1}\frac{du}{dx} + u^{4m} = 4x^6$

$\Rightarrow \frac{du}{dx} = \frac{4x^6 - u^{4m}}{2mx^4x^{2m-1}}$. Since it is homogeneous, the degree of $4x^6 - u^{4m}$ and $2mx^4u^{2m-1}$ must be same

$\Rightarrow 6 = 4m = 4 + 2m - 1 \Rightarrow m = \frac{3}{2}$.

7.(C) The given equation is reduced to $x = e^{xy(dy/dx)} \Rightarrow \log x = xy\frac{dy}{dx}$

$$\Rightarrow \quad \int y\,dy = \int \frac{1}{x}\log_e x\,dx$$

$$\frac{y^2}{2} = \frac{(\log_e x)^2}{2} + C \Rightarrow y = \pm\sqrt{\{(\log x)^2 + C\}}$$

8.(A) The given differential equation can be written as $\frac{dx}{dy} = xy[x^2\sin y^2 + 1]$

$\Rightarrow \frac{1}{x^3}\frac{dx}{dy} - \frac{1}{x^2}y = y\sin y^2$. This equation is reducible to linear equation, so putting $-1/x^2 = u$, the last equation can be written as $\frac{du}{dy} + 2uy = 2y\sin y^2$.

The integrating factor of this equation is e^{y^2}. So required solution is $ue^{y^2} = \int 2y\sin y^2.e^{y^2}dy + C$

$= \int(\sin t).e^t dt + C = \frac{1}{2}e^{y^2}(\sin y^2 - \cos y^2) + C$ where $(t = y^2)$

$\Rightarrow \quad 2u = (\sin y^2 - \cos y^2) + Ce^{-y^2} \quad \Rightarrow \quad 2 = x^2[\cos y^2 - \sin y^2 - 2Ce^{-y^2}]$

9.(A) The general solution will be obtained by replacing p by c, where c is an arbitrary constant. So, the solution is $y = cx + \log c$.

10.(D) Differentiating the given equation we get $\left(x + \frac{1}{p}\right)\frac{dp}{dx} = 0$. For singular solution, we take $x + \frac{1}{p} = 0 \Rightarrow p = -\frac{1}{x}$.

Putting in the given equation, we obtain $y = -1 + \log\left(-\frac{1}{x}\right) \Rightarrow y + 1 = -\log(-x)$.

11.(B) Putting $\frac{dy}{dx} = p$, the equation becomes $y = xp + p^2$, which is the Clairut's equation. So, the solution is obtained by replacing p by c, so, $y = cx + c^2$, where c is the arbitrary constant.

12.(C) Singular form is $x + f'(p) = 0 \Rightarrow x + 2p = 0$

Therefore $p = -\frac{x}{2}$ putting in the given equation,

We get $y = x\left(-\frac{x}{2}\right) + \frac{x^2}{4} \Rightarrow y = -\frac{x^2}{4}$

13.(C) Put $x^2 = u$ and $y^2 = v$, then $\frac{dy}{dx} = \frac{x}{y}\left(\frac{dv}{du}\right)$

The equation then becomes, $x^2\left(y - \frac{x^2}{y}\frac{dv}{du}\right) = y.\frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2 \Rightarrow y^2 - x^2\frac{dv}{du} = \left(\frac{dv}{du}\right)^2$ or $v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$

which is Clairut's equation in variables u and v, so the solution is $v = uc + c^2 \Rightarrow y^2 = cx^2 + c^2$

14.(D) Writing $p = \frac{dy}{dx}$ and differentiating w.r.t. *x*, we have $p = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx}$

$\Rightarrow \quad 0 = p(1 + 2xp^3) + 2x\frac{dp}{dx}(1 + 2p^3x) \quad \Rightarrow \quad p + 2x\frac{dp}{dx} = 0 \Rightarrow 2\frac{dp}{p} = -\frac{dx}{x}$

$\Rightarrow \quad 2\log p + \log x = \text{const.} \Rightarrow p^2x = c$ or $p = \pm\sqrt{\frac{c}{x}}$ for $p = \sqrt{\frac{c}{x}}$ we get option 'D'

15.(B) $x\left(\cos\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)(xdy - ydx)$

Dividing both sides by x^2dx we get : $\left(\cos\frac{y}{x}\right)\left(\frac{y}{x} + \frac{dy}{dx}\right) = \frac{y}{x}\sin\left(\frac{y}{x}\right)\left(\frac{dy}{dx} - \frac{y}{x}\right)$

Which is homogeneous equation

Putting $y = vx$ we get :

$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad$ Or $\quad \cos v\left(v + v + x\frac{dv}{dx}\right) = v\sin v\left(x\frac{dv}{dx}\right)$

$\Rightarrow \quad 2v\cos v = x\frac{dv}{dx}(v\sin v - \cos v)$

Separating the variables, we get :

$\Rightarrow \quad \frac{2dx}{x} = \left(\frac{v\sin v - \cos v}{v\cos v}\right)dv$

Integrating $2\ln x + \ln c = \ln\sec v - \ln v$

$\Rightarrow \quad cx^2 = \frac{\sec v}{v} \Rightarrow \sec v = cx^2v \quad \Rightarrow \quad \sec\frac{y}{x} = cxy$

Where '*c*' is an arbitrary constant.

16.(A) The given equation can be written as $\left(\frac{dx}{x} - \frac{dy}{y}\right) + \left(\frac{x^2dy - y^2dx}{(x-y)^2}\right) = 0$

or $\quad \left(\frac{dx}{x} - \frac{dy}{x}\right) + \left(\frac{\frac{dy}{y^2} - \frac{dx}{x^2}}{\left(\frac{1}{y} - \frac{1}{x}\right)^2}\right) = 0 \quad$ or $\quad \left(\frac{dx}{x} - \frac{dy}{y}\right) + \left(\frac{\frac{dy}{y^2} - \frac{dx}{x^2}}{\left(\frac{1}{x} - \frac{1}{y}\right)^2}\right) = 0$

Integrating, we get $\ln|x| - \ln|y| - \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)} = c \quad$ or $\quad \ln\left|\frac{x}{y}\right| + \frac{xy}{(x-y)} = c$

Where 'c' is an arbitrary constant.

17.(C) Put $x = r\sec\theta$ and $y = r\tan\theta$. So, $x^2 - y^2 = r^2$ . . . . .(1)

and $\sin\theta = \frac{y}{x}$ . . . . .(2)

Then differentiating (1) we get : $2x\,dx - 2y\,dy = 2r\,dr$

Or $x\,dx - y\,dy = r\,dr$ . . . . .(3)

And differentiating (2) we get : $\frac{x\,dy - y\,dx}{x^2} = \cos\theta\, d\,\theta$

Or $x\,dy - y\,dx = x^2\cos\theta\, d\,\theta = r^2\sec^2\theta\cos\theta d\theta = r^2\sec\theta d\theta$ . . . . .(4)

Substituting values from (3) and (4) in the given differential equation, we get :

$$\frac{rdr}{r^2\sec^2\theta d\theta} = \sqrt{\left(\frac{1+r^2}{r^2}\right)} = \frac{\sqrt{1+r^2}}{r} \quad \text{or} \quad \frac{dr}{\sqrt{(1+r^2)}} = \sec\theta\, d\,\theta$$

Integrating both sides $\ln(r + \sqrt{(1+r^2)}) = \ln(\sec\theta + \tan\theta) + \ln c$

Where c is an arbitrary constant.

or $(r + \sqrt{(1+r^2)}) = c(\sec\theta + \tan\theta)$ or $\sqrt{(x^2 - y^2)} + \sqrt{(1 + x^2 - y^2)} = c\left(\frac{x+y}{\sqrt{(x^2-y^2)}}\right)$

18.(D) Differentiating both sides w.r.t, x of the given equation

$$x.y(x) + \int_0^x y(t)dt.\,1 = (x+1)xy(x) + \int_0^x ty(t)dt \quad \text{or} \quad \int_0^x y(t)dt = x^2y(x) + \int_0^x ty(t)dt$$

Again differentiating both sides w.r.t., x

$$y(x) = x^2y(x) + y(x)2x + xy(x)$$

or $(1-3x)y(x) = x^2y'(x)$ *or* $\frac{y'(x)}{y(x)} = \left(\frac{1}{x^2} - \frac{3}{x}\right)$

Integrating, we get $\ln y(x) = -\frac{1}{x} - 3\ln x + \ln c$

or $\ln\left(\frac{x^3y(x)}{c}\right) = -\frac{1}{x}$ *or* $\frac{x^3y(x)}{c} = e^{-1/x}$ or $y(x) = \frac{ce^{-1/x}}{x^3}$

19.(B) Put $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$ …………….(1)

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ …………… (2)

then differentiating (1) we get :

$$2x\,dx + 2y\,dy = 2r\,dr$$

or $x\,dx + y\,dy = r\,dr$ …………….(3)

and differentiating (2) we get :

$\frac{x\,dy + y\,dy}{x^2} = \sec^2\theta d\theta$ or $x\,dy + y\,dx = x^2\sec^2\theta\, d\,\theta$ ……………..(4)

Substituting values from (3) and (4) in the given differential equation, we get :

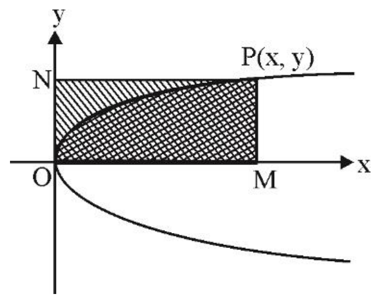
$$\frac{r\,dr}{r^2\,d\theta}=\sqrt{\left(\frac{a^2-r^2}{r^2}\right)}=\sqrt{\frac{(a^2-r^2)}{r}} \qquad \text{or} \qquad \frac{dr}{\sqrt{(a^2-r^2)}}=d\,\theta$$

Integrating both sides we gets : $\sin^{-1}\left(\frac{r}{a}\right)=\theta+c \Rightarrow r=a\sin(\theta+c)$

or $\sqrt{(x^2+y^2)}=a\sin\left(\tan^{-1}\left(\frac{y}{x}\right)+c\right)$ {from (1) and (2)} where c is an arbitrary constant.

20.(C) Let $P(x,y)$ be the point on the curve passing through the origin $O(0,0)$, and let PN and PM be the lines parallel to the x-and y-axes, respectively. If the equation of the curve is $y=y(x)$, then the area POM equals

$\int_0^x y\,dx$ and the area PON equals $xy-\int_0^x y\,dx.$



Assuming that 2(POM) = PON, we there fore have

$$2\int_0^x y\,dx=xy-\int_0^x y\,dx \quad \Rightarrow 3\int_0^x y\,dx=xy$$

Differentiating both sides of this gives

$$3y=x\frac{dy}{dx}+y \quad \Rightarrow 2y=x\frac{dy}{dx}\Rightarrow \frac{dy}{y}=2\frac{dx}{x}$$

$\Rightarrow \quad \log y=2\log x+C \Rightarrow y=Cx^2$,

With C being a constant. This solution represents a parabola. We will get a similar result if we has started instead with 2(PON) = POM.

21.(A) Differentiating the given equation, we have $2x+2y\frac{dy}{dx}+2g=0 \quad \Rightarrow g=-\left(x+y\frac{dy}{dx}\right)$

Putting this value in $x^2+y^2+2gx+C=0$, we have

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)+C=0$$

Replacing $\frac{dy}{dx}$ by $\frac{-dx}{dy}$, we have the differential equation of orthogonal trajectories as

$$y^2-x^2-2xy\frac{dx}{dy}+C=0 \qquad \Rightarrow \qquad 2x\frac{dx}{dy}-\frac{1}{y}x^2=\frac{-C}{y}-y$$

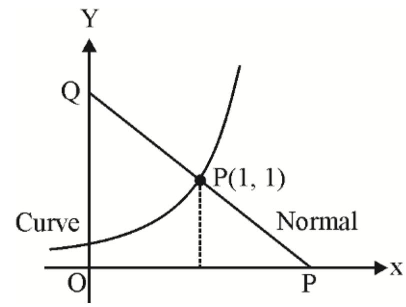
Putting $x^2=v$, we gave $\frac{dv}{dy}-\frac{1}{y}v=-\frac{C}{y}-y$, which is linear differential equation in v and y whose I.F. is $\frac{1}{y}$.

Hence $\frac{v}{y}=\int\left(-\frac{C}{y^2}-1\right)dy+C=\frac{C}{y}-y+k$ where 'k' is integration constant

$\Rightarrow \quad x^2+y^2-ky-C=0$, which represent system of circles with centre on y-axis.

22.(C) Given, equation of normal at $P(1,1)$ is $ay + x = a + 1$ $\therefore$ Slope of tangent at $P = a \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = a$

Given $\frac{dy}{dx} \propto y \Rightarrow \frac{dy}{dx} = ky$ $\Rightarrow$ $\left(\frac{dy}{dx}\right)_{(1,1)} = k = a$



$\therefore$ $\frac{dy}{dx} = ay \Rightarrow \frac{dy}{y} = a\,dx$ (Variables being separated)

$\Rightarrow$ $\ln y = ax + c$

It is passing through (1, 1) then $c = -a$

$\Rightarrow$ equation of the curve is $y = e^{a(x-1)}$.

23.(ABCD) Solving for $\frac{dy}{dx}$, we obtain $\frac{dy}{dx} = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2} = y(-\cot x \pm \operatorname{cosec} x)$

Thus, we have $\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)dx$

$\Rightarrow$ $\log y = -\log\sin x + \log\tan\frac{x}{2} + \log c$ $\Rightarrow$ $y = \frac{c\tan\frac{x}{2}}{\sin x} = \frac{c}{2\cos^2\frac{x}{2}} = \frac{c}{1+\cos x}$

Solving $\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)dx$, we get $y = \frac{c}{1-\cos x}$ $\Rightarrow$ $x = 2\sin^{-1}\sqrt{\frac{C}{2y}}$.

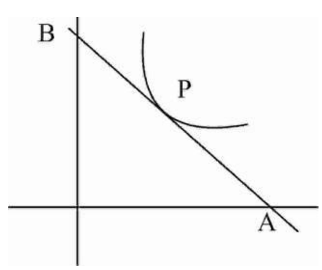
24.(AB) Rewriting the given equation, we get $\frac{dy}{dx} = x(e^{(n-1)y} - 1)$

$\Rightarrow$ $\frac{dy}{e^{(n-1)y} - 1} = xdx \Rightarrow \frac{1}{n-1}\int \frac{(n-1)e^{(n-1)y}}{(e^{(n-1)y} - 1)e^{(n-1)y}}dy = \frac{x^4}{2} + C$

$\Rightarrow$ $\frac{1}{n-1}\int \frac{du}{u(u-1)} = \frac{x^2}{2} + C$ (where $u = e^{(n-1)y}$)

$\Rightarrow$ $\frac{1}{n-1}\log\frac{u-1}{u} = \frac{x^2}{2} + C \Rightarrow \frac{1}{n-1}\log\left(\frac{e^{(n-1)y} - 1}{e^{(n-1)y}}\right) = \frac{x^2}{2} + C \Rightarrow e^{(n-1)y} = C\,e^{(n-1)y + (n-1)x^2/2} + 1$

25.(CD) $Y - y = \frac{dy}{dx}(X - x)$



$A\left(x - y\frac{dx}{dy}, 0\right), B\left(0, y - x\frac{dy}{dx}\right)$

$\frac{3\left(x - y\frac{dx}{dy}\right) + 0}{4} = x,$

$x\frac{dy}{dx} + 3y = 0$; $\frac{dy}{y} + 3\frac{dx}{x} = 0$

$\ell ny + 3\ell nx = \ell nc$ $\Rightarrow$ $yx^3 = c = 1$, as $f(1) = 1$

26.(AB) $f''(x) = g''(x) \Rightarrow f'(x) = g'(x) + c \Rightarrow f(x) = g(x) + cx + d$

$f'(1) = 4, g'(1) = 2 \Rightarrow c = 2$

$f(2) = 9, g(2) = 3 \Rightarrow d = 2$

$f(x) - g(x) = 2x + 2$

27.(BC) $\frac{dy}{dx} = \frac{6x^2}{2y + \cos y} \Rightarrow \int (2y + \cos y) dy = \int 6x^2 dx \Rightarrow (y^2 + \sin y) = 2x^3 + c$ and $c = \pi^2 - 2$

28.(AD) $(x^2+1)\frac{d^2y}{dx^2} = 2x\frac{dy}{dx} \Rightarrow \int \frac{y_2}{y_1} dx = \int \frac{2x}{x^2+1} dx \Rightarrow \ell n|y_1| = \ell n|x^2+1| + \ell nc$

$y_1 = c(x^2+1)$ and $c = 3 \Rightarrow \int dy = \int 3(x^2+1) dx \Rightarrow y = x^3 + 3x + 1$

29.(AB) Given equation of conic having its axis along the co-ordinate axes is $ax^2 + by^2 = 1$

$$\Rightarrow ax + by\frac{dy}{dx} = 0 \Rightarrow a + b\left(\frac{dy}{dx}\right)^2 + by\frac{d^2y}{dx^2} = 0 \Rightarrow y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2}$$

30.(AB) $3xydx + 4x^2dy + 2y^3dx + 6xy^2dy = 0 \Rightarrow 3xydx + 4x^2dy + 2d(xy^3) = 0$

$$\Rightarrow 3x^2y^4dx + 4x^3y^3dy + 2(xy^3)d(xy^3) = 0 \Rightarrow d(x^3y^4) + 2(xy^3)d(xy^3) = 0 \Rightarrow x^2y^6 + x^3y^4 = c$$

31.(AC) $\frac{dy}{dx} + Py = Q$ . . . .(i)

$\frac{dy_1}{dx} + Py_1 = Q$ . . . .(ii)

and $\frac{dy_2}{dx} + Py_2 = Q$ . . . .(iii)

$\frac{d(y_1 - y_2)}{dx} + P(y_1 - y_2) = 0$ . . . .(iv)

And $\frac{d(y - y_1)}{dx} + P(y - y_1) = 0 \quad \therefore \frac{d(y - y_1)}{d(y_1 - y_2)} = \frac{y - y_1}{y_1 - y_2}$

$$\Rightarrow \ln(y - y_1) = \ln(y_1 - y_2) + \ln c \Rightarrow y - y_1 = c(y_1 - y_2) \Rightarrow y = y_1 + c(y_1 - y_2)$$

Also $\alpha y_1 + \beta y_2$ is a solution $\Rightarrow \alpha y_1 + \beta y_2 = y_1(1+c) - cy_2$

$\alpha + \beta = 1$

32.(ABCD) $\frac{d\theta}{dt} = -k(\theta - \theta_a)$

$\ln|\theta - \theta_a|_{37}^{34} = -k \times 15 \Rightarrow k = \frac{1}{15}\ln\frac{7}{4}$ where $\theta_a = 30^\circ C$

$$t(\theta = 31^\circ) = \ln\left(\frac{37-30}{31-30}\right)\frac{15}{\ln 7/4} = 15\log_{7/4} 7$$

$$t(\theta = 32^\circ) = \ln\left(\frac{37-30}{32-30}\right)\frac{15}{\ln 7/4} = 15\log_{7/4}\frac{7}{2}$$

33.(BCD) $\frac{dv}{dt} = -kA \Rightarrow A\frac{dH}{dt} = -k.A \quad \Rightarrow \quad \int_H^0 dH = -k\int_0^T dt$

34.(CD) $x^2 + y^2 - kx = 0$

Diff. w.r.t. x, $2x + 2y\frac{dy}{dx} = k = \frac{x^2+y^2}{x}$

For orthogonal curve, $2x - 2y\frac{dx}{dy} = \frac{x^2+y^2}{x} \quad \Rightarrow \quad \frac{x^2-y^2}{2xy} = \frac{dx}{dy}$

35.(ABD) $\left|y\frac{dx}{dy}\right| = \frac{dy}{dx}$

$\Rightarrow y\frac{dx}{dy} = \frac{dy}{dx} \quad \because \frac{dy}{dx} > 0 \;\Rightarrow \frac{dy}{dx} = \sqrt{y} \;\Rightarrow 2\sqrt{y} = x + c$

$\because$ it passes through (2, 1) $\quad \therefore c = 0 \quad \therefore 2\sqrt{y} = x$ or $y^2 = \frac{x^2}{4}$

Area $= \int 2\sqrt{y}\,dy = 4\sqrt{3}\,sq.unit$

36.(ABD) $\frac{dy}{dx} - \frac{\tan 2x}{\cos^2 x}y = \cos^2 x$

I.F. = $e^{-\int \frac{\tan 2x}{\cos^2 x}dx} = e^{-\int \frac{2\sin 2x}{\cos 2x(1+\cos 2x)}dx} = e^{\log\frac{\cos 2x}{1+\cos 2x}} = \frac{\cos 2x}{1+\cos 2x}$

$y.\frac{\cos 2x}{1+\cos 2x} = \int \cos^2 x \times \frac{\cos 2x}{1+\cos 2x}dx + c = \frac{1}{2}\frac{\sin 2x}{2} + c$

$y = \frac{1}{2}\tan 2x \cos^2 x + c$

$\because y\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8} \therefore c = 0 \quad \therefore y = \frac{1}{2}\tan 2x \cos^2 x$

37.(CD) Equation of normal is $Y - y + \frac{dx}{dy}(X - x) = 0$

$\therefore$ Length of perpendicular from origin is $\frac{\left|y + x\frac{dx}{dy}\right|}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}} = y + x\frac{dx}{dy} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1$

38.(BC) $t = \frac{1}{y}$

$$\int_1^{1/x} \frac{-\ell ny}{1+\frac{1}{y}+\frac{1}{y^2}} \times \frac{-1}{y^2}dy = \int_1^{1/x} \frac{\ell ny}{y^2+y+1}dy$$

39.(AC) Diff. the equation we get : $x(1-x)f(x)+\int_0^x(1-t)f(t)dt = xf(x)$

$$\int_0^x(1-t)f(t)dt - x^2 f(x) \Rightarrow (1-x)f(x) = x^2 f'(x) + f(x).2x \Rightarrow \int\frac{f'(x)}{f(x)}dx = \int\left(\frac{1}{x^2}-\frac{3}{x}\right)dx$$

$$\ln f(x) = -\frac{1}{x} - 3\ln x + c$$

$\therefore$ $f(1)=1 \quad \therefore c=1$

. $x^2+y^2+2ax+2by+c=0$

$\because$ a, b, c are arbitrary constants. $\therefore$ order of the corresponding differential equation is 3.

Differentiating three times and eliminating the constant we get degree as 1.

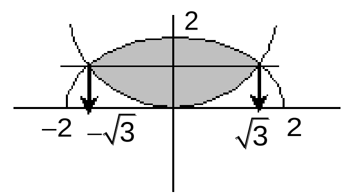
If $a+b=c^2$ and $a-2b=c \Rightarrow a=\frac{2c^2+c}{3}$ *and* $b=\frac{c^2-c}{3}$

40.(AB) Clearly, the constant function $y = 0$ is a solution. Differentiating the given equation with respect to x, we get.

$$p=\frac{dy}{dx}=\left(p+p^3\right)+x\left(\frac{dp}{dx}+3p^2\frac{dp}{dx}\right)$$

$$\Rightarrow -p^3 - x\left(1+3p^2\right)\frac{dp}{dx} \Rightarrow -\frac{dx}{x}=\left(\frac{1}{p^3}+\frac{3}{p}\right)dp \quad \Rightarrow \log xp^3 = \frac{1}{2p^2}+c \Rightarrow xp^3 = ke^{1/2p^2}$$

41.(BC) $\because$ $\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^3\frac{d^2x}{dy^2}$



$\therefore$ $y=f(x)$ is solution of $\frac{dy}{dx}=\frac{-x}{y}$

i.e. $f(x)=\sqrt{4-x^2}$ & $y=y(x)$ is solution of $\frac{dy}{dx}=\frac{2y}{x}$ i.e. $g(x)=\frac{1}{3}x^2$

42.(ABC) $y=e^{-x}\sin x$

$\Rightarrow$ $y_2 = e^{-x}\cos x - e^{-x}\sin x \Rightarrow$ $y_1=\sqrt{2}e^{-x}\sin\left(\frac{\pi}{4}-x\right)=-\sqrt{2}e^{-x}\sin\left(x-\frac{\pi}{4}\right)$

$\therefore$ $y_2=\left(\sqrt{2}\right)^2 e^{-x}\sin\left(x-\frac{\pi}{2}\right)$

$y_3=-\left(\sqrt{2}\right)^{2\backslash 3}e^{-x}\sin\left(x-\frac{3\pi}{4}\right)$ & $y_4=4e^{-x}\sin(x-\pi)$ $\Rightarrow y_4=-4y$

$\Rightarrow$ $y_4+4y=0 \Rightarrow y_8+4y_4=0 \Rightarrow y_8-16y_4=0$

43.(BCD) $\because$ $\int_0^x f(x)dx = kf^3(x) \Rightarrow \frac{1}{2}f^2(x)=\frac{x}{3k} \Rightarrow f(x)=3\sqrt{x}$

So, $f\left(\sin^2 x\right)=3|\sin x|$

44.(ACD) $\because$ Differential equation is $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0$

45.(AC) $\because V=\frac{1}{3}\pi r^2 h$ and $\frac{h}{27}=\frac{r}{18}$

$\therefore \quad V=\frac{1}{3}\pi\frac{27}{18}.r^3=\frac{\pi}{2}r^3 \quad ; \quad \frac{dV}{dr}=\frac{3\pi r^2}{2}\ ; \ \frac{1}{r^2}\frac{dV}{dr}=\frac{3\pi}{2}$

46.(ABCD) $\because \quad A\equiv\left(x-y\frac{dx}{dy},\ 0\right)$ and $B\equiv\left(0,\ y-x\frac{dy}{dx}\right)$

$\therefore \quad BP=\sqrt{x^2\left(1+\left(\frac{dy}{dx}\right)^2\right)}=|x|\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

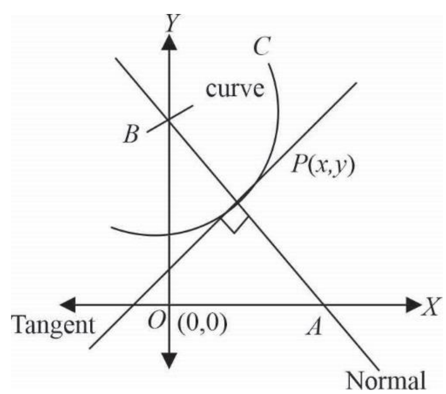
$AP=\sqrt{y^2\left(1+\left(\frac{dx}{dy}\right)^2\right)}=\frac{|y|\sqrt{\left(\frac{dy}{dx}\right)^2+1}}{\left|\frac{dy}{dx}\right|}$

$\therefore \quad \frac{BP}{AP}=\frac{|x|}{|y|}.\left|\frac{dy}{dx}\right|=\frac{3}{1} \quad \Rightarrow \quad \frac{dy}{dx}=\pm\frac{3y}{x} \quad \Rightarrow \quad \ln y=\pm\ln x^3 \quad \Rightarrow \quad y=x^3$ or $\frac{1}{x^3}$

47.(ABC) $\because \quad \frac{dy}{dx}+1=\frac{x^2-y+x+y}{x+y}$

$\Rightarrow \quad \frac{d(x+y)}{dx}=\frac{x^2+x}{x+y}\ ; \ \frac{(x+y)^2}{2}=\frac{x^3}{3}+\frac{x^2}{2}+c$ for $x=0=y \Rightarrow c=0$

48.(AD) The equation of normal of $P(x,y)$ is $(Y-y)=\frac{-1}{\frac{dy}{dx}}(X-x)$



$\therefore A\left(x+y\frac{dy}{dx},0\right)$ and $B\left(0,y+\frac{x}{\frac{dy}{dx}}\right)$

Now $\frac{1\left(x+y\frac{dy}{dx}\right)+2(0)}{1+2}=x\Rightarrow x+y\frac{dy}{dx}=3x$

$y\frac{dy}{dx}=2x$ .....(1)

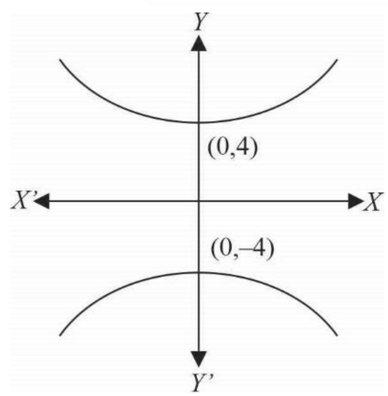
$\Rightarrow \quad \int y dy=\int 2x\ dx\Rightarrow \frac{y^2}{2}=x^2+C$

Also $(0,4)$ satisfy it, so $C=8$

$\therefore \quad y^2=2x^2+16$ (equation of curve)

Which represent a hyperbola

Also $\left.\frac{dy}{dx}\right]_{(4,4\sqrt{3})}=\frac{2(4)}{4\sqrt{3}}=\frac{2}{\sqrt{3}}$



49.(ABC) $f(x)=\int_0^x \{f(t)\cos t-\cos(t-x)\}dx$

$=\int_0^x f(t)\cos t dt-\int_0^x \cos(-t)dt \qquad \left[\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

$f(x)=\int_0^x f(t)\cos t dt-\sin x$

Differentiating both the sides, we get $f'(x)=f(x)\cos x-\cos x$

Let $f(x)=y;\ f'(x)=\frac{dy}{dx}$

$\frac{dy}{dx}-y\cos x=-\cos x$

If $=e^{-\int \cos x dx}=e^{-\sin x}$

Therefore, $y.e^{-\sin x}=-\int e^{-\sin x};\ y=Ce^{\sin x}+1$

If $s \qquad y=0 \qquad$ (from the given relation)

$\Rightarrow \qquad C=-1$

Therefore, $f(x)=1-e^{\sin x}$

Now, minimum value $=1-e \qquad$ (when $x=\pi/2$)

Maximum value $=1-e^{-1} \qquad$ (when $x=-\pi/2$)

$f'(x)=-e^{\sin x}\cos x$

Therefore, $f'(0)=-1$

$f''(x)=-\left[\cos^2 x.e^{\sin x}-e^{\sin x}.\sin x\right];\quad f''\left(\frac{\pi}{2}\right)=e$

50.(ABD) $\frac{dy}{dx}+y=f(x)\Rightarrow$ IF $=e^x$

$ye^x=\int e^x f(x)dx+C$

Now, if $0\le x\le 2$, then $ye^x=\int e^x e^{-x}dx+C$

$\Rightarrow \qquad ye^x=x+C$

$x=0, y(0)=1, C=1$

$\therefore \qquad ye^x=x+1 \qquad \ldots$(i)

$y=\frac{x+1}{e^x};\ y(1)=\frac{2}{e}\Rightarrow y'=\frac{e^x-(x+1)e^x}{e^{2x}}$

$y'(1)=\frac{e-2e}{e^2}=\frac{-e}{e^2}=-\frac{1}{e}$

If $x>2$, $ye^x=\int e^{x-2}dx$

$ye^x = e^{x-2} + C$

$y = e^{-2} + Ce^{-x}$

As y is continuous.

$\therefore \quad \lim\limits_{x\to 2} \dfrac{x+1}{e^x} = \lim\limits_{x\to 2}\left(e^{-2} + Ce^{-x}\right)$

$3e^{-2} = e^{-2} + Ce^{-2} \Rightarrow C = 2$

$\therefore \quad$ for $x > 2$

$y = e^{-2} + 2e^{-x}$

Hence, $y(3) = 2e^{-3} + e^{-2} = e^{-2}\left(2e^{-1} + 1\right)$

$y' = -2e^{-x}$; $\quad y'(3) = -2e^{-3}$

51.(CD) Here, $\left(f'(x)\right)^2 + 4f'(x).f(x) + \left(f(x)\right)^2 = 0$

$$\Rightarrow \quad \left(\frac{f'(x)}{f(x)}\right)^2 + 4\left(\frac{f'(x)}{f(x)}\right) + 1 = 0 \Rightarrow \frac{f'(x)}{f(x)} = \frac{-4 \pm \sqrt{16-4}}{2} \Rightarrow \frac{f'(x)}{f(x)} = -2 \pm \sqrt{3}$$

Integrating both the sides $\log\left|f(x)\right| = \left(-2 \pm \sqrt{3}\right)x + C_1 \Rightarrow f(x) = e^{\left(-2\pm\sqrt{3}\right)x}.C$

52.(ABCD)

(A) $\quad 32x + 2y\dfrac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = m_1 = -\frac{16x}{y} \text{ and } 16y^{15}\frac{dy}{dx} = k \Rightarrow \frac{dy}{dx} = m_2 = \frac{k}{16y^{15}}$$

$$m_1 m_2 = -\frac{16x}{y}.\frac{k}{16y^{15}} = -\frac{x}{y^{16}}.k = -\frac{x}{y^{16}}.\frac{y^{16}}{x} = -1$$

(B) $\quad \dfrac{dy}{dx} = 1 - ce^{-x} = 1 - (y - x) = -(y - x - 1)$ $\qquad$ [using $ce^{-x} = y - x$]

and $\dfrac{dy}{dx} - k.\dfrac{dy}{dx}e^{-y} = 1$

$\dfrac{dy}{dx}\left[1 - ke^{-y}\right] = 1$ or $\left[1 - (x + 2 - y)\right]\dfrac{dy}{dx} = 1$ $\qquad$ [using $ke^{-y} = x - y + 2$]

$\dfrac{dy}{dx} = m_2 = \dfrac{1}{y - x - 1} \Rightarrow m_1 m_2 = -1$

(C) $\quad \dfrac{dy}{dx} = 2cx = 2x.\dfrac{y}{x^2} = \dfrac{2y}{x} = m_1$

Also , $2x + 4y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{2y} = m_2$

Hence, $m_1 m_2 = -1$

(D) $x^2 - y^2 = c$

$2x - 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y} = m_1$

$xy = k$

$x\dfrac{dy}{dx} + y = 0 \qquad \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x} = m_2$

$\therefore \qquad m_1 m_2 = -1$

53.(BC) $x\dfrac{dy}{dx} = y\log\left(\dfrac{y}{x}\right)\dfrac{dy}{dx} = \dfrac{y}{x}$, put $y = vx$

$\Rightarrow \qquad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore \qquad v + x\dfrac{dv}{dx} = v\log v \Rightarrow x\dfrac{dv}{dx} = v(\log v - 1)$

$\Rightarrow \qquad \displaystyle\int \frac{dv}{v(\log v - 1)} = \int \frac{dx}{x} \Rightarrow \int \frac{dx}{t-1} = \int \frac{dx}{x},$

Let $\log v = t$

$\Rightarrow \qquad \log(t-1) = \log(x) + \log C \Rightarrow \qquad \log(t-1) = \log(xC)$

$\Rightarrow \qquad t = 1 + xC \Rightarrow \qquad \log\dfrac{y}{x} = 1 + xC$

$\Rightarrow \qquad y = x.e^{1+Cx}$ and $\log\dfrac{y}{x} = \log e + Cx \qquad \Rightarrow \qquad y = ex.e^{Cx}$

54.(AB) (A) $f(x, tx) = e^t + \tan^{-1}(t)$, independent of x. $\Rightarrow$ Homogeneous differential equation.

(B) $\log\left(\dfrac{y}{x}\right)dx + \dfrac{y^2}{x^2}\sin^{-1}\dfrac{y}{x}.dy = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{\log\left(\dfrac{y}{x}\right)}{\dfrac{y^2}{x^2}\sin^{-1}\left(\dfrac{y}{x}\right)}$; $f(x,y) = \dfrac{\log\left(\dfrac{y}{x}\right)}{\dfrac{y^2}{x^2}\sin^{-1}\left(\dfrac{y}{x}\right)}$

$\therefore \qquad f(x, tx) = \dfrac{\log(t)}{t^2\sin^{-1}(t)}$, independent of x $\Rightarrow$ Homogeneous differential equation.

(C) $f(x,y) = x^2 + \sin x.\cos y$

$f(x, tx) = x^2 + \sin x.\cos(tx)$, not independent of x. $\Rightarrow$ Not homogeneous differential equation.

(D) $f(x,y) = \dfrac{x^2 + y^2}{xy^2 - y^3}$

$\Rightarrow \qquad f(x, tx)$ is not independent of $f(x, tx)$

$\Rightarrow \qquad$ Not homogeneous differential equation.

55.(A,B, C)

$$\frac{dy}{dx} - y\cot x = -\frac{\sin x}{x^2}$$

Integrating Factor $= e^{\int -\cot x dx} = e^{-\log \sin x} = \frac{1}{\sin x}$

$\therefore$ Required solution is $y.\frac{1}{\sin x} = \int -\frac{\sin x}{x^2}.\frac{1}{\sin x}dx + C$

$$y.\frac{1}{\sin x} = \frac{1}{x} + C$$

As $y\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \Rightarrow C = 0$

$\therefore$ $y = \frac{\sin x}{x} \Rightarrow \lim_{x \to 0} f(x) = 1$

$\therefore$ $I = \int_0^{\pi/2} \frac{\sin x}{x} dx$

Since, $\frac{\sin x}{x}$ is decreasing, when $x > 0$

$\Rightarrow$ $f(x) < f(0) \Rightarrow \int_0^{\pi/2} f(x) < \frac{\pi}{2}$ and $x < \frac{\pi}{2} \Rightarrow f(x) > \int \left(\frac{\pi}{2}\right) \Rightarrow \int_0^{\pi/2} f(x)dx > 1$

56.(BC) (A) Order of the differential equation is 2.

(B) $\frac{xdy - ydx}{\sqrt{x^2+y^2}} = dx \Rightarrow \frac{\frac{xdy - ydx}{x^2}}{\sqrt{\left(1+\frac{y^2}{x^2}\right)}} = \frac{dx}{x}$

$\therefore$ $\ln\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = \ln|Cx|$; $\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} = Cx$ i.e. $y + \sqrt{x^2+y^2} = Cx^2$

(C) $y = e^x(A\cos x + B\sin x)$

$$\frac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) = y + e^x(-A\sin x + B\cos x)$$

$\therefore$ $\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) - y$

$$\frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y = 2\left(\frac{dy}{dx} - y\right)$$

(D) $(1+y^2)\frac{dy}{dx} + x = 2e^{\tan^{-1}y} \Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1}y}}{1+y^2}$

If $= e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y} \Rightarrow xe^{\tan^{-1}y} = 2\int e^{\tan^{-1}y}.\frac{e^{\tan^{-1}y}}{1+y^2}dy \Rightarrow xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + k$

57.(AD) The tangent at $P(x, y)$ is $Y - y - (X - x)y_1 = 0$

It meets x-axis at $A\left(x - \frac{y}{y_1}, 0\right)$ and y-aixs at $B(0, y - xy_1)$

$\Delta OAB = 2 \rightarrow OA.OB = 4$

$\left(x - \frac{y}{p}\right)(y - xp) = 4, p = \frac{dy}{dx}$

$(y - xp)^2 = -4p$

$y - xp = 2\sqrt{-p}$

$y = xp + 2\sqrt{-p}$

The general solution is $y = cx + 2\sqrt{-c}$

$x = 1, y = 1 \rightarrow c = -1, y = 2 - x$

$y(2) = 0$

Differentiating (1) w.r.t. c

$0 = x - \frac{1}{\sqrt{-c}}$

Eliminating c form (1) and (2)

$y = -\frac{1}{x} + \frac{2}{x} = \frac{1}{x}$

$y(2) = \frac{1}{2}$

58.(C) $f'(x) = 1 + \frac{f(x)}{x}$

$\frac{dy}{dx} - \frac{y}{x} = 1$

$\therefore$ Integrating factor $= e^{\int -\frac{1}{x} dx} = \frac{1}{x}$

$\therefore$ $y \times \frac{1}{x} = \int 1 \times \frac{1}{x} dx + c = \ln x + c$

$y = x \ln x + cx \quad \Rightarrow c = 1$

$\therefore$ $\frac{f(x) - x}{x} = \ln x \quad \Rightarrow \int_0^{\pi} \ln(\sin\theta) d\theta = -\pi \ln 2$

59.(CD) It is Clairauts's equation, $y = xp - p^2$

General solution $y = cx - c^2 \rightarrow$

$x + y + 1 = 0$ for $c = -1$

Singular solution is $y = \frac{x^2}{4}$.

60. **[A-u]** **[B-s]** **[C-q]** **[D-p]**

$$A=\int_{-2}^{2}\left(ax^3+bx+c\right)dx=4c\ ; \qquad B=\int_{-1}^{1}\left(ax^3+bx\right)dx=0$$

$$C=\sum_{n=1}^{10}\left(\int_{-2n-1}^{-2n}\sin^{27}x\,dx+\int_{2n}^{2n+1}\sin^{27}x\,dx\right)=\sum_{n=1}^{10}\left(\int_{2n+1}^{2n}\sin^{27}x\,dx+\int_{2n}^{2n+1}\sin^{27}x\,dx\right)=0$$

$$D=\left.\frac{\tan^{n-1}x}{n-1}\right|_0^{\pi/4}=\frac{1}{n-1}$$

61. **[A-r]** **[B-s]** **[C-p]** **[D-q]**

(A) $y\left(e^x+1\right)dy=(y+1)e^x dx \Rightarrow \int\frac{e^x}{e^x+1}dx=\int\frac{y}{y+1}dy \Rightarrow \ln\left(e^x+1\right)=y-\ln|y+1|+\ln c$

$\Rightarrow\left(e^x+1\right)|(y+1)|=ce^y$ which can also be written as $\left(e^x+1\right)(y+1)=\pm ce^y$

(B) $x\frac{dy}{dx}+y=xy^3 \Rightarrow \frac{1}{y^3}\frac{dy}{dx}+\frac{1}{y^2x}=1 \Rightarrow$ Let $\frac{1}{y^2}=t \Rightarrow \frac{dt}{dx}-\frac{2t}{x}=-2 \Rightarrow \frac{1}{y^3}\frac{dy}{dx}=-\frac{1}{2}\frac{dt}{dx}$

Which is linear in form. Also the equation can be written as $xdy+ydx=xy^3dx$

$$d(xy)=x^3y^3\frac{dx}{x^2} \Rightarrow \frac{1}{(xy)^3}d(xy)=\frac{dx}{x^2} \Rightarrow -\frac{1}{2x^2y^2}+\frac{1}{x}+c=0$$

(C) $\frac{dy}{dx}=\frac{xy+y}{xy+x} \Rightarrow \left(\frac{1+y}{y}\right)dy=\left(\frac{1+x}{x}\right)dx$

On integrating both sides, we get, $\log y+y=\log x+x+\log A$

$\log\left(\frac{y}{Ax}\right)=x-y \Rightarrow y=A\times xe^{x-y}$

(D) Put $3x-4y=X \Rightarrow 3-\frac{4dy}{dx}=\frac{dX}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{4}\left(3-\frac{dX}{dx}\right)$

Therefore the given equation reduced to

$$\frac{3}{4}-\frac{1}{4}\frac{dX}{dx}=\frac{X-2}{X-3} \Rightarrow \frac{-1dX}{4dx}=\frac{4X-8-3(X-3)}{4(X-3)}=\frac{X+1}{4(X-3)}$$

$$-\left(\frac{X-3}{X+1}\right)dX=dx \Rightarrow -\left(1-\frac{4}{X+1}\right)dX=dx=-X+4\log(X+1)=x+c'$$

62. **[A-r]** **[B-s]** **[C-q]** **[D-p]**

(A) $y=a\cos(x+b)+ce^x+d\sin x$ **(B)** $y=a\left(\frac{1-\cos 2x}{2}\right)+b\left(\frac{1+\cos 2x}{2}\right)+\sin 2x+d\cos 2x+e\sin x$

(C) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$ **(D)** differentiating, $\frac{dy}{dx}=c$ Order is 1

63. **[A-q, s]** **[B-p]** **[C-p]** **[D-q, r, s]**

(A) Equation of the required parabola is of the form

$y^2 = 4a(x-h)$. Differentiating, we have $2y\frac{dy}{dx} = 4a \Rightarrow y = \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$

(B) We have $y = a(x+a)^2$ .....(i) $\Rightarrow \frac{dy}{dx} = 2a(x+a)$ ......(ii)

Dividing equation (i) and (ii), we get $\frac{y}{\frac{dy}{dx}} = \frac{x+a}{2} \Rightarrow x+a = \frac{2y}{y_1}$, where $y_1 = \frac{dy}{dx}$

(C) The given equation is $\left(1+3\frac{dy}{dx}\right)^{2/3} = 4\frac{d^3y}{dx^3}$

Cubing, we get $\left(1+3\frac{dy}{dx}\right)^2 = 64\left(\frac{d^3y}{dx^2}\right)^3$

(D) We have $y^2 = 2c\left(x+\sqrt{c}\right)$ ...(i)

Diff. w.r.t. x, we get $2y\frac{dy}{dx} = 2c \Rightarrow c = y\frac{dy}{dx}$

Putting in equation, (i), we get $y^2 = 2\left(y\frac{dy}{dx}\right)x + 2\left(y\frac{dy}{dx}\right)^{3/2}$

64. **[A-q]** **[B-r]** **[C-p]** **[D-s]**

(A) $y = e^{4x} + 2e^{-x}; y_1 = 4e^{4x} - 2e^{-x}; y_2 = 16e^{4x} + 2e^{-x}$

$y_3 = 64e^{4x} - 2e^{-x}$

Now, $y_3 - 13y_1 = \left(64e^{4x} - 2e^{-x}\right) - 13\left(4e^{4x} - 2e^{-x}\right) = 12e^{4x} + 24e^{-x}$

$y_3 - 13y_1 = 12\left(e^{4x} + 2e^{-x}\right) = 12y$

65.(4) $\int_1^4 \frac{2e^{\sin x^2}}{x}dx = \int_1^{16}\frac{e^{\sin t}}{t}dt = \left[F(t)\right]_1^{16} = F(16) - F(1) \Rightarrow \frac{k}{4} = 4 \Rightarrow k = 16$

66.(2) Given y = tanz

$\frac{dy}{dx} = \sec^2 z.\frac{dz}{dx}$ ....(i)

Now $\frac{d^2y}{dx^2} = \sec^2 z.\frac{d^2z}{dx^2} + \frac{dz}{dx}\frac{d}{dx}\left(\sec^2 z\right)$ [using product rule]

$\frac{d^2y}{dx^2} = \sec^2 z.\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right).2\sec^2 z.\tan z$ .....(ii)

Now $1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2 = 1 + \frac{2(1+\tan z)}{\sec^2 z}.\sec^4 z.\left(\frac{dz}{dx}\right)^2$

$= 1 + 2(1+\tan z)\sec^2 z\left(\frac{dz}{dx}\right)^2 = 1 + 2\sec^2 z\left(\frac{dz}{dx}\right)^2 + 2\tan z.\sec^2 z\left(\frac{dz}{dx}\right)^2$ .....(iii)

From (ii) and (iii), we have RHS of (ii) = RHS of (iii)

67.(8) $\dfrac{dy}{dt}+2ty=t^2$

I.F. = e^{t^2} $\quad\therefore\quad$ Solution is $y\cdot e^{t^2}=\int t\cdot t e^{t^2}dt$

$$y.e^{t^2}=t.\frac{e^{t^2}}{2}-\frac{1}{2}\int e^{t^2}dt+C$$

$$y=\frac{t}{2}-e^{-t^2}\int\frac{e^{t^2}dt}{2}+Ce^{-t^2}$$

$$\lim_{t\to\infty}\frac{y}{t}=\frac{1}{2}-\lim_{t\to\infty}\frac{\int\frac{e^{t^2}}{2}}{te^{t^2}}=\frac{1}{2}$$

68.(2) $\dfrac{dy}{dx}=\dfrac{1}{x\cos y+2\sin y\cos y}$

$\therefore \dfrac{dx}{dy}=x\cos y+2\sin y\cos y$ $\qquad\therefore \dfrac{dx}{dy}+(-\cos y)x=2\sin y\cos y$

$\therefore$ I. F. = $e^{-\int\cos y dy}e=e^{-\sin y}$

The solution is $x.e^{-\sin y}=2\int e^{-\sin y}.\sin y\cos y dy=-2\sin y.e^{-\sin y}-2\int\left(-e^{-\sin y}\right).\cos y dy$

$$=-2\sin y.e^{-\sin y}+2\int\left(-e^{-\sin y}\right).\cos y dy$$

69.(1) $\dfrac{dy}{dx}=\dfrac{1}{dx/dy};\dfrac{d^2y}{dx^2}=\dfrac{d}{dy}\left(\dfrac{1}{dx/dy}\right).\dfrac{dy}{dx}=-\dfrac{1}{(dx/dy)^3}\dfrac{d^2x}{dy^2}$

Hence $x\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^3-\dfrac{dy}{dx}=0$

Become $-x\dfrac{1}{(dx/dy)^3}\dfrac{d^2x}{dy^2}+\dfrac{1}{(dx/dy)^3}-\dfrac{1}{(dx/dy)}=0$

Or $x\dfrac{d^2x}{dy^2}-1+\left(\dfrac{dx}{dy}\right)^2=0\Rightarrow x\dfrac{d^2x}{dy^2}+\left(\dfrac{dx}{dy}\right)^2=1$

70.(1) $\dfrac{dy}{dx}=-\dfrac{\sqrt{\left(x^2-1\right)\left(y^2-1\right)}}{xy}$

$$\int\frac{y}{\sqrt{y^2-1}}dy=-\int\frac{\sqrt{x^2-1}}{x}dx$$

Let $\quad y^2-1=t^2\Rightarrow 2ydy=2tdt$ $\qquad\therefore\int\dfrac{t}{t}dt=-\int\dfrac{x^2-1}{x\sqrt{x^2-1}}dx$

$$t=-\int\frac{x}{\sqrt{x^2-1}}dx+\int\frac{1}{x\sqrt{x^2-1}}dx$$

$$\sqrt{y^2-1}=-\sqrt{x^2-1}+\sec^{-1}x+c$$

Passing through (1, 1)

$\Rightarrow \quad \sqrt{y^2-1}=-\sqrt{x^2-1}+\sec^{-1}x \quad \Rightarrow \quad \sqrt{k^2-1}=-1+\dfrac{\pi}{4}$

$\Rightarrow \quad k^2-1=1+\dfrac{\pi^2}{16}-\dfrac{\pi}{2} \quad \Rightarrow \quad k^2=2+\dfrac{\pi^2}{16}-\dfrac{\pi}{2}=1.046\,; \quad [k]=\sqrt{1.046}=1$

71.(8) Equation the tangent at $P(x_1, y_1)$ of $y=f(x)$

$$y-y_1=\frac{dy}{dx}(x-x_1) \qquad \ldots\ldots\ldots\text{(i)}$$

This tangent cuts the x-axis so $x_2=x_1-\dfrac{y_1}{\left(\dfrac{dy}{dx}\right)}$

$\because \quad x_1, x_2, x_3, \ldots\ldots\ldots x_n$ are in AP

$$x_2-x_1=-\frac{y_1}{\frac{dy}{dx}}=\log_2 e \text{ given}$$

$$-y=\log_2 e\frac{dy}{dx}$$

$$\frac{dy}{y}\log_2 e=-dx \text{ Integrating both side}$$

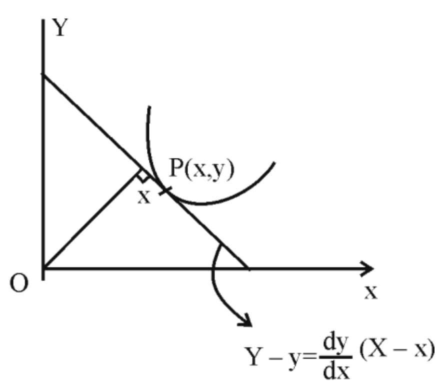
$$\log_e y=-x\log_e 2+c$$

$$y=ke^{-x\log_e 2}$$

Put $\quad x=-2$

$$y=2e^{+2\log_e^2}=2\times 4=8$$

72.(2)



Equation of tangent is $X\dfrac{dy}{dx}-y-Y\dfrac{dy}{dx}+y=0$ perpendicular distance from origin is

$\therefore$ $\perp$ from (0, 0) = x

$$\left|\frac{0-0-x\frac{dy}{dx}+y}{\sqrt{\left(\frac{dy}{dx}\right)^2+1}}\right|=x \quad \Rightarrow \quad \left|\frac{x\frac{dy}{dx}-y}{\sqrt{\left(\frac{dy}{dx}\right)^2+1}}\right|=x \Rightarrow \left(x\frac{dy}{dx}-y\right)^2=x^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

$$x^2\left(\frac{dy}{dx}\right)^2+y^2-2xy\frac{dy}{dx}=x^2+x^2\left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{y^2-x^2}{2xy}=\frac{dy}{dx}$$ ……(i) (homogenous)

Put y = vx in (i)

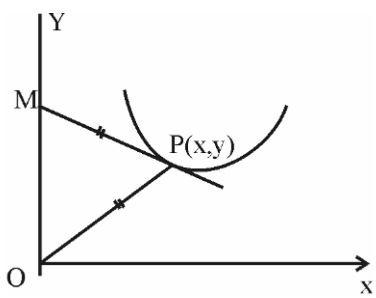
$$v+x\frac{dv}{dx}=\frac{v^2-1}{2v}$$

$$\int\frac{2v}{v^2+1}dv=-\int\frac{dx}{x} \quad \Rightarrow \quad \ln\left(v^2+1\right)=-\ln x+\ln c$$

$$v^2+1=\frac{c}{x} \quad \Rightarrow \quad \frac{y^2+x^2}{x^2}=\frac{c}{x}\Rightarrow y^2+x^2=cx$$

Passes through (1, 1), then c = 2 $\Rightarrow$ $x^2+y^2-2x=0$

73.(4) $\frac{dy}{dx}-y=1-e^{-x}$



P = –1

$Q=1-e^{-x}$

I.F. = $e^{\int pdx}=e^{\int -1dx}=e^{-x}$

$\therefore\ y.e^{-x}=\int e^{-x}\left(1-e^{-x}\right)dx+C$

$$y.e^{-x}=-e^{-x}+\frac{1}{2}e^{-2x}+C$$

$$y=-1+\frac{1}{2}e^{-x}+Ce^{x}$$

x = 0, y = y_0

So $\quad C=y_0+\frac{1}{2}$

$$y=-1+\frac{1}{2}e^{-x}+\left(y_0+\frac{1}{2}\right)e^{x}$$

$$x\to\infty;\ y\to \textit{finite value so } y_0+\frac{1}{2}=0$$

74.(1) $2x^2ydx-2y^4dx+2x^3dy+3xy^3dy=0$

Dividing throughout by x^3y, we get $2\frac{dx}{x}-\frac{2y^3}{x^3}dx+2.\frac{dy}{y}+\frac{3y^2}{x^2}dy=0$

$\Rightarrow \quad 2\dfrac{dx}{x}+2\dfrac{dy}{y}+\dfrac{3y^2x^4dy-2y^3x^3dx}{x^6}=0 \Rightarrow \quad 2\dfrac{dx}{x}+2\dfrac{dy}{y}+\dfrac{3y^2x^2dy-2y^3xdx}{x^4}=0$

$\Rightarrow \quad 2d(\ln x)+2d(\ln y)+d\left(\dfrac{y^3}{x^2}\right)=0 \Rightarrow 2\ln|x|+2\ln|y|+\dfrac{y^3}{x^2}=C$

When $x=1, y=1$. So, $C=1$

75.(8) Let the salt content at time $'t'$ be u lb, so that its rate of change is du/dt.

$=2gal\times 2lb=4lb/\min$

If c be the concentration of the brine at time t, the rate at which the salt content decreases due to the out flow

$=2\ gal\times c1b/\min\ =2c1b/\min$

$\therefore \quad \dfrac{du}{dt}=4-2c \qquad$ ….(i)

Also, since there is no increase in the volume of the liquid, the concentrations $c=\dfrac{u}{50}$

$\therefore \quad$ Eq. (i) becomes $\dfrac{du}{dt}=4-\dfrac{2u}{50}$

Separating the variables and integrating, we have

$\int dt=25\int\dfrac{du}{100-u}$ or $t=-25\ln(100-u)+K \qquad$ ….(ii)

initially when $t=0, u=0$

$0=-25\ln\ 100+K \qquad$ …(iii)

Eliminating $'K'$ from Eqs. (ii) and (iii), we get $t=25\ln\left(\dfrac{100}{100-u}\right)$

Taking $t=t_1$ when $u=40$ and $t=t_2$ when $u=80$, we have $t_1=25\ln\left(\dfrac{100}{60}\right)$ and $t_2=25\ln\left(\dfrac{100}{20}\right)$

$\therefore \quad$ The required time $(t_2-t_1)=25\left(\ln 5-\ln\dfrac{5}{3}\right)=25\ln 3$

$=25\times 1.0986=26\min 28s\ =1648s=206\times 8=206\times\lambda\ ;\ \lambda=8$

76.(1) $f(x+f(y)+xf(y))=y+f(x)+yf(x) \qquad$ ….(i)

Differentiating w.r.t. x and y is constant

$f'(x+f(y)+xf(y))(1+f(y))=f'(x)+yf'(x) \qquad$ ….(ii)

From Eq. (i) again differentiating w.r.t. y as x is constant

$f'(x+f(y)+xf(y))(1+x)f'(y)=1+f(x) \qquad$ …(iii)

From Eqs. (ii) and (iii)

$\dfrac{1+f(y)}{(1+x)f'(y)}=\dfrac{(1+y)f'(x)}{1+f(x)}$

$\dfrac{(1+y)f'(y)}{1+f(y)}=\dfrac{1+f(x)}{(1+x)f'(x)}=\lambda$

$\therefore \quad f'(y) = \frac{1+f(y)}{1+y}\lambda$ and $f'(x) = \frac{1+f(x)}{\lambda(1+x)}$

$\therefore \quad \lambda = \frac{1}{\lambda} \Rightarrow \lambda = \pm 1 \quad \therefore \quad \frac{f'(x)}{1+f(x)} = \pm\frac{1}{1+x}$

Integrating both the sides $f(x) = C(1+x)^{\pm 1} - 1$

From Eq. (i) put $x = y = 0$

$f(f(0)) = f(0)$

From Eq. (ii), $f(0) = C - 1$

So, $f(C-1) = C-1$

$\therefore \quad C = 0, 1 \quad$ (taking +ve sign)

So, $f(0) = -1$ and $f(x) = (1+x) - 1 = x$ and $C = 1$

$\therefore \quad f(x) = (1+x)^{-1} - 1 = \frac{-x}{1+x}$

$1 + f(x) = \frac{1}{1+x} \quad \therefore \quad 1 + f(2018) = \frac{1}{2019} \quad \therefore \quad (2019)(1+f(2018)) = 1$

77.(1) $\phi'(x) + 2\phi(x) \le 1$

$e^{2x}\phi'(x) + 2\phi(x)e^{2x} \le e^{2x}$

$\frac{d}{dx}\left(e^{2x}\phi(x) - \frac{1}{2}e^{2x}\right) \le 0$

$\therefore \quad \left(e^{2x}\phi(x) - \frac{1}{2}\right)$ is a non-increasing function of x.

$\therefore$ $2\phi(x)$ is always less than or equal to 1.

78.(1) $\frac{1}{2}(1+x)^{-1/2} - \frac{a}{2}(1+y)^{-1/2}\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{1}{\sqrt{1+x}} = \frac{a}{\sqrt{1+y}} \cdot \frac{dy}{dx} \Rightarrow a = \frac{\sqrt{1+y}}{\sqrt{1+x}} \cdot \frac{1}{dy/dx}$

Putting this value in the given equation.

$$\frac{dy}{dx}\sqrt{1+x} - \frac{1+y}{\sqrt{1+x}} = \frac{dy}{dx}$$

$\Rightarrow \quad (1+x)\frac{dy}{dx} = 1 + y + \sqrt{1+x}\frac{dy}{dx}$

$\Rightarrow \quad \left(1 + x - \sqrt{1+x}\right)\frac{dy}{dx} = 1 + y$

$\therefore$ Degree of the given equation is 1.

79.(3) Here, $2f^5(x).f'(x).\left|1+(f'(x))^2\right| = f''(x)$

$\Rightarrow \quad f^4(x)d\left(f^2(x)\right) = \dfrac{f''(x)}{1+\left(f'(x)\right)^2} \quad \Rightarrow \quad \left|f^4(x)d\left(f^2(x)\right)\right| = \int d\left(\tan^{-1}\left(f'(x)\right)\right)$

$\Rightarrow \quad \dfrac{f^6(x)}{3} + C = \tan^{-1}\left(f'(x)\right)$, as $f(0) = f'(0) = 0 \Rightarrow c = 0$

$\Rightarrow \quad \dfrac{f^6(x)}{3} = \tan^{-1}\left(f'(x)\right) \quad \Rightarrow \quad 0 \le \dfrac{f^6(x)}{3} < \dfrac{\pi}{2}$

$\therefore \quad -\left(\dfrac{3\pi}{2}\right)^{1/6} < f(x) < \left(\dfrac{3\pi}{2}\right)^{1/6} \quad \Rightarrow$ Number of integers between k_1 and k_2 are 3.

80.(1) Put, $dy/dx = t$

$\therefore \quad dt/dx - t + e^{2x} = 0, I.F. = e^{-\int dx}$

$\therefore$ solution is, $t.e^{-x} = \int -e^{2x}.e^{-x}dx + C$

$\Rightarrow \quad te^{-x} = -e^x + C: \; y'(0) = 1 \Rightarrow C = 2$

$\therefore \quad dy/dx = \dfrac{2-e^x}{e^{-x}} \int dy = \int \left(2e^x - e^{2x}\right)dx$

$\Rightarrow \quad y = 2e^x - \dfrac{e^{2x}}{2} + C^1, y(0) = 2 \Rightarrow C^1 = 1/2$

$\therefore \quad y(x) = 2e^x - \dfrac{e^{2x}}{2} + \dfrac{1}{2} \Rightarrow \quad y_{\max}(x) = \dfrac{5}{2}$ for $x = \log 2 \qquad \therefore \; [2x] = [2\log 2] = [\log 4] = 1$

81.(3) $\because \quad \dfrac{d}{dx}\left\{\dfrac{dy}{dx}\,.\,y\right\} = e^x$

$\Rightarrow \quad y\dfrac{dy}{dx} = e^x + c \quad \Rightarrow \quad 1\times 1 = e^0 + c \quad \Rightarrow c = 0 \quad \Rightarrow \quad ydy = e^x dx$

$\dfrac{1}{2}y^2 = e^x + c \quad \therefore \dfrac{1}{2} = 1 - c \quad \Rightarrow c = -\dfrac{1}{2}$

$y^2 = 2e^x - 1 \quad \Rightarrow f^2(x) = 2e^x - 1$

82.(7) Let $x+3 = X$ & $y-1 = Y$

$\therefore \dfrac{dY}{dX} = \dfrac{(X+Y)^2 - Y^2}{X^2} = \dfrac{X^2 + 2XY}{X^2}$

$\Rightarrow \dfrac{dY}{dX} - \dfrac{2Y}{X} = 1$

I.F. = $e^{\int -\frac{2}{X}dX} = e^{-2\ln X} = \dfrac{1}{X^2}$

$\therefore \quad Y \times \dfrac{1}{X^2} = \int \dfrac{1}{X^2} \times 1\, dX + c$

$\Rightarrow \quad \dfrac{(y-1)}{(x+3)^2} = -\dfrac{1}{(x+3)} + c \quad \Rightarrow (y-1) = c(x+3)^2 - (x+3)$

$\therefore \quad 0 = 9c - 3 \quad \Rightarrow \quad c = \dfrac{1}{3}$

$y = 1 + \dfrac{1}{3}(x+3)^2 - (x+3)$

$y(3) = 1 + 12 - 6 = 7$

83.(8) $\because \quad \dfrac{dy}{dx^2} = \dfrac{dy}{dx} \times \dfrac{dx}{dx^2} = \dfrac{1}{2x}\dfrac{dy}{dx}$

$\therefore \quad$ equation, $\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{\sin^{-1} x}{x}$

I.F. $= e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$\therefore \quad y \times x^2 = \int \dfrac{\sin^{-1} x}{x} \times x^2 dx$

$= \sin^{-1} x \,.\, \dfrac{x^2}{2} - \int \dfrac{1}{\sqrt{1-x^2}} \times \dfrac{x^2}{2} dx$

$= \dfrac{1}{2}x^2 \sin^{-1} x + \dfrac{1}{2}\int \dfrac{1-x^2-1}{\sqrt{1-x^2}} dx$

$= \dfrac{1}{2}x^2 \sin^{-1} x + \dfrac{1}{2}\int \sqrt{1-x^2}\,dx - \dfrac{1}{2}\int \dfrac{dx}{\sqrt{1-x^2}}$

$= \dfrac{1}{2}x^2 \sin^{-1} x + \dfrac{1}{4}\left[x\sqrt{1-x^2} + \sin^{-1} x\right] - \dfrac{1}{2}\sin^{-1} x + c$

$y \times x^2 = \dfrac{1}{2}\left(2x^2 - 1\right)\sin^{-1} x + \dfrac{1}{4}x\sqrt{1-x^2} + c \quad \Rightarrow \quad y(1) \times 1^2 = \dfrac{1}{4}(2-1) \times \dfrac{\pi}{2} + 0 = \dfrac{\pi}{8}$

84.(8) If $f'(x) < 2f(x)$

$\dfrac{dy}{dx} - 2y < 0$

$e^{-2x} . \dfrac{dy}{dx} - 2e^{-2x} y < 0$

$\dfrac{d}{dx}\left(y \,.\, e^{-2x}\right) < 0$

$g(x) = f(x)e^{-2x}$ is decreasing function

$\therefore g(x) \le g\left(\dfrac{1}{2}\right) \Rightarrow f(x) \le 2e^{2x}$

85.(2) $\because \quad \dfrac{dy}{dx} = \dfrac{x\left(3y^2 - x^2\right)}{y\left(3x^2 - y^2\right)}$

$\therefore \quad m = 3 \;\&\; n = -1$

86.(3) $\because$ differentiations, $xf(x)+1\times\int_0^x f(t)\,dt=(x+1)xf(x)+\int_0^x tf(t)\,dt$

Again differentiate, we have $f(x)=x^2 f'(x)+f(x)\,.\,2x+xf(x)$

$\dfrac{f'(x)}{f(x)}=\dfrac{1}{x^2}-\dfrac{3}{x} \Rightarrow \quad \ln f(x)=-\dfrac{1}{x}-3\ln x+c \qquad \because \qquad f(1)=\dfrac{1}{e}$

$-1=-1-0+c$

$c=0$

$\therefore \qquad \ln f(x)=\ln e^{-\frac{1}{x}}+\ln x^{-3}$

$f(x)=e^{-\frac{1}{x}}\times\dfrac{1}{x^3}$

$e^{\frac{1}{x}} f(x)=\dfrac{1}{x^3}; \ \ g(x)=\dfrac{1}{x^3}; \ \ g'(x)=\dfrac{-3}{x^4}; \ \ g'(1)=-3$

87.(2) $\because \qquad \left(x\dfrac{dy}{dx}\right)^2-2xy\dfrac{dy}{dx}+y^2+\left(x\dfrac{dy}{dx}-y\right)-2=0 \qquad \Rightarrow \qquad \left(x\dfrac{dy}{dx}-y\right)^2+\left(x\dfrac{dy}{dx}-y\right)-2=0$

$\therefore \qquad x\dfrac{dy}{dx}-y=-2$ or $1 \ \Rightarrow \qquad \dfrac{dy}{dx}-\dfrac{y}{x}=-\dfrac{2}{x}$ or $\dfrac{1}{x}$

I.F. $=e^{\int -\frac{1}{x}dx}=\dfrac{1}{x}$

$\therefore$ Solution is $y\times\dfrac{1}{x}=\int\dfrac{-2}{x}\times\dfrac{1}{x}dx$ or $\int\dfrac{1}{x}\times\dfrac{1}{x}dx$

$\dfrac{y}{x}=+\dfrac{2}{x}+c$ or $-\dfrac{1}{x}+c$

$y=2+cx$ or $-1+cx$

$\therefore$ Either $(y-cx)=2$ or $(y-cx)=-1$ for $c=2015$

We have two answers

Either $(y-2015\,.\,x)=2$ or $(y-2015\,.\,x)=-1$

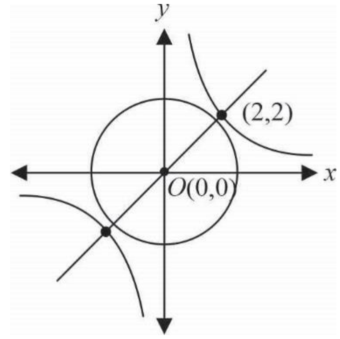
88.(5) $\because$ Differential equation is $\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}\dfrac{d^3y}{dx^3}-3\dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right)^2=0$

So, $m=3$ and $n=2$

89.(62) Given $y\left(\dfrac{d^2y}{dx^2}\right)=2\left(\dfrac{dy}{dx}\right)^2 \Rightarrow \dfrac{y''}{y'}=\dfrac{2y'}{y}\Rightarrow \ln\ y'=2\ln y+\ln a$

$\Rightarrow \qquad \dfrac{y'}{y^2}=a\Rightarrow\int\dfrac{dy}{y^2}=\int adx\Rightarrow\dfrac{-1}{y}=ax+b$

Since curve is passing through $(2,2)$ and $\left(8,\dfrac{1}{2}\right)$



So, $2a+b=\frac{-1}{2}$ ...(i)

$8a+b=-2$ ...(ii)

On solving (i) and (ii), we get $a=\frac{-1}{4}, b=0$

Hence, $C_1 : xy=4$ and curve C_2 is $x^2+y^2=4$.

$\therefore$ Shortest distance between C_1 and $C_2 = 2\sqrt{2}-2=\sqrt{8}-2$

Hence, $\left(p^2-q\right)=64-2=62$

90.(8) Given, $\frac{xdy}{dx}-2y=x^4y^2$ or $\frac{dy}{dx}-\frac{2}{x}y=x^3y^2$. Now dividing by y^2, we get

$\frac{1}{y^2}\frac{dy}{dx}-\frac{2}{y}\frac{1}{x}=x^3$ ...(i)

This is a Bernouli's differential equation, substituting $\frac{-2}{y}=t$, we get

$\Rightarrow \quad \frac{2}{y^2}\frac{dy}{dx}=\frac{dt}{dx}$. So, equation (i) becomes

$\Rightarrow \quad \frac{1}{2}\frac{dt}{dx}+\frac{t}{x}=x^3 \Rightarrow \frac{dt}{dx}+\frac{2}{x}t=2x^3$

$I.F.=e^{\int \frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$

So, general solution is given by $x^2t=\frac{2x^6}{6}+C \Rightarrow \frac{-x^2.2}{y}=\frac{x^6}{3}+C \Rightarrow \frac{-2}{y}=\frac{x^4}{3}+\frac{C}{x^2}$

If $x=1, y=-6, \Rightarrow C=0$

$\therefore \quad \frac{-2}{y}=\frac{x^4}{3} \Rightarrow y=\frac{-6}{x^4}$ i.e. $t(x)=\frac{-6}{x^4}$

Now, $\frac{dy}{dx}=24x^{-5}=\frac{24}{x^5}$. Hence $\left.\frac{dy}{dx}\right]_{x=3^{1/5}}=\frac{24}{3}=8.$

91.(1) $y(x+y)=x \Rightarrow (x+2y)\frac{dy}{dx}=1-y \Rightarrow \frac{dx}{x+2y}=\frac{dy}{1-y} \Rightarrow \int\frac{dx}{x+2y}=-\ln(1-y)+c \Rightarrow k=1$

92.(1) Equation of normal is

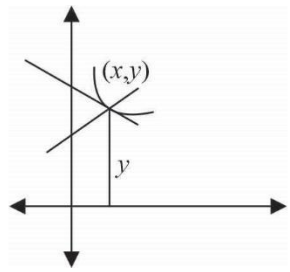
$Y-y=-\frac{1}{m}(X-x)$

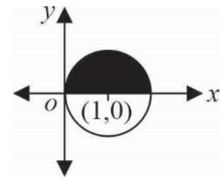
$X+mY-(x+my)=0$ ....(1)

Perpendicular distance from (0, 0) to equation (1) is $\left|\frac{x+my}{\sqrt{1+m^2}}\right|=|y|$

$\Rightarrow \quad (x+my)^2 = y^2(1+m^2) \Rightarrow x^2+2mxy = y^2 \Rightarrow m = \dfrac{y^2-x^2}{2xy}$ … (2)

Put $y^2 = t \Rightarrow 2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$





$\therefore$ Equation (2) becomes

$x\dfrac{dt}{dx} = t - x^2 \Rightarrow \dfrac{dt}{dx} - \dfrac{1}{x}t = -x$

$\therefore \quad I.F. = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \dfrac{1}{x}$

Now general solution is given by $t\left(\dfrac{1}{x}\right) = -x + C \Rightarrow y^2\left(\dfrac{1}{x}\right) = -x + C$

As $(1,1)$ satisfy it, so $C = 2$

$\Rightarrow \quad y^2 = -x^2 + 2x \Rightarrow x^2 + y^2 - 2x = 0$ Hence required area $= \dfrac{k\pi}{2}$ $\quad \therefore \quad k = 1$

93.(0) $(2ny + xy\log_e x)dx - x\log_e x\,dy = 0$

$\Rightarrow \quad \dfrac{dy}{y} = \left(\dfrac{2n}{x\log_e x} + 1\right) \Rightarrow \log y = 2n\log|\log x| + x + c$ & $c = 0$ [$\because$ curve passes through (e, e^e)]

$\therefore \quad y = e^{x+\log(\log x)^{2n}} = e^x(\log x)^{2n} \quad \Rightarrow \quad f(x) = e^x(\log x)^{2n}$

$$g(x) = \lim_{x\to\infty} f(x) = \begin{cases} \to \infty & if \;\; x < \dfrac{1}{e} \\ 0 \; if \;\; \dfrac{1}{e} < x < e \\ \to \infty & if \;\; x > e \end{cases}$$

Now, $\therefore \quad \displaystyle\int_{1/e}^{e} g(x)\,dx = 0$

VECTORS

1.(D) $\left[\hat{a}\ \vec{p}\ \vec{q}\right]$ = projection of $\vec{p}\times\vec{q}$ in the direction of $\hat{a}$. Hence the given vector is $\vec{p}\times\vec{q}$

2.(A) Any vector $\vec{r}$ can be written as $\vec{r}=\left(\vec{r}\cdot\vec{a}\right)\vec{a}+\left(\vec{r}\cdot\vec{b}\right)\vec{b}+\left(\vec{r}\cdot\vec{c}\right)\vec{c}$ where $\vec{a},\vec{b},\vec{c}$ are three mutually perpendicular unit vectors.

$\Rightarrow$ $\left(\vec{r}\cdot\vec{a}\right)^2+\left(\vec{r}\cdot\vec{b}\right)^2+\left(\vec{r}\cdot\vec{c}\right)^2 = \left[\left(\vec{r}\cdot\vec{a}\right)\vec{a}+\left(\vec{r}\cdot\vec{b}\right)\vec{b}+\left(\vec{r}\cdot\vec{c}\right)\vec{c}\right]\cdot\vec{r}\Rightarrow\vec{r}\cdot\vec{r}=\left|\vec{r}\right|^2$

$\Rightarrow$ Clearly **(A)** is the answers.

3.(B) $\left(\bar{a}\times\bar{x}\right)+\bar{b}=\bar{x}\Rightarrow\bar{a}\times\left(\bar{a}\times\bar{x}\right)+\left(\bar{a}\times\bar{b}\right)=\bar{a}\times\bar{x}$

$\left(\bar{a}\cdot\bar{x}\right)\bar{a}-\left(\bar{a}\cdot\bar{a}\right)\bar{x}+\left(\bar{a}\times\bar{b}\right)=\bar{x}-\bar{b}$

Projection of $\bar{x}$ along $\bar{a}$ is 2 units $\Rightarrow\dfrac{\left(\bar{a}\cdot\bar{x}\right)}{|\bar{a}|}=2$ $\Rightarrow\bar{a}\cdot\bar{x}=2$ So $\bar{x}=\dfrac{1}{2}\left[2\bar{a}+\bar{b}+\left(\bar{a}\times\bar{b}\right)\right]$.

4.(A) Given that $\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}a+3\sqrt{2}b+4c}{6.m}$, m = $|\bar{\alpha}|$ $\Rightarrow$2(a + 3b) + 4$\sqrt{2}$ c = 6m

Since a, b, c are rational, so $c = 0$, therefore vector lies in the xy plane.

5.(D) $\bar{d}.\bar{a}=\cos y[\bar{a}\,\bar{b}\,\bar{c}]=-\bar{d}.(\bar{b}+\bar{c})$ [as $\bar{d}.(\bar{a}+\bar{b}+\bar{c})=0$] $\Rightarrow$ $\cos y==-\dfrac{\bar{d}\cdot(\bar{a}+\bar{b})}{[\bar{a}\,\bar{b}\,\bar{c}]}$

Similarly $\sin x=-\dfrac{\bar{d}\cdot\left(\bar{a}+\bar{b}\right)}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}$ and $2=-\dfrac{\bar{d}\cdot\left[\bar{a}+\bar{c}\right]}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}$

$\sin x + \cos y + 2 = 0 \Rightarrow \sin x + \cos y = -2 \Rightarrow \sin x = -1,\ \cos y = -1$

Since we want minimum value of $x^2 + y^2$, so $x=-\dfrac{\pi}{2}$, $y = \pi$.

6.(C) $\overrightarrow{AB}\times\overrightarrow{AC}=(\vec{i}+\vec{k})\times(2\vec{i}+\vec{j})=\vec{k}+2\vec{j}-\vec{i}$.

Height of tetrahedron = $\dfrac{\left|\overrightarrow{AD}\cdot\overrightarrow{AB}\times\overrightarrow{AC}\right|}{6\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}=\dfrac{\left|(\vec{i}+2\vec{j}+\vec{k})\cdot(-\vec{i}+2\vec{j}+\vec{k})\right|}{6\left|-\vec{i}+2\vec{j}+\vec{k}\right|}=\left|\dfrac{-1+4+1}{6\sqrt{6}}\right|=\dfrac{4}{6\sqrt{6}}=\dfrac{1}{3}\sqrt{\dfrac{2}{3}}$.

7.(C) $\bar{d}\cdot\hat{a}=\bar{d}\cdot\hat{b}=\bar{d}\cdot\hat{c}\Rightarrow\lambda(1+\hat{b}\cdot\hat{c})=\lambda(1+\hat{b}\cdot\hat{c})=\lambda(\hat{a}\cdot\hat{b}+\hat{a}\cdot\hat{c})$

$\Rightarrow$ $1+\hat{b}\cdot\hat{c}=\hat{a}\cdot\hat{b}+\hat{a}\cdot\hat{c}$ $\Rightarrow$ $1-\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}-\hat{a}\cdot\hat{c}=0$

$\Rightarrow$ $1-\hat{a}\cdot\hat{b}+(\hat{b}-\hat{a})\cdot\hat{c}=0$ $\Rightarrow$ $\hat{a}\cdot(\hat{a}-\hat{b})+(\hat{b}-\hat{a})\cdot\hat{c}=0$

$\Rightarrow$ $(\hat{a}-\hat{c})\cdot(\hat{a}-\hat{b})=0\Rightarrow\hat{a}-\hat{c}$ is perpendicular to $(\hat{a}-\hat{b})$

$\Rightarrow$ The triangle is right angled.

8.(D) Since $\bar{d}$ makes equal angles with the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ $\Rightarrow$ d = $\dfrac{\mu(\bar{a}+\bar{b}+\bar{c})}{3}$

[$\bar{d}$ passes through the centroid of the triangle with position vectors $\bar{a}$, $\bar{b}$, $\bar{c}$] **… (1)**

Again $[\bar{a}\ \ \bar{b}\ \ \bar{c}]\bar{d}=[\bar{d}\ \ \bar{b}\ \ \bar{c}]\bar{a}+[\bar{d}\ \ \bar{c}\ \ \bar{a}]\bar{b}+[\bar{d}\ \ \bar{a}\ \ \bar{b}]\bar{c}$ **… (2)**

From (1) and (2), we get $[\bar{d}\ \ \bar{b}\ \ \bar{c}]=[\bar{d}\ \ \bar{c}\ \ \bar{a}]=[\bar{d}\ \ \bar{a}\ \ \bar{b}]$.

9.(A) $\bar{a}+\bar{b}+\bar{c}-3\bar{d}=0$ and $1+1+1-3=0 \Rightarrow \bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ are coplanar.

10.(B) $\vec{\alpha}\cdot\vec{\beta} = ab + bc + ca = \sqrt{a^2+b^2+c^2}\,\sqrt{b^2+c^2+a^2}\,\cos\theta \Rightarrow \cos\theta = \dfrac{ab+bc+ca}{\left(a^2+b^2+c^2\right)}$

Now $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$

$a^2+b^2+c^2 \geq ab+bc+ca \Rightarrow \dfrac{ab+bc+ca}{a^2+b^2+c^2} \leq 1$

Also $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geq 0$

$\Rightarrow \dfrac{ab+bc+ca}{a^2+b^2+c^2} \geq -1/2 \Rightarrow -\dfrac{1}{2} \leq \cos\theta \leq 1 \Rightarrow \theta \in [0, 2\pi/3]$.

11.(B) For the point of intersection of the lines

$\vec{a}+t\left(\vec{b}\times\vec{c}\right)=\vec{b}+s\left(\vec{c}\times\vec{a}\right) \Rightarrow \vec{a}.\vec{c}+t\left(\vec{b}\times\vec{c}\right).\vec{c}=\vec{b}.\vec{c}+s\left(\vec{c}\times\vec{a}\right).\vec{c} \Rightarrow \vec{a}.\vec{c}=\vec{b}.\vec{c}$.

Hence **(B)** is the correct answer.

12.(C) $4\bar{a}+5\bar{b}+9\bar{c}=0 \Rightarrow$ Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar

$\Rightarrow \vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are collinear $\Rightarrow \left(\vec{b}\times\vec{c}\right)\times\left(\vec{c}\times\vec{a}\right)=\vec{0}$.

Hence **(C)** is the correct answer

13.(C) We have $p\bar{r}+(\bar{r}\cdot\bar{b})\bar{a}=\bar{c} \Rightarrow p\bar{r}\cdot\bar{b}+(\bar{r}\cdot\bar{b})\bar{a}\cdot\bar{b}=\bar{c}\cdot\bar{b}$

$\Rightarrow p\bar{r}\cdot\bar{b}+0=\bar{c}\cdot\bar{b} \Rightarrow \bar{r}.\bar{b}=\dfrac{\bar{c}\cdot\bar{b}}{p} \Rightarrow p\bar{r}+\dfrac{\bar{c}\cdot\bar{b}}{p}\bar{a}=\bar{c} \Rightarrow \bar{r}=\dfrac{\bar{c}}{p}-\dfrac{\bar{c}.\bar{b}}{p^2}\bar{a}$.

14.(D) $\ell x+my=0$ **. . . .(i)**, $my+nz=0$ **. . . .(ii)**, $\ell x+nz=0$ **. . . .(iii)** and $\ell x+my+nz=p$ **. . . .(iv)**

Solving (i), (ii), (iv) $A\left(\dfrac{p}{\ell}, \dfrac{-p}{m}, \dfrac{p}{n}\right)$

Similarly other points are $B\left(\dfrac{p}{\ell}, \dfrac{p}{m}, \dfrac{-p}{n}\right)$, $C\left(\dfrac{-p}{\ell}, \dfrac{p}{m}, \dfrac{p}{n}\right)$ and O(0, 0, 0)

$\overline{OA}=\dfrac{p}{\ell}\hat{i}-\dfrac{p}{m}\hat{j}+\dfrac{p}{n}\hat{k}$, $\overline{OB}=\dfrac{p}{\ell}\hat{i}+\dfrac{p}{m}\hat{j}-\dfrac{p}{n}\hat{k}$ and $\overline{OC}=-\dfrac{p}{\ell}\hat{i}+\dfrac{p}{m}\hat{j}+\dfrac{p}{n}\hat{k}$

Volume of parallelopiped $=\dfrac{p^3}{\ell mn}\begin{vmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ -1 & 1 & 1\end{vmatrix}=\dfrac{4p^3}{\ell mn}$ $\therefore$ Volume of tetrahedron $=\dfrac{1}{6}\times\dfrac{4p^3}{\ell mn}$

15.(C) Given $\vec{r}\times\vec{a}=\vec{b}\times\vec{a} \Rightarrow \vec{r}\times\vec{a}-\vec{b}\times\vec{a}=0 \Rightarrow (\vec{r}-\vec{b})\times\vec{a}=0 \Rightarrow (\vec{r}-\vec{b})$ is parallel/collinear to $\vec{a}$

$\Rightarrow \vec{r}-\vec{b}=p\vec{a} \Rightarrow \vec{r}=b+p\vec{a}$ **... (1)** (Equation of the line)

Similarly, $\vec{r}\times\vec{b}-\vec{a}\times\vec{b}=0 \Rightarrow \vec{r}=a+s\vec{b}$ **... (2)** (Equation of the line)

The point of intersection of (1) and (2) gives $\vec{b}+p\vec{a}=\vec{a}+s\vec{b} \Rightarrow \vec{b}(1-s)+\vec{a}(p-1)=0$

Since $\vec{a}$ and $\vec{b}$ are non collinear $\Rightarrow 1-s=0$ and $p-1=0 \Rightarrow p=s=1$

Hence point of intersection is $\vec{a}+\vec{b}=3\hat{i}+\hat{j}-\hat{k}$.

16.(A) As $\bar{a}$ and $\bar{b}$ are $\perp \Rightarrow \bar{a}\cdot\bar{b}=0$

$$\bar{r}\times\bar{a}=\bar{b} \qquad \dots \textbf{(i)}$$

Taking cross product with $\bar{a}$ $\Rightarrow$ $\bar{a}\times(\bar{r}\times\bar{a})=\bar{a}\times\bar{b}$ $\Rightarrow$ $\bar{r}=\dfrac{(\bar{a}\cdot\bar{r})}{\bar{a}\cdot\bar{a}}\bar{a}+\dfrac{(\bar{a}\times\bar{b})}{\bar{a}\cdot\bar{a}}$

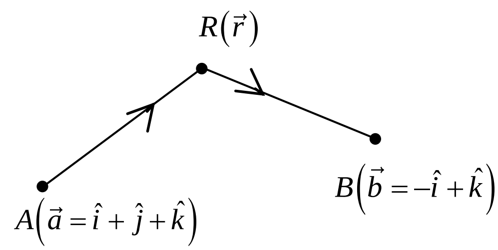
17.(A) Any vector $\perp$ to plane of PQR is $\overline{PQ}\times\overline{PR}=(\bar{q}-\bar{p})\times(\bar{r}-\bar{p})=\bar{q}\times\bar{r}+\bar{p}\times\bar{q}+\bar{r}\times\bar{p}$

If OP makes an angle α with face PQR. Then it will make an angle $\left(\dfrac{\pi}{2}-\alpha\right)$ with its normal.

18.(B) Taking dot product with $\bar{a},\bar{b}\,\&\,\bar{c}$ respectively $\Rightarrow$ $l=\dfrac{\bar{r}\cdot\bar{a}}{[\bar{a}\,\bar{b}\,\bar{c}]}=\dfrac{\bar{r}\cdot\bar{a}}{2}$; $m=\dfrac{\bar{r}\cdot\bar{b}}{2},n=\dfrac{\bar{r}\cdot\bar{c}}{2}$

19.(A) $\bar{b}\cdot\bar{c}=0$ and $\bar{a}\cdot\hat{k}<0$

20.(D) $\overrightarrow{AR}=(\vec{r}-\vec{a})$, $\overrightarrow{BR}=\left(\vec{r}-\vec{b}\right)$.



Since $\overrightarrow{AR}$ is perpendicular to $\overrightarrow{BR}$

$\Rightarrow (\vec{r}-\vec{a})\cdot\left(\vec{r}-\vec{b}\right)=0 \Rightarrow |\vec{r}|^2-\left(\vec{a}+\vec{b}\right)\vec{r}+\vec{a}\cdot\vec{b}=0$

$\Rightarrow \vec{r}\cdot\left(\vec{r}-\left(\vec{a}+\vec{b}\right)\right)+0=0 \Rightarrow$ Either $\vec{r}$ is perpendicular to $\vec{r}-\left(\vec{a}+\vec{b}\right)$ or $\vec{r}=\left(\vec{a}+\vec{b}\right) \Rightarrow \vec{r}=\hat{j}+2\hat{k}$.

21.(B) Volume of tetrahedron $V=\dfrac{1}{6}\left[\bar{a}\bar{b}\bar{c}\right]$

Here $\left[\bar{a}\bar{b}\bar{c}\right]^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}=\begin{vmatrix}4 & 3 & 4\sqrt{3}\\ 3 & 9 & 6\sqrt{2}\\ 4\sqrt{3} & 6\sqrt{2} & 16\end{vmatrix}=4[144-72]+3\left[24\sqrt{6}-48\right]+4\sqrt{3}\left[18\sqrt{2}-36\sqrt{3}\right]$

$=288+72\sqrt{6}-144+72\sqrt{6}-432=144\sqrt{6}-288=144\left(\sqrt{6}-2\right)$

$V=\dfrac{1}{6}\times 12\sqrt{\left(\sqrt{6}-2\right)}=2\sqrt{\sqrt{6}-2}$ Since $V=\dfrac{1}{3}\times$ base area $\times$ height $\Rightarrow$ height $=\dfrac{6\sqrt{\sqrt{6}-2}}{2}=3\sqrt{\left(\sqrt{6}-2\right)}$.

22.(B) $[(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})]\cdot(\bar{c}\times\bar{b})=0 \Rightarrow ([\bar{a}\,\bar{b}\,\bar{d}]\bar{c}-[\bar{a}\,\bar{b}\,\bar{c}]d)\cdot(\bar{c}\times\bar{b})=0 \Rightarrow [\bar{a}\,\bar{b}\,\bar{c}]\,[\bar{d}\,\bar{c}\,\bar{b}]=0$

23.(C) Take cross product with $\hat{\alpha}$, we get : $\hat{\alpha}\times\bar{x}=\hat{\alpha}\times\hat{\beta}-\hat{\alpha}\times(\hat{\alpha}\times\bar{x}) \Rightarrow \hat{\alpha}\times\bar{x}=\hat{\alpha}\times\hat{\beta}+\bar{x}-(\hat{\alpha}\cdot\bar{x})\hat{\alpha}$

But if we take dot product with $\hat{\alpha}$ we get $\hat{\alpha}\cdot\bar{x}=0$ So $2\bar{x}=\hat{\beta}-(\hat{\alpha}\times\hat{\beta})$

Now $4|\bar{x}|^2=|\hat{\beta}|^2+|\hat{\alpha}\times\hat{\beta}|^2-2\hat{\alpha}\cdot(\hat{\alpha}\times\hat{\beta})=2 \Rightarrow |x|=\dfrac{1}{\sqrt{2}}$.

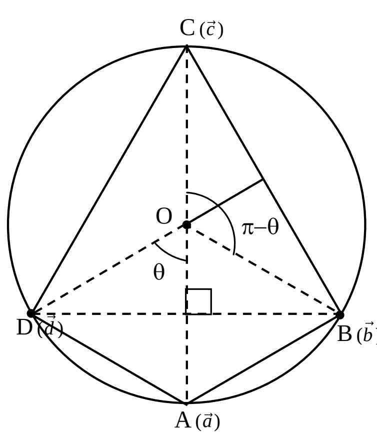
24.(D) $|\vec{d}-\vec{a}|^2+|\vec{b}-\vec{c}|^2=(\vec{d}-\vec{a})\cdot(\vec{d}-\vec{a})+(\vec{b}-\vec{c})\cdot(\vec{b}-\vec{c})$

$=|\vec{d}|^2+|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2-2\vec{d}\cdot\vec{a}-2\vec{b}\cdot\vec{c}$

$=4R^2-2(\vec{d}\cdot\vec{a}+\vec{b}\cdot\vec{c})$

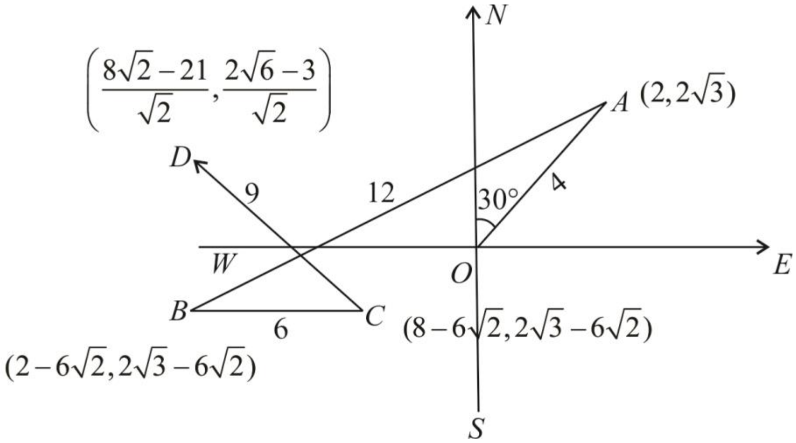
$=4R^2-2R^2(\cos\angle AOB+\cos\angle BOC$

$=4R^2\,(\because \angle AOB=\pi-\angle BOC)$



25.(A) The internal bisector will be parallel to $\hat{a}+\hat{b}$

26-30. **26.(C)** **27.(D)** **28.(B)** **29.(B)** **30.(A)**



31.(A) The vector along the bisector of the given angle is $\vec{c} = t\left(\frac{7\hat{i}-4\hat{j}-4\hat{k}}{\sqrt{81}} + \frac{-2\hat{i}-\hat{j}+2k}{\sqrt{9}}\right) = t\left(\frac{\hat{i}-7\hat{j}+2\hat{k}}{9}\right)$

Since $|c| = 5\sqrt{6}, 5\sqrt{6} = t\frac{\sqrt{54}}{9} \Rightarrow t = 15.$

32.(B) Vectors along *AB*, *BC* and *CA* are $\vec{b}-\vec{a}, \vec{c}-\vec{b}$ and $\vec{a}-\vec{c}$ respectively. Hence the bisectors of the angle *B* and *C* respectively are

$\vec{r} = \vec{b} + \mu\left(\frac{\vec{b}-\vec{a}}{c} + \frac{\vec{c}-\vec{b}}{a}\right)$ and $\vec{r} = \vec{c} + t\left(\frac{\vec{b}-\vec{c}}{a} + \frac{\vec{c}-\vec{a}}{b}\right)$ so that for the point *P*

$\vec{b} + \mu\left(\frac{\vec{b}-\vec{a}}{c} + \frac{\vec{c}-\vec{b}}{a}\right) = \vec{c} + t\left(\frac{\vec{b}-\vec{c}}{a} + \frac{\vec{c}-\vec{a}}{b}\right) \Rightarrow t = \frac{b\mu}{c}$ and $\mu = \frac{ac}{b+c+a} \Rightarrow$ the p.v. of *P* is $\frac{b\vec{b}+c\vec{c}-a\vec{a}}{b+c-a}$.

33.(C) Let *A* be the initial point and the p.v. of *B* and C be $\vec{b}$ and $\vec{c}$ respectively. Hence, let $AB = c$ and $AC = b.$ The internal bisector is $r = t\left(\frac{\vec{b}}{c} + \frac{\vec{c}}{b}\right)$ and the equation of the line *BC* is $\vec{r} = \vec{b} + k(\vec{c}-\vec{b})$ and hence the p.v. of *D* is $\frac{b\vec{b}+c\vec{c}}{b+c}$

$$\Rightarrow \quad \overline{BD} = \frac{b\vec{b}+c\vec{c}}{b+c} - \vec{b} = \frac{c(\vec{c}-\vec{b})}{b+c} \quad \Rightarrow \quad BD = \frac{c|\vec{c}-\vec{b}|}{b+c} = \frac{ca}{b+c}.$$

Similarly $BE = \frac{c|\vec{c}-\vec{b}|}{c-b} = \frac{ca}{c-b}$ and hence $\frac{1}{BE} + \frac{1}{BD} = \frac{c-b}{ac} + \frac{c+b}{ac} = \frac{2c}{ac} = \frac{2}{a} = \frac{2}{BC}.$

34.(C) Incentre is the point of intersection of the internal bisectors of angles *A*, *B* and *C*. Equation of the bisectors of angles *A* and *B* are $\vec{r} = \vec{a} + t\left(\frac{\vec{c}-\vec{a}}{b} + \frac{\vec{b}-\vec{a}}{c}\right)$ and $\vec{r} = \vec{b} + \mu\left(\frac{\vec{a}-\vec{c}}{c} + \frac{\vec{c}-\vec{b}}{a}\right)$

Equating the coefficient of $\vec{a}, \vec{b}$ and $\vec{c},$ we find that the *p.v* of the incentre is $\frac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c}.$

35.(A) Since the three vertices A, B, C of the triangle are coplanar, all the points lying as the linear AB, BC, CD are naturally coplanar.

36.(B) $(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}-\vec{b})\vec{c} \Rightarrow \quad \vec{b}\cdot\vec{c}=0$ and $\vec{a}\cdot\vec{b}=0$ or $\vec{a}=\lambda\vec{c}$

37.(A)

38.(B) $(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=\dfrac{\vec{b}}{2} \Rightarrow \vec{a}\cdot\vec{c}=\dfrac{1}{2}$ & $\vec{a}\cdot\vec{b}=0 \Rightarrow$ angle between $\vec{a}$ & $\vec{c}=\dfrac{\pi}{3}$

39.(D) **40.(C)**

39-40 $\vec{r}\cdot(10\hat{j}-8\hat{i}-\vec{r})=40$**;** $\vec{r}=x\hat{i}+y\hat{j}$

Curve is circle $x^2+y^2+8x-10y+40=0$; Centre $=(-4,5)$; $r=1$, We need to find distance from (–2, 3)

41.(CD) $|\vec{a}+\vec{b}+\vec{c}+\vec{d}|^2=\Sigma|\vec{a}|^2+2\Sigma\,\vec{a}\cdot\vec{b}=4+2\,\vec{d}\,.(\vec{a}+\vec{b}+\vec{c})$

Let $\vec{d}=\lambda\vec{a}+\mu\vec{b}+\nu\vec{c}$, then $\vec{d}\cdot\vec{a}=\vec{d}\cdot\vec{b}=\vec{d}\cdot\vec{c}=\cos\theta \quad \Rightarrow \quad \lambda=\mu=\nu=\cos\theta$

Also $\lambda^2+\mu^2+\nu^2=1 \Rightarrow 3\cos^2\theta=1 \Rightarrow \cos\theta=\pm\dfrac{1}{\sqrt{3}} \qquad \therefore \qquad |\vec{a}+\vec{b}+\vec{c}+\vec{d}|^2=4\pm\dfrac{2\cdot 3}{\sqrt{3}}=4\pm 2\sqrt{3}$.

42.(AB) As $\bar{d}$ is perpendicular to $\bar{a}+\bar{b}+\bar{c}$

$\Rightarrow \quad \bar{d}\cdot\bar{a}+\bar{d}\cdot\bar{b}+\bar{d}\cdot\bar{c}=0$ **… (1)**

Given $\bar{d}=x(\bar{a}\times\bar{b})+y(\bar{b}\times\bar{c})+z(\bar{c}\times\bar{a})$ **… (2)**

$\Rightarrow \quad y[\bar{b}\,\bar{c}\,\bar{a}]+z[\bar{c}\,\bar{a}\,\bar{b}]+x[\bar{a}\,\bar{b}\,\bar{c}]=0 \quad \Rightarrow \quad (x+y+z)[\bar{a}\,\bar{b}\,\bar{c}]=0$

As $\bar{a},\bar{b}$ & $\bar{c}$ are non coplanar $\quad x+y+z=0$ **… (3)**

$\Rightarrow \quad x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=0$

$(x+y+z)^2=0$; $x^2+y^2+z^2=-2(xy+yz+zx) \quad \Rightarrow \quad (xy+yz+zx)=\dfrac{x^2+y^2+z^2}{-2}$

43.(AC) $\bar{u}\cdot\bar{b}=0$ **… (1)**

$\bar{u}=\bar{a}-(\bar{a}\cdot\bar{b})\bar{b}\;=(\bar{b}\cdot\bar{b})\bar{a}-(\bar{a}\cdot\bar{b})\bar{b}$

$\Rightarrow \quad |\bar{u}|=\bar{b}\times(\bar{a}\times\bar{b}) \quad \Rightarrow \quad |\bar{u}|=|\bar{b}||\bar{a}\times\bar{b}|\sin\dfrac{\pi}{2}=|\bar{a}\times\bar{b}| \quad \Rightarrow \quad |\bar{u}|=|\bar{v}|$ **… (2)**

44.(AC) $\bar{a}\times(\bar{b}\times\bar{c})=(x^2-2x+6)\bar{b}+\sin y\,\bar{c}$

$\Rightarrow \quad (\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}=(x^2-2x+6)\bar{b}+\sin y\bar{c} \quad \Rightarrow \quad \bar{a}\cdot\bar{c}=x^2-2x+6$ and $\bar{a}\cdot\bar{b}=-\sin y$

As $\bar{a}\cdot\bar{b}+\bar{a}\cdot\bar{c}=4 \Rightarrow x^2-2x+6-\sin y=4$, $(x-1)^2+1=\sin y \Rightarrow x=1\;\&\;y=(4n+1)\dfrac{\pi}{2}$

45.(ABC) As $\bar{\alpha},\bar{\beta}$ & $\bar{\gamma}$ are coplanar $\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$

or $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=\dfrac{(a+b+c)}{2}((a-b)^2+(b-c)^2+(c-a)^2)=0$

$\Rightarrow$ Either $a+b+c=0$ or $a=b=c$

46.(ABCD) Given $\bar{\alpha}\cdot\bar{\beta}=0,\ \bar{\alpha}\cdot\bar{\gamma}=\bar{\beta}\cdot\bar{\gamma}=\cos\theta$ **… (1)**

By taking dot product by $\hat{\alpha}\,\&\,\hat{\beta}$ we get $x=y=\cos\theta$ **… (2)**

$z(\hat{\alpha}\times\hat{\beta})=(\hat{\gamma}-x\hat{\alpha}-y\hat{\beta}) \Rightarrow z^2|\hat{\alpha}\times\hat{\beta}|^2=|(\hat{\gamma}-x\hat{\alpha}-y\hat{\beta})|^2 \Rightarrow z^2=1-2\cos^2\theta$ **… (3)**

47.(AB) $\hat{a}\cdot\hat{b}=\cos 2\theta$ and $|\hat{a}-\hat{b}|<1 \Rightarrow 1+1-2\hat{a}\cdot\hat{b}<1$; $2\hat{a}\cdot\hat{b}>1 \Rightarrow \cos 2\theta > 1/2 \Rightarrow 1/2 < \cos 2\theta \le 1$

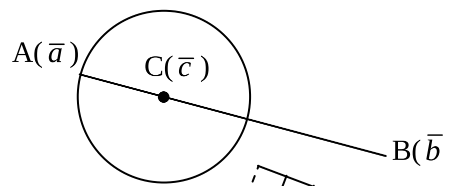
48.(BC) Let $\vec{r}=\overline{AB}+\lambda\overline{AC}=\hat{k}-\hat{i}+\lambda\left(\hat{k}-\hat{j}\right)=-\hat{i}-\lambda\hat{j}+(1+\lambda)\hat{k}$ **….(i)**

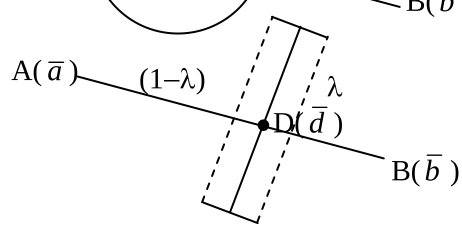
Again since $|AB|=|BC|=|CA|$,

Incentre is same as circumcentre, and hence IA is perpendicular to $BC \Rightarrow \vec{r}$ is parallel to $BC \Rightarrow \vec{r}=\lambda\left(\hat{i}-\hat{j}\right)$.

Hence unit vector $\vec{r}=\pm\dfrac{1}{\sqrt{2}}\left(\hat{i}-\hat{j}\right)$

49.(ABC) Take $\bar{c}=\dfrac{2\bar{a}+\bar{b}}{3}$ **… (1)**





then first equation becomes $|\bar{r}-\bar{c}|=\dfrac{1}{3}|\bar{a}-\bar{b}|$ **… (2)**

Locus of variable point P is a sphere whose centre is at point C ($\bar{c}$)

and radius $\dfrac{1}{3}|\bar{a}-\bar{b}|=\dfrac{1}{3}AB$

Taking $\lambda\bar{a}+(1-\lambda)\bar{b}=\bar{d}$. D ($\bar{d}$) lie on the line joining A ($\bar{a}$)

and B ($\bar{b}$) and the equation becomes $(\bar{r}-\bar{d})\cdot(\bar{a}-\bar{b})=0$

$\Rightarrow$ In this case locus of the variable point is a plane passing through point $D(\bar{d})$ and normal to the plane containing $(\bar{a}-\bar{b})$

Obviously the system of equation will not have any solution if point D ($\bar{d}$) lies outside the sphere

$\Rightarrow$ $|3\bar{d}-2\bar{a}-\bar{b}|-|\bar{a}-\bar{b}|>0$ … (3)

$\Rightarrow$ $|3\lambda-2|>1$ $\Rightarrow$ $\lambda<\dfrac{1}{3}$ or $\lambda>1 \Rightarrow \lambda\in(-\infty, 1/3)\cup(1,\infty)$.

50.(AC) Let vector $\bar{d}$ be $\bar{d}=\dfrac{\bar{a}+\bar{b}+a\times\bar{b}}{3}$ then

$$\bar{d}.\bar{a}=\frac{|\bar{a}|^2+\bar{a}.\bar{b}}{3}=|\bar{d}||\bar{a}|\cos\alpha \qquad \textbf{… (1)}$$

$$\bar{d}.\bar{b}=\frac{|\bar{b}|^2+\bar{a}.\bar{b}}{3}=|\bar{d}||\bar{b}|\cos\alpha \qquad \textbf{… (2)}$$

$$\bar{d}.(\bar{a}\times\bar{b})=\frac{|\bar{a}\times\bar{b}|^2}{3}=|\bar{d}||\bar{a}\times\bar{b}|\cos\alpha \qquad \textbf{… (3)}$$

(i) From equation (1) and (2) we get $\dfrac{|\bar{a}|^2+\bar{a}.\bar{b}}{3|\bar{a}|}=\dfrac{|\bar{b}|^2+\bar{a}.\bar{b}}{3|\bar{b}|}$

$\Rightarrow$ $(|\bar{a}|-|\bar{b}|)\left(1-\dfrac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right)=0$ $\Rightarrow$ $|\bar{a}|=|\bar{b}|$, as $\bar{a}$ and $\bar{b}$ are non-collinear

(ii) From equation (2) and (3)

$|\bar{b}|=\dfrac{|\bar{b}|^2+\bar{a}.\bar{b}}{|\bar{a}\times\bar{b}|}\Rightarrow|\bar{b}|^3\sin\theta=|\bar{b}|^2(1+\cos\theta)$ (where θ is the angle between $\bar{a}$ and $\bar{b}$)

$\Rightarrow$ $|\bar{b}|=|\bar{a}|=\dfrac{1+\cos\theta}{\sin\theta}$

Now $|d|^2 = \dfrac{(\bar{a}+\bar{b}+\bar{a}\times\bar{b})}{3}\cdot\dfrac{(\bar{a}+\bar{b}+\bar{a}\times\bar{b})}{3}$

$\Rightarrow \quad |d|^2 = \dfrac{|\bar{a}|^2+|\bar{b}|^2+2\bar{a}.\bar{b}+|\bar{a}\times\bar{b}|^2}{9} \quad \Rightarrow \quad 9|\bar{d}|^2 = |\bar{a}|^2 (2 + 2\cos\theta + |\bar{a}|^2 \sin^2\theta)$

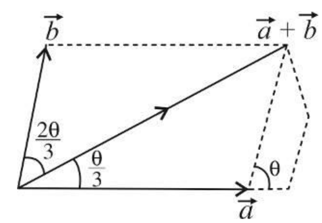
$\Rightarrow \quad 9|\bar{d}|^2 = |\bar{a}|^2 (2 + 2\cos\theta + (1+\cos\theta)^2) \quad \Rightarrow \quad 9|\bar{d}|^2 = |\bar{a}|^2 (1+\cos\theta)(3+\cos\theta)$

Now $|\bar{d}|\cos\alpha = \dfrac{|\bar{a}\times\bar{b}|}{3} \Rightarrow |\bar{d}|\cos\alpha = \dfrac{|\bar{a}||\bar{b}|\sin\theta}{3}$

$\Rightarrow \quad \cos\alpha = \dfrac{|\bar{a}|^2\sin\theta}{|\bar{a}|\sqrt{(1+\cos\theta)(3+\cos\theta)}} \quad \Rightarrow \quad \cos\alpha = \sqrt{\dfrac{1-\cos\theta}{3+\cos\theta}}$

$\Rightarrow \quad \cos\alpha = \dfrac{1}{\sqrt{3}}$ or $\alpha = \cos^{-1}\dfrac{1}{\sqrt{3}}$ when $\bar{a}.\bar{b} = 0$.

51.(BD) $\tan\dfrac{\theta}{3} = \dfrac{b\sin\theta}{a+b\cos\theta} \Rightarrow a\sin\dfrac{\theta}{3} + b\sin\dfrac{\theta}{3}\cos\theta = b\sin\theta\cos\dfrac{\theta}{3}$



$\Rightarrow \quad a\sin\dfrac{\theta}{3} = b\left[\sin\theta\cos\dfrac{\theta}{3} - \cos\theta\sin\dfrac{\theta}{3}\right] \Rightarrow a\sin\dfrac{\theta}{3} = b\sin\dfrac{2\theta}{3}$

$\Rightarrow \quad a\sin\dfrac{\theta}{3} = 2b\sin\dfrac{\theta}{3}\cos\dfrac{\theta}{3} \quad \Rightarrow \cos\dfrac{\theta}{3} = \dfrac{a}{2b} \Rightarrow \theta = 3\cos^{-1}\left(\dfrac{a}{2b}\right)$

$\left|\vec{a}+\vec{b}\right| = \sqrt{a^2+b^2+2ab\cos\theta} = \sqrt{a^2+b^2+2ab\left(4\cos^3\dfrac{\theta}{3} - 3\cos\dfrac{\theta}{3}\right)}$

$= \sqrt{a^2+b^2+2ab\left(\dfrac{4a^3}{8b^3}\right) - 6ab\left(\dfrac{a}{2b}\right)} = \sqrt{b^2+\dfrac{a^4}{b^2}-2a^2} = \dfrac{a^2-b^2}{b}$

52.(BD) $\bar{a}\times(\bar{b}\times\bar{c})$ is coplanar with $\bar{b}\,\&\,\bar{c}$ and $\perp \bar{a}$.

53.(AB) Equation of line in paramagnetic form is $\bar{r} = (\hat{i}+\hat{j}-2\hat{k}) + \lambda\left(\dfrac{4}{5}\hat{j} - \dfrac{3}{5}\hat{k}\right)$ Place $\lambda = \pm 1$

54.(AD) Knowing the volume of the prism we find its altitude $H = (AA_1) = \sqrt{6}$ and

Let Vertex $A_1(x, y, z)$

$\vec{AA_1} = (x-1, y, z-1)$. We get the other equations and from the condition $\vec{AA_1}$

Perpendicular to $\vec{AC}$ or compute $\pm\dfrac{\vec{AB}\times\vec{AC}}{|\vec{AB}\times\vec{AC}|} = \hat{n}$

$\sqrt{6}\hat{n} = \pm\left(\hat{i} + 2\hat{j} + \hat{k}\right) = (x-1)\,\hat{i} + (y)\,\hat{j} + (z-1)\,\hat{k}$ Compare to get A & D

55.(ABCD) $|(\bar{a}\times\bar{b})\cdot\bar{c}| = |\bar{a}||\bar{b}|\sin\alpha(\hat{n}\cdot\bar{c}) = |\bar{a}||\bar{b}||\bar{c}|\sin\alpha\cos\beta$

Where α is angle between $\bar{a}\,\&\,\bar{b}$, $\&\,\beta$ is angle between $\bar{c}$ & normal to $\bar{a}\,\&\,\bar{b}$.

If $|(\bar{a}\times\bar{b})\cdot\bar{c}| = |\bar{a}||\bar{b}||\bar{c}| \Rightarrow \alpha = \dfrac{\pi}{2}, \beta = 0$

56.AC (A) Section Formula (C) $(\vec{b}-\vec{a})\times(\vec{c}-\vec{a}) = 0 \Rightarrow \vec{b}\times\vec{c} + \vec{a}\times\vec{b} + \vec{c}\times\vec{a} = 0$

57.BC (B) Line of intersection of 2 planes (C) $\perp$ to both $\vec{a}\times\vec{b}$ and $\vec{c}\times\vec{d}$

58.(CD) $\left|\vec{a}+\vec{b}\right|<1 \Rightarrow \left|\vec{a}+\vec{b}\right|^2<1 \;\Rightarrow\; 1+1+2\cos\alpha<1 \Rightarrow \cos\alpha<-\dfrac{1}{2}$

59.CD Let vector be $l\vec{b}+m\vec{c}$; $\left|\left(l\vec{b}+m\vec{c}\right)\cdot\vec{a}\right|=\dfrac{\sqrt{2}}{\sqrt{3}}$

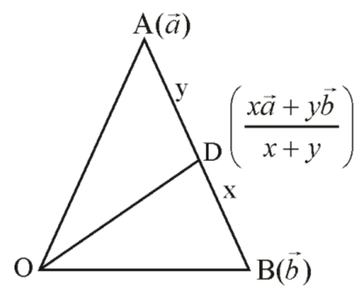
60.AC use angle bisector formula $\dfrac{\vec{a}+\vec{b}}{\left|\vec{a}+\vec{b}\right|}\times 2$

61.BC $\vec{\lambda}+\vec{\mu}+\vec{v}=0 \;\Rightarrow$ coplanar

62.AC

63. **[A-q, r, s] [B-p, s] [C-p, r] [D-p, q]**

$\vec{c}=(x+y)\overline{OD}$



64. **[A-r] [B-p] [C-s] [D-q]**

(B) $[\vec{a}\;\; \vec{b}+\vec{a}\times\vec{b}\;\; \vec{b}]=[\vec{a}\;\vec{b}\;\vec{b}]+[\vec{a}\;\; \vec{a}\times\vec{b}\;\; \vec{b}]=[\vec{a}\;\; \vec{a}\times\vec{b}\;\; \vec{b}]=\vec{a}\times(\vec{a}\times\vec{b}).\,\vec{b}]=((\vec{a}.\vec{b})\vec{a}-(\vec{a}.\vec{a})\vec{b}).\vec{b}$

$(\vec{a}.\vec{b})^2-(\vec{a}.\vec{a})(\vec{b}.\vec{b})$

(C) $[\vec{a}+\vec{b}+\vec{c}\qquad \vec{a}+\vec{b}\qquad \vec{b}+\vec{c}]=[\vec{a}\;\vec{b}\;\vec{c}]=|a|^2=1$

(D) As $\vec{x}$ & $\vec{y}$ are in the plane of $\vec{a}$ & $\vec{b}$ & $\vec{a}$, $\vec{b}$ & $\vec{c}$ are coplanar $\Rightarrow [\vec{x}\;\vec{y}\;\vec{c}]=0$

65. **[A-p, q, r, s] [B-p, q] C-p, r] [D-r]**

(A) $\begin{vmatrix}1 & 2 & -1\\ -1 & 1 & a\\ 0 & 1 & 0\end{vmatrix}\neq 0$

(B) $\vec{a}.\vec{b}>0$ **… (1)**

As $\vec{b}$ makes obtuse angle $\Rightarrow \lambda<0$ **… (2)**

(C) $\begin{vmatrix}2 & -1 & 1\\ 1 & 2 & 1+a\\ 3 & a & 5\end{vmatrix}=0$

66. **[A-r, s] [B-q, r] [C-q, r] [D-p, q, r]**

(A) $\vec{a}\cdot\vec{c}=\dfrac{1}{4}$ Also $\vec{a}\cdot\vec{b}-2\vec{a}\cdot\vec{c}=\lambda\vec{a}\cdot\vec{a}$ $\vec{a}\cdot\vec{b}=\lambda+\dfrac{1}{2}$

Also $\vec{b}\cdot\vec{b}-2\vec{b}\cdot\vec{c}=\lambda\vec{a}\cdot\vec{b}$ $\vec{b}\cdot\vec{c}=8-\dfrac{\lambda^2}{2}-\dfrac{\lambda}{4}$

Also $\vec{b}\cdot\vec{c}-2\vec{c}\cdot\vec{c}=\lambda\vec{a}\cdot\vec{c}$ $8-\dfrac{\lambda^2}{2}-\dfrac{\lambda}{4}-2=\dfrac{\lambda}{4} \;\Rightarrow\; \lambda^2+\lambda-12=0 \Rightarrow \lambda=3,-4$

(B) $xy-2zy+3zx=2zx+3xy-zy$

$\Rightarrow \quad 2xy + yz - zx = 0 \quad$ ......(i) and $x - y + 2z = 0 \quad$ ......(ii)

$x^2 + y^2 + z^2 = 12 \quad$ .....(iii) and $y < 0 \quad$ ....(iv)

On solving (i) and (ii) we get $x + y = 0$

Solving with $x - y = 2z$ [from (ii)]

$2x = -2z \Rightarrow x = -z$

$\therefore \quad x^2 + y^2 + z^2 = 3z^2 = 12 \quad \Rightarrow \quad z = 2, -2$

Given $y < 0 \Rightarrow x > 0$ and $z < 0 \quad \therefore \quad z = -2 \quad \therefore \quad x + y - z = 2 \quad \therefore \quad q, r$ are correct option

(C) $G = \dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$, $\vec{H} = \vec{a} + \vec{b} + \vec{c}$

Now $\overrightarrow{AD} + \overrightarrow{BD} + \overrightarrow{CH} + 3\overrightarrow{HG} = \lambda \overrightarrow{HD}$

$\vec{\alpha} - \vec{a} + \vec{\alpha} - \vec{b} + \vec{a} + \vec{b} - 3\left(\dfrac{2}{3}(\vec{a} + \vec{b} + \vec{c})\right) = \lambda(\vec{\alpha} - (\vec{a} + \vec{b} + \vec{c}))$

$\Rightarrow \quad 2(\vec{d} - \vec{a} + \vec{b} + \vec{c}) = \lambda(\vec{d} - (\vec{a} + \vec{b} + \vec{c})) \quad \Rightarrow \quad \lambda = 2 \quad \therefore \quad q, r$ are correct option

(D) If angle between $\vec{b}$ and $\vec{c}$ is α, then $\left|\vec{b} \times \vec{c}\right| = \sqrt{15}$

$\Rightarrow \quad \left|\vec{b}\right|\left|\vec{c}\right| \sin \pm = \sqrt{15} \Rightarrow \sin \pm = \dfrac{\sqrt{15}}{4}$; $\cos \pm = \dfrac{1}{4}$

$\Rightarrow \quad \vec{b} - 2\vec{c} = 4\lambda\vec{a} \Rightarrow \left(\vec{b} - 2\vec{c}\right)^2 = 16\lambda^2\left(\vec{a}\right)^2 \Rightarrow \left|\vec{b}\right|^2 + 4\left|\vec{c}\right|^2 - 4\vec{b}.\vec{c} = 16\lambda^2\left|\vec{a}\right|^2$

$\Rightarrow \quad 16 + 4 - 4\left|\vec{b}\right|\left|\vec{c}\right| \cos\alpha = 16\lambda^2 \Rightarrow \lambda = \pm 1$

67. **[A-r] [B-p] [C-p, q] [D-s]**

(A) $\because \left|\vec{a} \times \vec{b}\right| = \left|\vec{b} \times \vec{c}\right| = \left|\vec{c} \times \vec{a}\right|$ = twice the area of ΔABC having same direction. $\therefore \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

(B) As all the edges of a regular tetrahedron are of equal lengths.

$\therefore \quad \left|\vec{a}\right| = \left|\vec{b}\right| = \left|\vec{c}\right|$ and $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a}$

(C) $\vec{a} \times \vec{b} = \vec{c} \Rightarrow \vec{a} \perp \vec{c}$ and $\vec{b} \perp \vec{c}$; $\vec{b} \times \vec{c} = \vec{a} \Rightarrow \vec{b} \perp \vec{a}$ and $\vec{c} \perp \vec{a}$

$\Rightarrow \quad \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular $\Rightarrow \vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

Further $\left|\vec{a} \times \vec{b}\right| = \left|\vec{c}\right| \Rightarrow \left|\vec{a}\right|\left|\vec{b}\right| = \left|\vec{c}\right|$ and $\left|\vec{b} \times \vec{c}\right| = \left|\vec{a}\right| \Rightarrow \left|\vec{b}\right|\left|\vec{c}\right| = \left|\vec{a}\right| \Rightarrow \left|\vec{c}\right| = \left|\vec{a}\right|$ and $\left|\vec{b}\right| = 1$

(D) $\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow a^2 + b^2 + c^2 + 2\left(\vec{a}.\vec{b} + \vec{b}\vec{c} + \vec{c}\vec{a}\right) = 0 \Rightarrow \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = -\dfrac{3}{2}$

Also $\left|\vec{a}\right| = \left|\vec{b}\right| = \left|\vec{c}\right| = 1$

68.(5) Let $P(x_1, y_1), Q(x_2, y_2)$ be the two points on $y = 2^{x+2}$

$\overrightarrow{OP}.\hat{i}$ = projection of $\overrightarrow{OP}$ on x – axis, so $x_1 = -1 \Rightarrow y_1 = 2$

$\overrightarrow{OQ}.\hat{i}$ = projection of $\overrightarrow{OQ}$ on x – axis, so $x_2 = 2 \Rightarrow y_2 = 16$

If $\hat{j}$ is a unit vector along y-axis, then $\overrightarrow{OP} = -\hat{i} + 2\hat{j}$, $\overrightarrow{OQ} = 2\hat{i} + 16\hat{j}$

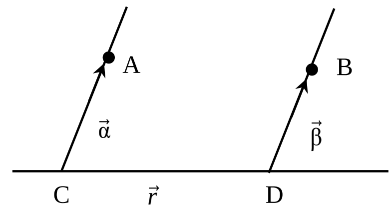
$\Rightarrow \quad \overrightarrow{OQ} - 4\,\overrightarrow{OP} = 6\hat{i} + 8\hat{j} \Rightarrow |\overrightarrow{OQ} - 4\,\overrightarrow{OP}| = \sqrt{36 + 64} = 10$

69.(9) *P.V.* of *C*

$\vec{r}_1 = (7\hat{i} + 6\hat{j} + 2\hat{k}) + a(3\hat{i} + 2\hat{j} + 4\hat{k}), a \in R$

P.V. of D, $\vec{r}_2 = (5\hat{i} + 3\hat{j} + 4\hat{k}) + b(2\hat{i} + \hat{j} + 3\hat{k}), b \in R$

$\overrightarrow{CD} = \vec{r}_2 - \vec{r}_1$ and we know that $\overrightarrow{CD} \parallel \vec{r} \Rightarrow \overrightarrow{CD} = c(2\hat{i} + 2\hat{j} + \hat{k})$



Hence by comparing both $\overrightarrow{CD}$ we get $3a - 2b - 2c + 2 = 0$, $2a - b - 2c + 3 = 0$

and $4a - 3b - c - 2 = 0 \Rightarrow a = 2, b = 1, c = 3 \Rightarrow \left|\overrightarrow{CD}\right| = 3\sqrt{2^2 + 2^2 + 1^2} = 9$

70.(3) $\left|\left(\vec{a} \times \vec{b}\right) \times \vec{c}\right| = \left|\vec{a} \times \vec{b}\right| |\vec{c}| \sin 30^0$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2i - 2j + k \Rightarrow |\vec{a} \times \vec{b}| = 3$$

$|\vec{c} - \vec{a}|^2 = 8 \Rightarrow |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c}.\vec{a} = 8 \Rightarrow |\vec{c}|^2 - 2|\vec{c}| + 1 = 0 \Rightarrow |\vec{c}| = 1 \Rightarrow \left|\left(\vec{a} \times \vec{b}\right) \times \vec{c}\right| = 1.\, 3.\, \frac{1}{2} = \frac{3}{2}$

71.(1) $\vec{a} \times (\vec{b} \times \vec{c}) + (\vec{a}.\vec{b})\vec{b} = (4 - 2\beta - \sin\alpha)\vec{b} + (\beta^2 - 1)\vec{c}$ **… (1)**

$\Rightarrow (\vec{a}.\vec{b} + \vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = (4 - 2\beta - \sin\alpha)\vec{b} + (\beta^2 - 1)\vec{c}$

$\Rightarrow \vec{a}.\vec{b} + \vec{a}.\vec{c} = 4 - 2\beta - \sin\alpha$ **… (2)** $\vec{a}.\vec{b} = 1 - \beta^2$ **… (3)** As $(\vec{c}.\vec{c})\vec{a} = \vec{c} \Rightarrow \vec{a}.\vec{c} = 1$ **… (4)**

$\Rightarrow 2 - \beta^2 = 4 - 2\beta - \sin\alpha \Rightarrow \sin\alpha = \beta^2 - 2\beta + 2 \Rightarrow \sin\alpha = (\beta - 1)^2 + 1 \Rightarrow \beta = 1, \alpha = (4n+1)\frac{\pi}{2}$

72.(6) $\vec{w} + (\vec{w} \times \vec{u}) = \vec{v}$ **… (1)**

Taking dot product with $\vec{v}$ & $\vec{w}$ we get :

$[\vec{u}\,\vec{v}\,\vec{w}] = 1 - \vec{w}.\vec{v}$ **… (2)** $|w|^2 = \vec{w}.\vec{v}$ **… (3)** $\Rightarrow [\vec{u}\,\vec{v}\,\vec{w}] = 1 - |w|^2$ **… (4)**

Now $|\vec{w} \times \vec{u}|^2 = |\vec{u} - \vec{w}|^2 \Rightarrow |w|^2 = \frac{1}{1 + \sin^2\theta} \Rightarrow$ Max $[\vec{u}\ \vec{v}\ \vec{w}] = \frac{1}{2}$

73.(7) Given $\vec{v}.\vec{u} = \hat{w}.\hat{u}$ & $\vec{v}.\vec{w} = 0 \Rightarrow |\vec{u} - \vec{v} + \vec{w}|^2 = 14$

74.(5) $\vec{a}.\vec{b} = 0$ $x_1 + x_2 + x_3 = 0$

Thus, we have to obtain the number of integral solutions of this equation.

Coefficient of x^0 *in* $(x^{-3} + x^{-2} + x^{-1} + x^0 + x + x^2)^3$

$$= x^0 \left| \left(\frac{1 + x + x^2 + x^3 + x^4 + x^5}{x^3}\right)^3 = x^9 \,|\, (1 - x^6)^3 (1 - x)^{-3} = {}^{11}C_9 - 3.{}^5C_3 = 25.\right.$$

75.(0) $\overrightarrow{BC} = \frac{\vec{u}}{|\vec{u}|} - \frac{\vec{v}}{|\vec{v}|}, \overrightarrow{AC} = \frac{2\vec{u}}{|\vec{u}|} \Rightarrow \overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC} = \frac{\vec{u}}{|\vec{u}|} + \frac{\vec{v}}{|\vec{v}|}$

$$\Rightarrow \quad \overrightarrow{AB} \cdot \overrightarrow{BC} = \left(\frac{\vec{u}}{|\vec{u}|} + \frac{\vec{v}}{|\vec{v}|}\right) \cdot \left(\frac{\vec{u}}{|\vec{u}|} - \frac{\vec{v}}{|\vec{v}|}\right) = \left(\frac{\vec{u} \cdot \vec{u}}{|\vec{u}|^2} - \frac{\vec{v} \cdot \vec{v}}{|\vec{v}|^2}\right) = \frac{|\vec{u}|^2}{|\vec{u}|^2} - \frac{|\vec{v}|^2}{|\vec{v}|^2} = 1 - 1 = 0$$

$$\Rightarrow \quad B = \frac{\pi}{2} \quad \Rightarrow \quad 1 + \cos 2A + \cos 2B + \cos 2C = 1 - 1 - 4\cos A \cos B \cos C = 0.$$

76.(1) Given that $\bar{u}\times\bar{v}+\bar{u}=\bar{w}$ and $\bar{w}\times\bar{u}=\bar{v}$

$\Rightarrow(\bar{u}\times\bar{v}+\bar{u})\times\bar{u}=\bar{v} \Rightarrow (\bar{u}\times\bar{v})\times\bar{u}=\bar{v} \Rightarrow \bar{v}-(\bar{u}.\bar{v})\bar{u}=\bar{v} \Rightarrow (\bar{u}.\bar{v})\bar{u}=0\Rightarrow(\bar{u}.\bar{v})=0$

Now, $[u\ v\ w] = u.(v\times w) = u.\ (\bar{v}\times(\bar{u}\times\bar{v}+\bar{u}) = u.(\bar{v}\times(\bar{u}\times\bar{v})+\bar{v}\times\bar{u}) = u.\ (\bar{v}^2\bar{u}-(\bar{u}.\bar{v})\bar{v}+\bar{v}\times\bar{u}) = v^2u^2 = 1$.

77.(4) $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c}\\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c}\\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c}\end{vmatrix}=16$

78.(0) $\vec{r}_1=4\vec{a}+m_1\vec{b};\ \vec{r}_2=l_2\vec{a}+m_2\vec{b}\ ;\ \vec{r}_3=l_3\vec{a}+m_3\vec{b}$

Since $\vec{r}_1,\vec{r}_2$ and $\vec{r}_2$ are co-linear $(\vec{r}_1-\vec{r}_2)\times(\vec{r}_2-\vec{r}_3)=0$

$\left((l_1-l_2)\vec{a}+(m_1-m_2)\vec{b}\right)\times\left((l_2-l_3)\vec{a}+(m_2-m_3)\vec{b}\right)=0$

$(l_1-l_2)(m_2-m_3)(\vec{a}\times\vec{b})-(m_1-m_2)(l_2-l_3)(\vec{a}\times\vec{b})=0$ or $(l_1-l_2)(m_2-m_3)-(m_1-m_2)(l_2-l_3)=0$

$\begin{vmatrix}1&1&1\\ l_1&l_2&l_3\\ m_1&m_2&m_3\end{vmatrix}=\begin{vmatrix}1&1&1\\ l_1-l_2&l_2&l_3\\ m_1-m_2&m_2&m_3\end{vmatrix}=\begin{vmatrix}0&1&1\\ l_1-l_2&l_2-l_3&l_3\\ m_1-m_2&m_2-m_3&m_3\end{vmatrix}$

$(l_1-l_2)(m_2-m_3)-(m_1-m_2)(l_2-l_3)=0$

79.(2) The p.v. of mid-point of BC is $\left(\frac{\lambda-1}{2}\right)\hat{i}+4\hat{j}+\left(\frac{\lambda+2}{2}\right)\hat{k}$

Since it is equally inclined to the axes, therefore its d.c. are $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

80.($2\sqrt{\frac{5}{7}}$) The equation of plane OAC

$(\vec{r}-\vec{0})=\vec{r}_1\times\vec{r}_3-(say\ \pi)$ Now, find the perpendicular distance of point B from the plane π

81.(7) $\vec{r}_1=\hat{i}-6\hat{j}+10\hat{k}$, $\vec{r}_2=-\hat{i}-3\hat{j}+7\hat{k}$, $\vec{r}_3=5\hat{i}-\hat{j}+h\hat{k}$, $\vec{r}_4=7\hat{i}-4\hat{j}+7\hat{k}$

The vectors corresponding to the edges are $\vec{a}=\vec{r}_1-\vec{r}_2=2\hat{i}-3\hat{j}+3\hat{k}$

$\vec{b}=\vec{r}_3-\vec{r}_2=6\hat{i}+2\hat{j}+(h-7)\hat{k}$; $\vec{c}=\vec{r}_4-\vec{r}_1=6\hat{i}+2\hat{j}-3\hat{k}$ Volume of the tetrahedron $=\left|\frac{1}{2}[\vec{a}\ \vec{b}\ \vec{c}]\right|=11$

82.(1) $\vec{a}\cdot\vec{b}=0=\vec{a}\cdot\vec{c}$; $\left|\vec{a}\times\vec{b}-\vec{a}\times\vec{c}\right|=\left|\vec{a}\times(\vec{b}-\vec{c})\right|=|\vec{a}|\cdot\left|\vec{b}-\vec{c}\right|$

Since $\vec{a}$ is perpendicular to $\vec{b}-\vec{c}$ as $\vec{a}\cdot(\vec{b}-\vec{c})=0$

$|\vec{a}|=1$ as $\left|\vec{b}-\vec{c}\right|=\sqrt{|b|^2+|c|^2-2\vec{b}\cdot\vec{c}}$

83.(3) $\vec{a}+\vec{b}=\lambda\vec{c}$ and $\vec{b}+\vec{c}=\mu\vec{a}$

$(\vec{a}-\vec{c})=\lambda\vec{c}-\mu\vec{a}\Rightarrow\vec{a}(1+\mu)=\vec{c}(\lambda+1) \Rightarrow \mu=1,\lambda=-1 \Rightarrow \vec{a}+\vec{b}+\vec{c}=0$

$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$; $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-3$

84.(2) Shortest distance $=\left|(\vec{a}-\vec{c})\cdot(\vec{a}\times\vec{d})\right|$

THREE DIMENSIONAL GEOMETRY

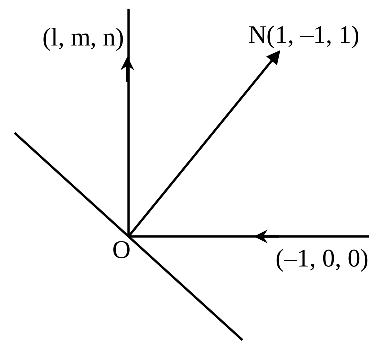
1.(D) Let $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$

$x = 2\lambda,\ y = 2 + 3\lambda,\ z = 3 + 4\lambda$

Now $(2\lambda + 2).2 + (7 + 3\lambda).3 + (-4 + 4\lambda).4 = 0$; $4\lambda + 4 + 21 + 9\lambda - 16 + 16\lambda = 0$; $29\lambda + 9 = 0$; $\lambda = -\frac{9}{29}$.

2.(A) The line is $\frac{x-1}{2}=\frac{y}{-1}=\frac{z+2}{1}=r$. There is no point of intersection of line and the plane hence line is parallel to the plane so direction cosines of the projected line remain same.

3.(D) The $d.c$'s of incident ray are $(-1, 0, 0)$. Let the $d.c$'s of reflected ray be (l, m, n), then the direction ratios of the normal to the plane of mirror will be $(l + 1, m, n)$.



So, $\frac{l+1}{1}=\frac{m}{-1}=\frac{n}{1}=k$ (say)

$\Rightarrow$ $l = k - 1,\ m = -k,\ n = k$

$\Rightarrow$ $(k - 1)^2 + (-k)^2 + (k)^2 = l^2 + m^2 + n^2 = 1 \Rightarrow k = 2/3$.

So, (l, m, n) is $\left(\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$.

4.(C) Since $\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \neq 0$, the given equations has a unique solution. Therefore the given planes meet in a unique point.

5.(A) Let $(\lambda, \lambda, \lambda)$ be the point of intersection of two lines.

$\Rightarrow$ $\lambda(\sin A + \sin B + \sin C) = 2d^2$ and $\lambda(\sin 2A + \sin 2B + \sin 2C) = d^2$

$\Rightarrow$ $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=\frac{1}{2}\Rightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{16}$.

6.(D) Point A is (a, b, c)

$\Rightarrow$ Points P, Q, R are $(a, b, -c), (-a, b, c)$ and $(a, -b, c)$ respectively.

$\Rightarrow$ Centroid of triangle PQR is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ $\Rightarrow$ $G \equiv \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

$\Rightarrow$ A, O, G are collinear $\Rightarrow$ area of triangle AOG is zero.

7.(A) Let the equation to the plane be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$

$\Rightarrow$ $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$ ($\because$ The plane passes through a, b, c)

Now the points of intersection of the plane with the coordinate axes are $A(\alpha, 0, 0)$, $B(0, \beta, 0)$ and $C(0, 0, \gamma)$

$\Rightarrow$ Equation to planes parallel to the coordinate planes and passing through A, B and C are $x = \alpha$, $y = \beta$ and $z = \gamma$.

$\therefore$ The locus of the common point is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$ (by eliminating α, β, γ from above equation)

Hence **(A)** is the correct answer.

8.(A) $a_1a_1 + b_1b_2 + c_1c_2 = 6 + 2 + 12 =$ positive

So acute angle bisector is $\dfrac{2x-y+2z+3}{3} = -\dfrac{3x-2y+6z+8}{7}$ $\Rightarrow$ $23x - 13y + 32z + 45 = 0$

Hence **(A)** is the correct answer.

9.(A) Let l, m, n be direction ratio of the normal

$l + m + n = 0$

$l + m + nd = 0 \quad \Rightarrow \quad n(1-d) = 0 \Rightarrow n = 0$

$\left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, 0\right), \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0\right).$

10.(C) Equation of straight line through origin is $\dfrac{x-0}{l} = \dfrac{y-0}{m} = \dfrac{z-0}{n}$

Where $l((b+c) + m(c+a) + n(a+b) = 0$ and $l(b-c) + m(c-a) + n(a-b) = 0$

On solving, $\dfrac{l}{2\left(a^2-bc\right)} = \dfrac{m}{2\left(b^2-ca\right)} = \dfrac{n}{2\left(c^2-ab\right)}$

$\therefore$ Equation of the straight line is $\dfrac{x}{a^2-bc} = \dfrac{y}{b^2-ca} = \dfrac{z}{c^2-ab}$

Hence **(C)** is the correct answer.

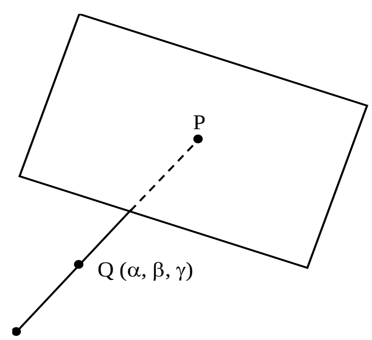
11.(D) Let P (α, β, γ) be the foot of perpendicular from origin to the plane.

Equation of plane is $\alpha(x-\alpha) + \beta(y-\beta) + \gamma(z-\gamma) = 0$

$$\frac{x}{\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}} + \frac{y}{\frac{\alpha^2+\beta^2+\gamma^2}{\beta}} + \frac{z}{\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}} = 1.$$

$(\alpha^2 + \beta^2 + \gamma^2)^2(\alpha^{-2} + \beta^{-2} + \gamma^{-2}) = k^2$

So locus is $(x^2 + y^2 + z^2)^2 (x^{-2} + y^{-2} + z^{-2}) = k^2$.



12.(A) Let $OP : PQ = \lambda : 1$

$P \equiv \left(\dfrac{\lambda\alpha}{\lambda-1}, \dfrac{\lambda\beta}{\lambda-1}, \dfrac{\lambda\gamma}{\lambda-1}\right)$

P lies on plane $lx + my + nz = p$

$\Rightarrow \quad \dfrac{\lambda}{\lambda-1}(\alpha l + \beta m + \gamma n) = p \quad \Rightarrow \quad \dfrac{\lambda}{\lambda-1} = \dfrac{p}{l\alpha + m\beta + n\gamma}$

Also $OP \cdot OQ = p^2 \Rightarrow \sqrt{\alpha^2+\beta^2+\gamma^2}\left(\dfrac{\lambda}{\lambda-1}\right)\sqrt{\alpha^2+\beta^2+\gamma^2} = p^2 \quad \Rightarrow \quad \dfrac{p(\alpha^2+\beta^2+\gamma^2)}{l\alpha+m\beta+n\gamma} = p^2$

So locus is $x^2 + y^2 + z^2 = (lx + my + nz)p$.

13.(A) Equation of the plane passing through the given line is $3x - 2y + 1 + \lambda(2x - z + 1) = 0$.

Since the plane is orthogonal to the given plane,

$3(3 + 2\lambda) + 4(-2) + 5(-\lambda) = 0$ or $\lambda = -1$.

Hence the plane thorough the given line and orthogonal to the given plane is $x - 2y + z = 0$.

Thus the equation to the required line is $3x + 4y + 5z = 0 = x - 2y + z$.

Writing in symmetrical form, we find that the orthogonal projection of the given line is $\dfrac{x}{7} = \dfrac{y}{1} = \dfrac{z}{-5}$.

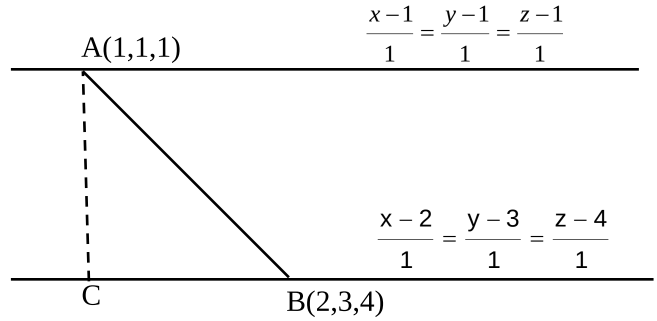
14.(C) Since the given lines are parallel. From figure,

$$BC = \frac{(2-1)\cdot 1}{\sqrt{3}} + \frac{(3-1)\cdot 1}{\sqrt{3}} + \frac{(4-1)\cdot 1}{\sqrt{3}}$$

$$= \frac{1+2+3}{\sqrt{3}} = 2\sqrt{3}.$$

$$AB = \sqrt{1+4+9} = \sqrt{14}$$

Shortest Distance = $AC = \sqrt{14-12} = \sqrt{2}$.



15.(A) Equation of line $x + 2y + z - 1 + \lambda(-x + y - 2z - 2) = 0$. **… (1)**

$x + y - 2 + \mu(x + z - 2) = 0$ **…(2)**

(0, 0, 1) lies on it $\Rightarrow \lambda = 0,\ \mu = -2$

For point of intersection point $z = 0$ and solve (1) and (2).

16.(ABC) Mid-point of object and image lies on mirror. Line joining object and image is $\perp$ to mirror.

17.(A) Any point on line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda$ is $P(\lambda, 2\lambda+1, 3\lambda+2)$, $A(1, 6, 3)$

d.r. of AP $\equiv <\lambda-1, 2\lambda-5, 3\lambda-1>$

AP 1 to the line

$\Rightarrow \quad 1(\lambda-1)+2(2\lambda-5)+3(3\lambda-1)=0 \quad \Rightarrow \quad 14\lambda-14=0 \quad \Rightarrow \quad \lambda=1 \quad \therefore \quad P\equiv(1,3,5)$

18.(B) Family of planes containing the given line is $x + y + z - 4 + \lambda\,[\,2x + y + 3z - 1] = 0$. If it passes through origin then $\lambda = -4$.

$\Rightarrow$ Equation of plane is $7x + 3y + 11z = 0$ **… (i)**

Equation of plane containing the given line and perpendicular to plane (i) is $x + y + z - 4 + \lambda_1 (2x + y + 3z - 1) = 0$**… (ii)**

where $7(1 + 2\lambda_1) + 3(1 + \lambda_1) + 11(1 + 3\lambda_1) = 0$

$\Rightarrow \quad \lambda_1 = \frac{-21}{50}$ = Line OA is perpendicular to plane (ii)

Hence direction ratios of OA are (8, 29, –13) $\quad \Rightarrow \quad$ Equation of line OA is $\frac{x}{8} = \frac{y}{29} = \frac{z}{-13}$.

19.(D) Let direction cosines of the line be l, m and $\cos 60^0$. Then its equation is

$$\frac{x}{l} = \frac{y}{m} = \frac{z-2}{1/2} = \lambda \qquad \textbf{... (1)}$$

$\Rightarrow \quad x = \ell\lambda,\ y = m\lambda$ and $z = 2 + \frac{\lambda}{2}$ putting in plane

$\ell\lambda + m\lambda + 2 + \frac{\lambda}{2} = 1 \Rightarrow \lambda\,(l + m + 1/2) = -1 \quad \Rightarrow \quad \lambda = \frac{-2}{2l+2m+1}$

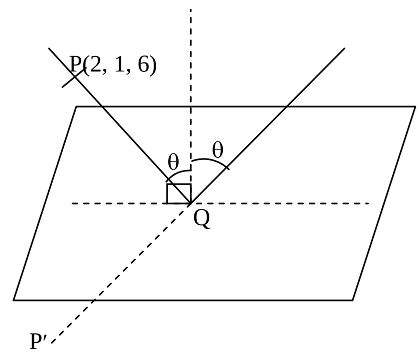
$x = \frac{-2l}{2l+2m+1},\ y = \frac{-2m}{2l+2m+1}$ and $z = 2 + \frac{-1}{2l+2m+1}$

Are the point of intersection of the line and plane

$\Rightarrow \quad 2l + 2m + 1 = \frac{-1}{z-2}$ putting in x and y we get :

$x = 2\,l\,(z - 2) \Rightarrow l = \dfrac{x}{2(z-2)}$ and $m = \dfrac{y}{2(z-2)}$ $\quad\Rightarrow\quad \left(\dfrac{x}{2(z-2)}\right)^2 + \left(\dfrac{y}{2(z-2)}\right)^2 + \dfrac{1}{4} = 1$

$\Rightarrow \quad x^2 + y^2 - 3\,(z-2)^2 = 0 \quad\quad \Rightarrow \quad x^2 + y^2 = 3\,(z-2)^2$ is the locus



20.(A) Equation of plane passing though (2, 1, 0), (5, 0, 1) and (4, 1, 1) is

$$\begin{vmatrix} x-2 & y-1 & z-0 \\ 5-2 & 0-1 & 1-0 \\ 4-2 & 1-1 & 1-0 \end{vmatrix} = 0$$

$\Rightarrow (x-2)(-1) - (y-1)(3-2) + z(0+2) = 0$

or, $\quad x + y - 2z = 3 \quad\quad$ **…(i)**

A point on the line $\dfrac{x-2}{3} = \dfrac{y-1}{4} = \dfrac{z-6}{5}$ is (2, 1, 6). Now the reflection point of this point (2, 1, 6) w. r. t mirror kept along the plane $x + y - 2z = 3$ is $\dfrac{\alpha'-2}{1} = \dfrac{\beta'-1}{1} = \dfrac{\gamma'-6}{-2} = \dfrac{-2(2+1-2\times 6-3)}{1^2+1^2+(-2)^2}$

$\Rightarrow \quad \alpha' = 2+4 = 6, \quad \beta' = 1+4 = 5 \quad \& \quad \gamma' = -8+6 = -2$.

Also the line $\dfrac{x-2}{3} = \dfrac{y-1}{4} = \dfrac{z-6}{5} = r$,

Let it meets the plane $x + y - 2z = 3$ at $(3r + 2, 4r + 1, 5r + 6)$

$\Rightarrow \quad 3r + 2 + 4r + 1 - 2(5r + 6) = 3 \quad\quad \Rightarrow \quad -3r = 12 \Rightarrow r = -4$

$\therefore \quad$ Point $Q = (-10, -15, -14)$

$\therefore \quad$ Equation of reflected ray is line joining P' and Q.

$\Rightarrow \quad \dfrac{x+10}{16} = \dfrac{y+15}{20} = \dfrac{z+14}{12}$

$\therefore \quad$ Reflected line $\dfrac{x+10}{4} = \dfrac{y+15}{5} = \dfrac{z+14}{3}$.

21.(C) Let any point on the given line $\dfrac{x-1}{2} = \dfrac{y-1}{1} = \dfrac{z-1}{1} = r$ is $(2r + 1, r + 1, r + 1)$

On putting this point on the plane $x + y + z = 1$

we get $r = -\dfrac{1}{2}$

$\Rightarrow \quad$ Given line cuts the plane $x + y + z = 1$ at the point $Q = \left(0, \dfrac{1}{2}, \dfrac{1}{2}\right)$

Now the line from point (1, 1, 1) and perpendicular to the given plane is $\dfrac{x-1}{1} = \dfrac{y-1}{1} = \dfrac{z-1}{1} = \lambda$ **… (1)**

$\Rightarrow \quad$ Any point on the line (1) is $(\lambda+1, \lambda+1, \lambda+1)$.

Therefore line (1) cuts the given plane at point R when $\lambda = \dfrac{1}{3} \Rightarrow \quad$ point R is $\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

Hence the equation of required line QR is $\dfrac{x-\frac{1}{3}}{\frac{1}{3}-0}=\dfrac{y-\frac{1}{3}}{\frac{1}{3}-\frac{1}{2}}=\dfrac{z-\frac{1}{3}}{\frac{1}{3}-\frac{1}{2}}$

or $\dfrac{x-\frac{1}{3}}{\frac{1}{3}}=\dfrac{y-\frac{1}{3}}{-\frac{1}{6}}=\dfrac{z-\frac{1}{3}}{-\frac{1}{6}}\Rightarrow\dfrac{3x-1}{2}=\dfrac{3y-1}{-1}=\dfrac{3z-1}{-1}$.

22.(B) Any point on the given line is $2r+1, -r-1, 4r+3$.

If this point lies on the planes, then $2r+1-2r-2+4r+3=12\Rightarrow r=\dfrac{5}{2}$.

Hence the point of intersection of the given line and that of the plane is $\left(6, -\dfrac{7}{2}, 13\right)$. Also a point on the line is $(1, -1, 3)$.

Let (α, β, γ) be its image in the given plane. In such a case $\dfrac{\alpha-1}{1}=\dfrac{\beta+1}{2}=\dfrac{\gamma-3}{1}=\lambda$

$\Rightarrow$ $\alpha=\lambda+1$, $\beta=2\lambda-1$, $\gamma=\lambda+3$. Now the midpoint of the image and the point $(1, -1, 3)$ lies on the plane i.e. $\left(1+\dfrac{\lambda}{2}, \lambda-1, 3+\dfrac{\lambda}{2}\right)$ lies in the plane

$\Rightarrow$ $\lambda=\dfrac{10}{3}$. Hence the image of $(1, -1, 3)$ is $\left(\dfrac{8}{3}, \dfrac{7}{3}, \dfrac{14}{3}\right)$.

Hence the equation of the required line is $\dfrac{x-6}{\frac{10}{3}}=\dfrac{y+\frac{7}{2}}{\frac{-35}{6}}=\dfrac{z-13}{\frac{25}{3}}$ or $\dfrac{x-6}{4}=\dfrac{y+\frac{7}{2}}{-7}=\dfrac{z-13}{10}$.

23.(A) x-axis and line of intersection of two plane must be coplanar which means point (x_1, 0, 0) will satisfy both the planes

i.e. $a_1x_1=d_1$ and $a_2x_1=d_2$ $\Rightarrow$ $\dfrac{d_1}{a_1}=\dfrac{d_2}{a_2}\Rightarrow d_1a_2=d_2a_1$.

24.(D) The equation of plane is $x+\lambda z=0$

As it is making an angle α from $x=0$ $\Rightarrow$ $\dfrac{1}{\sqrt{1+\lambda^2}}=\cos\alpha$ $\Rightarrow$ $\lambda=\pm\sqrt{\sec^2\alpha-1}$

25.(C) $P(x, y, z)$ lies on line through (0, 0, 2) making 60° with z-axis

i.e. $\dfrac{(xi+yj+(z-2)k)\cdot k}{\sqrt{x^2+y^2+(z-2)^2}}=\cos 60^\circ$ $\Rightarrow$ $x^2+y^2-3(z-2)^2=0.$

26.(B) The plane does not pass through (0, 0, 2) and its inclination with z-axis is θ, where $\sin\theta=\dfrac{1}{\sqrt{3}}$

i.e. $\theta<60^\circ\Rightarrow$ it represents a hyperbola.

27.(A) The plane passes through (0, 0, 2) and its inclination with z-axis is θ, where $\sin\theta=\dfrac{1}{\sqrt{3}}$

i.e. $\theta<60^\circ\Rightarrow$ it represents a pair of straight lines.

28.(B) The plane is parallel to z-axis $\Rightarrow$ locus is a hyperbola.

29.(A) The plane passes through (0, 0, 2) and its inclination with z-axis is θ, where $\sin\theta = \frac{1}{\sqrt{6}}$

i.e. $\theta < 60^\circ \Rightarrow$ it represents a pair of straight lines.

30.(D) In the new definition $l = \frac{x}{|x|+|y|+|z|}$, etc

31.(B) $d(O,P) = k \Rightarrow |x|+|y|+|z| = k$

which represents a set of 8 planes making intercepts of lengths k on positive as well as negative sides of all three axes.

32.(D) $d(O,P) = d(A,P)$

$\Rightarrow |x|+|y|+|z| = |x-1|+|y-2|+|z-3| \Rightarrow x+y+z = |x-1|+|y-2|+|z-3|$.

Now the sign-scheme is

$x-1$	$y-2$	$z-3$	Resulting equation	Inference
+	+	+	$0=-6$	No solution
–	+	+	$x=-2$	No solution
+	–	+	$y=-1$	No solution
+	+	–	$z=0$	xy-plane
–	–	+	$x+y=0$	$x=0, y=0 \Rightarrow z$-axis
–	+	–	$x+z=1$	Plane | | to y-axis
+	–	–	$y+z=2$	Plane | | to x-axis
–	–	–	$x+y+z=3$	Part of an oblique plane

33.(D) Side length of triangle = 2

So, the coordinates of vertices are $(\sqrt{2}, 0, 0), (0, \sqrt{2}, 0), (0, 0, \sqrt{2})$ and hence the centroid is $\left(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right)$.

34.(D) Both lines are coplanar

$$\begin{vmatrix} 2 & 3 & \lambda \\ 3 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix} = 0 \Rightarrow 2(-2+3)+3(3+3)+\lambda(-3-2) = 0 \Rightarrow \lambda = 4$$

$$\sin^{-1}\sin 4 = \sin^{-1}\sin(\pi-4) = \pi-4$$

35.(B) Let $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{4} = r_1$

$\Rightarrow x = 3+2r_1,\ y = 2+3r_1,\ z = 1+4r_1$ it will line on $\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-2}{3} \Rightarrow r_1 = 1$

So, point of intersection is (5, 5, 5)

36.(C) Equation of plane contains both lines $\begin{vmatrix} x-2 & y-2 & z-1 \\ 2 & 3 & 4 \\ 3 & 2 & 3 \end{vmatrix} = 0$

$(x-3)(1)+(y-2)(12-6)+(z-1)(4-9)=0$

$x+6y-5z=10$

37.(B) The equation of the line of intersection of the planes $y+z=0$ and $z+x=0$ are $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ ……(1)

Similarly the equation of edge of intersection of the planes $x+y=0$ and $x+y+z=a$ are $\frac{x}{1}=\frac{y}{-1}=\frac{z-a}{0}$ ……(2)

If l, m, n be the d.c's of the shortest distance between the lines (1) and (2)

$\therefore \quad 1\cdot l+1\cdot m-1\cdot n=0$ and $1\cdot l-1\cdot m+0\cdot n=0$

Solve these we get $l=-\frac{1}{\sqrt{6}}, m=-\frac{1}{\sqrt{6}}, n=-\frac{1}{\sqrt{6}}$

38.(A) It is clear from (1) and (2) that $A(0, 0, 0)$ is a point on lien (1) and $B(0, 0, a)$ is a point on line (2)

$\therefore \quad S.D=\left|(0-0)+m(0-0)+n(0-a)\right|=-\frac{2}{\sqrt{6}}(-a)=\frac{2}{\sqrt{6}}a$

39.(BC) The locus is the pair of planes passing through the point of intersection of the lines and having their bisectors as the normals.

40.(AB) The equation of the required plane is $(x - y + 1) + \lambda\,(2y + z - 6) = 0$

$\Rightarrow \quad x + (2\lambda - 1)\,y + \lambda z + 1 - 6\lambda = 0$

Since it makes an angle of 30^0 with $x + y + z = 5$

$\Rightarrow \quad \frac{|1+(2\lambda-1)+\lambda|}{\sqrt{3}\cdot\sqrt{1+\lambda^2+(2\lambda-1)^2}}=\frac{\sqrt{3}}{2} \quad \Rightarrow |6\lambda| = 3\sqrt{5\lambda^2-4\lambda+2}$

$\Rightarrow \quad 4\lambda^2 = 5\lambda^2 - 4\lambda + 2 \Rightarrow \lambda^2 - 4\lambda + 2 = 0 \Rightarrow \lambda = (2\pm\sqrt{2})$

$\Rightarrow \quad (x - y + 1) + (2 \pm \sqrt{2})\,(2y + z - 6) = 0$ are two required planes.

41.(A) Equation of planes $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Three mutually perpendicular planes are

$x = 0, y = 0$ and $z = 0$.

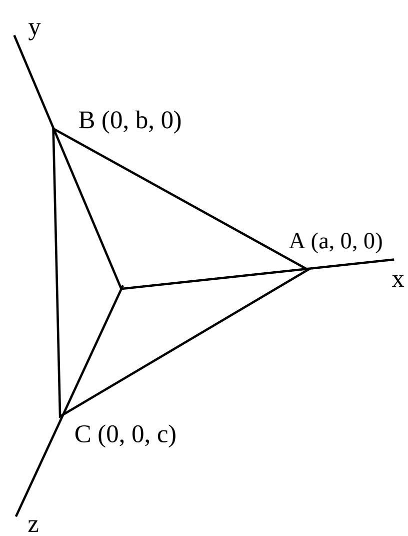
$d.r.$ of AB is $(a, -b, 0)$

$d.r.$ of AC is $(a, 0, -c)$

So, $\cos A=\frac{a^2}{\sqrt{a^4+\sum a^2b^2}} \Rightarrow \cot A=\frac{a^2}{\sqrt{\sum a^2b^2}}$

similarly $\cot B=\frac{b^2}{\sqrt{\sum a^2b^2}}$, $\cot C=\frac{c^2}{\sqrt{\sum a^2b^2}}$.

Let α be the angle between the planes z = 0 and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ then



$$\cos\alpha = \frac{\frac{1}{c}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = \frac{ab}{\sqrt{\sum a^2b^2}} \Rightarrow \cos^2\alpha = \frac{a^2b^2}{\sum a^2b^2} = \cot A \cot B.$$

Similarly other results can be proved.

42.(A) $l = \left(\frac{-bm-cn}{a}\right) \Rightarrow fmn + (gn+hm)\left(\frac{-bm-cn}{a}\right) = 0 \quad \Rightarrow \quad -\frac{hb}{a}\left(\frac{m}{n}\right)^2 + \left(f - \frac{gb}{a} - \frac{ch}{a}\right)\frac{m}{n} - \frac{gc}{a} = 0.$

If $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ are the roots then $\frac{m_1}{n_1}\cdot\frac{m_2}{n_2} = \frac{gc}{hb}$.

Similarly $\frac{l_1}{n_1}\cdot\frac{l_2}{n_2} = \frac{fc}{ha}$.

Two lines are perpendicular then

$\Rightarrow \quad l_1l_2 + m_1m_2 + n_1n_2 = 0 \Rightarrow \frac{l_1l_2}{n_1n_2} + \frac{m_1m_2}{n_1n_2} + 1 = 0 \Rightarrow \quad \frac{gc}{hb} + \frac{fc}{ha} + 1 = 0 \Rightarrow \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0.$

43.(ABCD)

As perpendicular distances are in *A.P*.

Let $|a-d|, |a|, |a+d|$ be these distances.

So, the coordinates of point P are $(\pm(a-d), \pm a, \pm(a+d))$

Now, distances from P of z–axis, y–axis and x–axis are respectively,

$2a^2 + d^2 - 2ad = 5$ **…(1)**

$2a^2 + 2d^2 = 10$ **… (2)**

$2a^2 + d^2 + 2ad = 13$ **…(3)**

Solving (1) and (3), we get : $4ad = 8$

Substituting in (2) : $\frac{4}{d^2} + d^2 = 5 \Rightarrow d^2 = 1$ or 4

$\Rightarrow \quad d = \pm 1, \; a = \pm 2$

So, Points are $(\pm 1, \pm 2, \pm 3)$. (There will be 8 such points)

44.(AB) The equation of a plane through the line of intersection of the plane $3y + 4z = 0$ and $x = 0$ is $3y + 4z + \lambda x = 0$.

This plane makes an angle 60^0 with the plane $3y + 4z = 0$

$\therefore \quad \cos 60^0 = \frac{3^2+4^2}{\sqrt{3^2+4^2}\sqrt{3^2+4^2+\lambda^2}} \Rightarrow \cos^2 60^0 = \frac{3^2+4^2}{3^2+4^2+\lambda^2} \quad \Rightarrow \quad \frac{1}{4} = \frac{25}{25+\lambda^2} \Rightarrow \lambda = \pm 5\sqrt{3}$

Hence required plane is $3y + 4z \pm 5\sqrt{3}\,x = 0$.

45.(CD) For the points where the line intersects the curve, we have $z = 0$. Therefore $\frac{x-2}{3} = \frac{y+1}{2} = \frac{0-1}{-1}$

$\Rightarrow \frac{x-2}{3} = 1$ and $\frac{y+1}{2} = 1 \Rightarrow x = 5$ and $y = 1$

Put these values in $xy = c^2$, we get $c = \pm\sqrt{5}$.

46.(C) If the two lines have direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 then the direction ratio of their bisectors will be $l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2$

47.(AD) Let the equation of plane be $Ax + By + Cz = 0$ **… (1)**

As line is parallel $\Rightarrow 2A - B - 2C = 0$ **… (2)**

As distance is $\frac{5}{3}$ and the point (1, – 3, –1) is lying on the line $\Rightarrow |A - 3B - C| = \frac{5}{3}$ **… (3)**

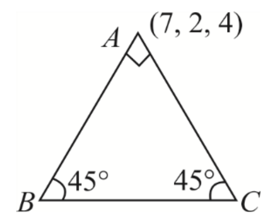
& $A^2 + B^2 + C^2 = 1$ **… (4)**

Solve (2), (3) and (4) for *A*, *B*, *C*.

48.(BD) Let the point *B* be $(-6 + 5\lambda, -10 + 3\lambda, -14 + 8\lambda)$

$\Rightarrow$ direction ratio of AB $\quad 5\lambda - 13, 3\lambda - 12, 8\lambda - 18$

The value of $\lambda (\lambda = 2,\ \lambda = 3)$ can be found as angle between *AB* and *BC* is 45°



49.(AB) Let the point on the line be $(2\lambda, 2 + 2\lambda, 1)$

Now as it strikes $x = 0 \Rightarrow \lambda = 0 \quad \Rightarrow \quad A(0, 2, 1)$

Find the reflection of any point *P* on the line along $x = 0$ let it be *Q* then *AQ* will be reflected ray.

50.(AB) Let the required line passes through the point (*h*, 0, 0). Then its equation is

$$\frac{x-h}{1} = \frac{y-0}{0} = \frac{z-0}{1} \text{ or, } \frac{x-h}{1} = \frac{y-0}{0} = \frac{z-0}{-1}$$

Let MN be a line perpendicular to given line and required line and its direction cosines be *l* : *m* : *n*, then

$l \times 1 + m \times 1 + n \times 1 = 0$

$l \times 1 + m \times 0 + n \times 1 = 0$

Solving we get $l = -n$ and $m = 0 \quad \Rightarrow \quad l = \frac{1}{\sqrt{2}}, n = -\frac{1}{\sqrt{2}}, m = 0$

Let point *A* be (*h*, 0, 0) and *B* be (1, 1, 0) then projection of *AB* on the line *MN* is $l(h-1) + m(0-1) + n(0-0) = 1$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}(h-1) + 0 - \left(\frac{1}{\sqrt{2}} \times 0\right) = 1 \Rightarrow h - 1 = \sqrt{2} \Rightarrow h = 1 + \sqrt{2}$$

$$\Rightarrow \quad \frac{x-1-\sqrt{2}}{1} = \frac{y-0}{0} = \frac{z-0}{1} \text{ or, } \frac{x-h}{1} = \frac{y-0}{0} = \frac{z-0}{-1} \quad \Rightarrow \quad l + m + n = 0 \text{ and } -l + 0 + n = 0$$

Solving, we get $n = \frac{1}{\sqrt{6}}$, $l = \frac{1}{\sqrt{6}}$ and $m = -\frac{2}{\sqrt{6}}$

$$\Rightarrow \quad \frac{1}{\sqrt{6}}(h-1) - \frac{2}{\sqrt{6}}(0-1) + \frac{1}{\sqrt{6}}(0-0) = 1 \Rightarrow h = \sqrt{6} - 1$$

$$\therefore \quad \frac{x-\sqrt{6}+1}{1} = \frac{y-0}{0} = \frac{z}{-1}.$$

Hence $\frac{x-\sqrt{6}+1}{1} = \frac{y-0}{0} = \frac{z}{-1}$ and $\frac{x-1-\sqrt{2}}{1} = \frac{y-0}{0} = \frac{z-0}{1}$ are the required lines.

51.(AC) Find the point of intersection of line and plane let it be *A*.

Find the foot of normal of point (–1, –1, –3) along the plane let it be *B* then *AB* is required line.

52.(ABCD) Three planes meet at two points it means they have infinite many solution, so $\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ \alpha & -1 & 3 \end{vmatrix} = 0$ on solving $\alpha = 4$

$P_1 : 2x + y + z = 1$ **;** $P_2 : x - y + z = 2$ **;** $P_3 : 4x - y + 3z = 5$

P on XOY plane $\equiv (1, -1, 0)$ (Which can be obtained by putting $z = 0$ in any two equations)

Q on YOZ plane $\equiv \left(0, -\frac{1}{2}, \frac{3}{2}\right)$ (By putting $x = 0$ in any two equations)

53.(AC) Any point on the line $(3\lambda - 2, 2\lambda - 1, 2\lambda + 3)$, then

$(3\lambda - 3)^2 + (2\lambda - 3)^2 + (2\lambda)^2 = 18 \quad \Rightarrow \quad \lambda = 0, \frac{30}{17} \Rightarrow$ points are $(-2, -1, 3)$ and $\left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17}\right)$.

54.(ABC) Let $P(2r_1, -3r_1, r_1)$ and $Q(3r_2 + 2, -5r_2 + 1, 2r_2 - 2)$ be the points on the given lines so that PQ is the line of shortest distance

d.r.s of PQ $2r_1 - 3r_2 - 2, -3r_1 + 5r_2 - 1, r_1 - 2r_2 + 2$

Since it is perpendicular to given lines $2(2r_1 - 3r_2 - 2) - 3(3r_1 + 5r_2 - 1) + (r_1 - 2r_2 + 2) = 0$

and $(2r_1 - 3r_2 - 2) - 5(-3r_1 + 5r_2 - 1) + 2(r_1 - 2r_2 + 2) = 0$

$\Rightarrow r_1 = 31/3,\ r_2 = 19/3 \Rightarrow$ P is $\left(\frac{62}{3}, -31, \frac{31}{3}\right)$ and Q is $\left(21, \frac{-92}{3}, \frac{32}{3}\right)$ d. r. s of PQ is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

55.(BC) Equation of planes bisecting the angle between given planes are

$$\frac{2x - y + 2z + 3}{\sqrt{4 + 1 + 4}} = \pm \frac{3x - 2y + 6z + 8}{\sqrt{9 + 4 + 36}}$$

$\Rightarrow 5x - y - 4z - 3 = 0$ and $23x - 13y + 32z + 45 = 0$ are required planes.

56.(BC)

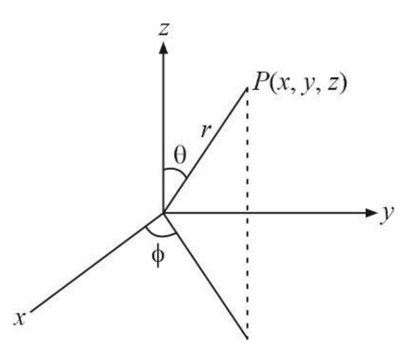
57.(AC) Since point of intersection of the given lines is (0, 0, 0). It must lie on the angle bisector

So, $\frac{0-2}{-1} = \frac{0+2}{1} = \frac{0 + k^2 - 1}{4} \Rightarrow k \pm 3$

58.(ABC) If P be (x, y, z) then from the figure, $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

$\Rightarrow \quad 1 = r\sin\theta\cos\phi,\ 2 = r\sin\theta\sin\phi,\ 3 = r\cos\theta \quad \Rightarrow \quad 1^2 + 2^2 + 3^2 = r^2 = \pm\sqrt{14}$

$\therefore \quad \sin\theta\cos\phi = \frac{1}{\sqrt{14}},\ \sin\theta\sin\phi = \frac{1}{\sqrt{14}}$



$\sin\theta\sin\phi = \frac{2}{\sqrt{14}},\ \cos\theta = \frac{3}{\sqrt{14}}$

(neglecting negative sign as θ and ϕ are acute)

$\therefore \quad \frac{\sin\theta\sin\phi}{\sin\theta\cos\phi} = \frac{2}{1} \Rightarrow \tan\phi = 2$

Also, $\tan\theta = \frac{\sqrt{5}}{3}$

59.(AB)

60. **[A-p, q] [B-p, q] [C-r, s] [D-q]**

(A) Let the point on the line be $(-1-\lambda, 12+5\lambda, 7+2\lambda)$, place it on the surface.

(B) Let the points on the two lines be $A(\lambda_1, \lambda_1 - 5, \lambda_1 - 1)$ and $B(\lambda_2 - 5, \frac{\lambda_2}{3}, \frac{\lambda_2}{2})$ respectively the variable line will be parallel to *AB*.

(C) Any plane through the intersection of the two planes is $(2x-y)+\lambda(3z-y)=0$.

Find λ using the condition that it is perpendicular to third plane

61. **[A-q] [B-p] [C-s] [D-r]**

(B) The line will be parallel to normal of the plane

(C) Let the points of shortest distance on the two lines be

$A(2\lambda_1, -3\lambda_1, \lambda_1)$ & $B(2+3\lambda_2, 1-5\lambda_2, -2+2\lambda_2)$

Now use the condition that AB will be perpendicular to the two lines to find the value of λ_1 & λ_2.

(D) The direction ratio of line of intersection will be cross product of $(3i - j + k)$ & $(i + 4j - 2k)$

62. **[A-t] [B-p] [C-s] [D-r]**

(A) The line is $\dfrac{x-5}{\frac{4}{\sqrt{26}}} = \dfrac{y}{\frac{1}{\sqrt{26}}} = \dfrac{z+6}{\frac{3}{\sqrt{26}}} = \approx$. Place $\lambda = \pm 3$

(B) The equation of plane is $\begin{vmatrix} x-2 & y-4 & z-6 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$

(C) The equation of line in parametric form is $\dfrac{x-2}{\frac{6}{7}} = \dfrac{y+3}{\frac{2}{7}} = \dfrac{z+1}{\frac{3}{7}} = \approx$

Now place $\lambda = \pm 14$ (Check which is nearer to origin)

(D) Let foot of perpendicular of the point $A(3, -1, 11)$ on line be $B(2\lambda, 2+3\lambda, 3+4\lambda)$.

Now as *AB* is perpendicular to line, find the value of λ.

63. **[A-q, s] [B-s] [C-r] [D-p]**

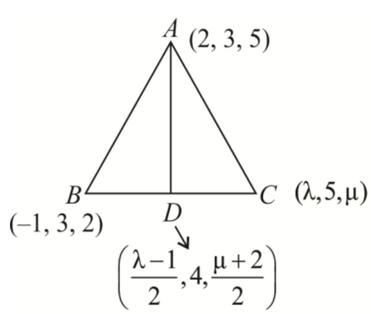
(A) Let d.c of the line be l, m, n then

$2l - 3m - n = 0$ **… (1)**, $-l + 2m - 2n = 0$ **… (2)** $l^2 + m^2 + n^2 = 1$ **… (3)**

64. **[A-q] [B-p] [C-p] [D-r]**

(A) Find the point of intersection of the line & plane. Let it be *A*. Find the foot of perpendicular of the point (1, 3, 4) on the given plane. Let it be *B*. *AB* will be the line *L* = 0

(B)



As AD is equally inclined from axes its direction ratio will be 1, 1, 1.

(D) As $A(a,b,c)$ is lying on the plane $\Rightarrow$ $3a+2b+c=7$

If $\vec{p}\ \&\ \vec{q}$ be two vectors then $\vec{p}\cdot\vec{q}\le|\vec{p}||\vec{q}|$

$$\Rightarrow \quad (3i+2j+k)(ai+bj+ck)\le\sqrt{14}\sqrt{a^2+b^2+c^2}$$

$$\sqrt{a^2+b^2+c^2}\ge\frac{7}{\sqrt{14}}$$

65.(2) Any plane passing through first line $2x + y + z - 1 + \lambda(3x + y + 2z - 2) = 0$. If it is parallel to second line

$(2 + 3\lambda)1 + (1 + \lambda)1 + (1 + 2\lambda)1 = 0 \Rightarrow \lambda = -2/3$.

Plane is $y - z + 1 = 0$

Distance from (0, 0, 0) $=\frac{1}{\sqrt{2}}$.

66.(1) Intersection of line $y - z - 1 = 0$, $x = 0$ with xy plane gives $x = 0$, $y = 1$ and $z = 0$ so point A is (0, 1, 0). Similarly point B is (– 1, 0, 0) and point C is (1, 0, 0). Here $AB^2 + AC^2 = BC^2$.So triangle ABC is right angled at A. Hence orthocentre is point A (0, 1, 0).

67.(4) Equation of plane passing through (1, 1, 1) and perpendicular to the given line is $x + 2y + 3z = 6$.

The minimum distance of the point (1, 1, 1) from the plane $x + y + z = 1$ measured perpendicular to $\frac{x-x_1}{1}=\frac{y-y_1}{2}=\frac{z-z_1}{3}$ is the perpendicular distance of the point (1, 1, 1) from the line of intersection of the above two planes.

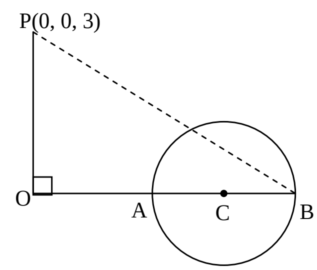
The equation of line of intersection of these planes is

$$\frac{x+4}{1}=\frac{y-5}{-2}=\frac{z-0}{1}=r \qquad \textbf{... (1)}$$

The minimum distance of (1, 1, 1) from the line (1) is $\frac{2}{3}\sqrt{21}$ units.

68.(5) Let O be the foot of perpendicular from the point P(0, 0, 3) to the $x - y$ plane. Hence it is evident that PB is the maximum distance of point P from the circle

$$\Rightarrow \quad PB^2 = OP^2 + OB^2 = 25 \ \Rightarrow PB = 5$$



69.(6) Equation of plane which contains the line $\frac{x-2}{3}=\frac{y-3}{1}=\frac{z-1}{2}$ is

$(x - 2 - 3(y - 3)) + \lambda(2(y - 3) - (z - 1)) = 0$

$x + (2\lambda - 3)y - \lambda z + 7 - 5\lambda = 0$

If the plane is parallel to $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$

$\Rightarrow 1 + (2\lambda - 3)2 - 3\lambda = 0 \Rightarrow \lambda = 5$.

So, equation of plane which contains second line and parallel to first is $x + 7y - 5z - 18 = 0$.

Similarly, equation of plane which contains first line and parallel to second is $(x - 1) + 7(y - 2) - 5(z - 3) = 0$

$\Rightarrow \quad x + 7y - 5z = 0$

Hence equidistant plane is $\frac{(x+7y-5z-18)+(x+7y-5z)}{2}=0$

$\Rightarrow \quad x + 7y - 5z - 9 = 0$.

Alternate Solution:

General points on lines are $P(\lambda + 1, 2\lambda + 2, 3\lambda + 3)$ and $Q(3\mu + 2, \mu + 3, 2\mu + 1)$

Let the required plane be $ax + by + cz = 1$

Then mid point of PQ will always lie on the plane for all values of λ and μ

$\Rightarrow \quad a\left(\frac{3\mu+\lambda+3}{2}\right)+b\left(\frac{2\lambda+\mu+5}{2}\right)+c\left(\frac{3\lambda+2\mu+4}{2}\right)=1$

$\Rightarrow \quad \mu(3a+b+2c)+\lambda(a+2b+3c)+3a+5b+4c-2=0 \quad \Rightarrow \quad 3a + b + 2c = 0, \quad a + 2b + 3c = 0$

$3a + 5b + 4c = 2 \quad \Rightarrow \quad a = \frac{1}{9}, b = \frac{7}{9}, c = -\frac{5}{9}$.

So, plane is $x + 7y - 5z = 9$.

70.(3) New plane will pass through the line of intersection of the planes $2x + 3y + 4z = 5$ and $z = 0$. This plane can be written as $2x + 3y + 4z - 5 + \lambda z = 0$. Now (1, 1, 1) lies on it

So, $\lambda = -4$

So, the equation of plane in new position is $2x + 3y - 5 = 0$

Angle between these planes is given by $\cos\alpha = \frac{2^2+3^2}{\sqrt{29}\sqrt{13}} \Rightarrow \alpha = \cos^{-1}\sqrt{\frac{13}{29}}$.

71.(4) Let the equation of plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ **… (1)**

$\Rightarrow \quad \left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| = p \quad \Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$ **… (2)**

The coordinate of centroid `is $\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right) \Rightarrow$ locus is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$

72.(1) The given planes are $x-cy-bz=0,\ cx-y+az=0,\ bx+ay-z=0$

$\therefore$ The rectangular array is $\left\|\begin{matrix}1 & -c & -b & 0\\ c & -1 & a & 0\\ b & a & -1 & 0\end{matrix}\right\| \quad \therefore \quad \Delta_4 = \begin{vmatrix}1 & -c & -b\\ c & -1 & a\\ b & a & -1\end{vmatrix} = \begin{vmatrix}1 & -c & -b\\ 0 & c^2-1 & bc+a\\ 0 & a+bc & b^2-1\end{vmatrix}$

(Subtracting c times 1st row from 2nd and b times from third row.)

$$= \begin{vmatrix}c^2-1 & bc+a\\ a+bc & b^2-1\end{vmatrix} = (c^2-1)(b^2-1)-(bc+a)(bc+a)$$

$$= (c^2b^2-c^2-b^2+1)-(b^2c^2+a^2+2abc) = 1-a^2-b^2-c^2-2abc$$

and $\quad \Delta_1 = \begin{vmatrix}-c & -b & 0\\ -1 & a & 0\\ a & -1 & 0\end{vmatrix} = 0$

Now, if the given planes intersect in a line, then Δ_4 must be zero.

i.e., $\quad 1-a^2-b^2-c^2-2abc=0 \quad$ i.e., $\quad a^2+b^2+c^2+2abc=1$

73.(4) The equations of the given planes can be written as

$$x-y+z+1=0$$
$$\lambda x+3y+2z-3=0$$
$$3x+\lambda y+z-2=0$$

$\therefore$ The rectangular array is $\begin{vmatrix}1 & -1 & 1 & 1\\ \lambda & 3 & 2 & -3\\ 3 & \lambda & 1 & -2\end{vmatrix}=0$

$\therefore$ $$\Delta_4=\begin{vmatrix}1 & -1 & 1\\ \lambda & 3 & 2\\ 3 & \lambda & 1\end{vmatrix}=(\lambda-4)(\lambda+3) \quad \textbf{... (i)}$$

$$\Delta_1=\begin{vmatrix}-1 & 0 & 0\\ 3 & 5 & 0\\ \lambda & \lambda+1 & \lambda-2\end{vmatrix}=-5(\lambda-2) \quad \textbf{... (ii)}$$

$$\Delta_2=\begin{vmatrix}1... & 1... & 0\\ \lambda & 2-\lambda & -3-\lambda\\ 3 & -2 & -5\end{vmatrix}=3\lambda-16 \quad \textbf{... (iii)}$$

$$\Delta_3=\begin{vmatrix}1 & -1 & 1\\ \lambda & 3 & -3\\ 3 & \lambda & -2\end{vmatrix}=(\lambda+3)(\lambda-2) \quad \textbf{... (iv)}$$

If, the given planes form a triangular prism, then we know that $\Delta_4=0$ and none of $\Delta_1,\Delta_2,\Delta_3$ is zero. Here from equations (i), (ii), (iii) and (iv) we find that if $\lambda=4,$ then $\Delta_4=0$ and none of $\Delta_1,\Delta_2,\Delta_3$ is zero.
Consequently for $\lambda=4,$ the given planes form a triangular prism.

74.(7) Equation of the required plane is $2x+y+2z-9+\lambda(4x-5y-4z-1)=0$

It passes through (3, 2, 1)

$\therefore$ $6+2-9+\lambda(12-10-4-1)=0 \quad\Rightarrow\quad 1-3\lambda=0 \Rightarrow \lambda=\dfrac{1}{3}$

$\therefore$ Equation of the required plane is $2x+y+2z-9+\dfrac{1}{3}(4x-5y-4z-1)=0$

$\Rightarrow$ $10x-2y+2z=28 \Rightarrow \dfrac{x}{14/15}+\dfrac{y}{14}+\dfrac{z}{14}=1 \Rightarrow a=\dfrac{14}{5}, b=-14, c=14 \Rightarrow (14+14-14)^2=196$

75.(4) The given line passes through $A(5,-2,6)$

$(PQ)^2=(AP)^2-(AQ)^2$

$AP^2=19$

Now AQ is the projection of AP on the given line

$\therefore$ $AQ=(5-4)\dfrac{3}{\sqrt{50}}+(-2+5)\left(\dfrac{-4}{\sqrt{50}}\right)+(6-3)\left(\dfrac{-5}{\sqrt{50}}\right)=\dfrac{6}{\sqrt{50}} \Rightarrow PQ^2=19-\dfrac{36}{50}=\dfrac{427}{25}$

$\therefore$ $100(PQ)^2=1828$

76.(5) The plane passes through the point of intersection of lines (0, 1, 1)

The point (2, 3, 2) lines on the line, whose projection on the second line is (1, 2, 0)

$\therefore$ the required plane has normal (1, 1, 2)

$\therefore$ Equation of the plane is $1(x-0)+1(y-1)+2(z-1)=0$

$x+y+2z-3=0$; $(k, -2, 0)$ lies on the plane $\therefore$ $k=5$

77.(2) Let $x\hat{i}+y\hat{j}+z\hat{k}=\vec{r}$

$\vec{r}\cdot(10\hat{j}-8\hat{j}-\vec{r})=41 \Rightarrow -\vec{r}\cdot(10\hat{i}-8\hat{i})+r^2+40=0$ … (1)

$P_1 > \max\left\{\vec{r}+2\hat{i}-3\hat{j}\right\}, P_2=\max\left\{\vec{r}+2\hat{i}-3\hat{j}\right\}$

Equation (1) is a sphere of centre $\dfrac{10\hat{j}-8\hat{i}}{2}$ and radius 1 unit $P_1-P_2=2\times1=2.$

78.(4) $\hat{i}(x+3y-4z)+\hat{j}(x-3y+5z)+\hat{k}(3x+y)=\lambda(x\hat{i}+y\hat{j}+z\hat{k})$

$(1-\lambda)x+3y-4z=0$; $x-(3+\lambda)y+5z=0$; $3x+y-\lambda z=0$

$$\begin{vmatrix}1-\lambda & 3 & -4\\ 1 & -(\lambda+3) & 5\\ 3 & 1 & -\lambda\end{vmatrix}=0 \Rightarrow \lambda=-1$$

$2x+3y=4z$ … (1)

$x-2y=-5z$ … (2)

$3x+y=-z$

Solving equation (1) and (2), we get

$y=\dfrac{4z-2(-5z)}{3-2(-2)}=\dfrac{14z}{7}=2z$; $x=-z,\ y=2z$; $\dfrac{x-y-z}{x}=\dfrac{-z-2z-z}{-z}=4.$

79.(6) Let the equation of the plane containing the lines $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$ and $\dfrac{x-2}{3}=\dfrac{y-3}{4}=\dfrac{z-4}{5}$ be

$a(x-1)+b(y-2)+c(z-3)=0$ … (1)

where $2a+3b+4c=0$ and $3a+4b+5c=0$

Solving last two equations, we get $\dfrac{a}{1}=\dfrac{b}{-2}=\dfrac{c}{1}$

$\therefore$ From equation (1), equation of the plane is $(x-1)-2(y-2)+(z-3)=0 \Rightarrow$ $x-2y+z=0$

Now the distance between the planes $Ax-2y+z=d$ and $x-2y+z=0$

(Clearly two planes must be parallel so $A=1$) ; $\dfrac{|d|}{\sqrt{1^2+2^2+1^2}}=\sqrt{6} \Rightarrow |d|=6$

80.(2) $4+\lambda_1=1+2\lambda_2 \Rightarrow \lambda_1-2\lambda_2=-3$ … (1)

$-3-4\lambda_1=-1-3\lambda_2 \Rightarrow 4\lambda_1-3\lambda_2=-2$ … (2)

From (1) and (2), $-5\times2=-10$

$\lambda_2=2, \lambda_1=1 \Rightarrow$ point of intersection $(4+1, -3-4, -1+7)$ i.e. $(5, -7, 6)$.

Distance of (5, –7, 6) from (1, –4, 7)

$=\sqrt{16+9+1}=\sqrt{26}$.

PROBABILITY

1.(C) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0 \Rightarrow$ whether $ad = 1$, $bc = 1$ or $ad = -1$, $bc = -1$

Which occur in eight ways. Total number of 2×2 determinants from $\{-1, 1\}$ is 16. Thus required probability is $\frac{8}{16} = \frac{1}{2}$.

2.(D) There can be four such numbers i.e. 43, 34, 62, 26.
Whose product of digit is 12.

$\Rightarrow$ Probability that the man will laugh by seeing the chosen numbers $= \frac{4}{90} = \frac{2}{45}$

$\Rightarrow$ Required probability $= 1 - \left(1 - \frac{2}{45}\right)^3 = 1 - \left(\frac{43}{45}\right)^3$

3.(D) Let A be the event of P_1 winning in 3rd round and B be the event of P_2 winning in first but loosing in second.

To find $P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$

$P(A) = \frac{{}^{8n-1}C_{n-1}}{{}^{8n}C_n} = \frac{1}{8}$ and $P(B \cap A)$

= Probability of both P_1 and P_2 winning in first round × probability of P_1 winning and P_2 loosing in second round × probability of P_1 winning in 3rd round.

$= \frac{{}^{8n-2}C_{4n-2}}{{}^{8n}C_{4n}} \times \frac{{}^{4n-2}C_{2n-1}}{{}^{4n}C_{2n}} \times \frac{{}^{2n-1}C_{n-1}}{{}^{2n}C_n} = \frac{n}{4(8n-1)}$. Hence, $P\left(\frac{B}{A}\right) = \frac{2n}{8n-1}$.

Alternate:

Probability that P_2 wins in first round given, P_1 wins $= \frac{{}^{8n-2}C_{4n-2}}{{}^{8n-1}C_{4n-1}} = \frac{4n-1}{8n-1}$.

In second round probability that P_2 looses in second round given P_1 wins $= 1 - \frac{2n-1}{4n-1} = \frac{2n}{4n-1}$.

Hence, probability that P_2 looses in second round. Given P_1 wins in 3rd round $\frac{4n-1}{8n-1} \times \frac{2n}{4n-1} = \frac{2n}{8n-1}$.

4.(D) The number of favorable cases is ${}^nC_0 \times 2^n + {}^nC_1 2^{n-1} + \ldots + {}^nC_n 2^0 = 3^n$. Total number of cases is $2^n \times 2^n = 4^n$.

Thus required probability is $\left(\frac{3}{4}\right)^n$.

5.(D) Probability that 3 different dices shows different faces is $\frac{{}^6P_3}{6^3} = \frac{5}{9}$, so required probability

${}^3C_2 \left(\frac{5}{9}\right)^2 \times \left(\frac{4}{9}\right)^1 = \frac{100}{243}$.

6.(A) 4 players will reach in semi final. Since all the players are of equal strength so this is equivalent to select 4 out of 2^n players

Total number of ways = ${}^{2^n}C_4$

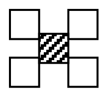
Favourable number of ways = ${}^{2^n-2}C_3 \times {}^2C_1$

Therefore required probability $= \dfrac{{}^{(2^n-2)}C_3 \times 2}{{}^{2^n}C_4} = \dfrac{(2^n-2)\times(2^n-3)\times(2^n-4)\times 8}{2^n(2^n-1)(2^n-2)(2^n-3)} = \dfrac{8\times(2^n-4)}{2^n(2^n-1)}$

7.(B) They will say same thing in two ways i.e. either both speak truth or both tell a lie.
So probability = $(0.6 \times 0.7) + (1 - 0.6)(1 - 0.7) = 0.42 + 0.12 = 0.54$.

8.(C) If the shaded square is chosen we can chose 2 squares out of 4 other at the corner in ${}^4C_2$ ways such shaded squares are 6^2.

Hence probability $= \dfrac{{}^4C_2 6^2}{{}^{64}C_3}$.



9.(C) Given that 5 and 6 have appeared on two of the dice the sample space reduces to $6^4 - 2\times 5^4 + 4^4$ (inclusion exclusion principle) also favourable cases are 4! = 24.

So, the required probability is $\dfrac{24}{302} = \dfrac{12}{151}$.

10.(C) Let a_n be the number of strings of H and T of length n with no two adjacent H's, then $a_1 = 2, a_2 = 3$.

Also, $a_{n+2} = a_{n+1} + a_n$ ($\because$ the string must begin with T or HT)

So, $a_3 = 5, a_4 = 8, a_5 = 8 + 5 = 13 \Rightarrow$ the required probability = $\dfrac{13}{2^5} = \dfrac{13}{32}$.

11.(B) A number has exactly 3 factors if the number is square of a prime number.
Squares of 11, 13, 17, 19, 23, 29, 31 are 3-digit numbers.

Required probability = $\dfrac{7}{900}$

12.(A) Favorable cases $m = {}^6C_2(2\,{}^4C_4 + 2\,{}^4C_3\,{}^1C_1 + {}^4C_2\,{}^2C_2)$

Total cases $\rightarrow$ $n = {}^{12}C_8$

Required probability = $\dfrac{m}{n} = \dfrac{16}{33}$

13.(B) The problem is of conditional probability. Total cases in which at least one of the cubes is red painted is 125 – 27 = 98 out of which 8 are painted on three sides.

$\Rightarrow$ Probability = $\dfrac{8}{98} = \dfrac{4}{49}$

14.(C) Either first ball is black or first three balls are black or first five and so on…. all being equally likely.

Total probability of both balls being odd is $\frac{1}{k}\left[\frac{1}{1!(2n-1)!}+\frac{1}{3!(2n-3)!}+........\right]$ where k is number of ways both can be odd. Probability of exactly one black ball is

$$\frac{\frac{1}{k}\left(\frac{1}{1!(2n-1)!}\right)}{\frac{1}{k}\left(\frac{1}{1!(2n-1)!}+\frac{1}{3!(2n-3)!}+........\right)} = \frac{2n}{{}^{2n}C_1+{}^{2n}C_3+........{}^{2n}C_{2n-1}} = \frac{2n}{2^{2n-1}} = \frac{n}{2^{2n-2}}$$

15.(D) Total ways in which they can seated in the buses are 3^{10}. Applying the principle of Inclusion-exclusion, we get number of favourable ways as $3^{10}-3.2^{10}+3$. Hence, requires probability = $\frac{3^{10}-3.2^{10}+3}{3^{10}}$.

16.(C) Let us assume that '*A*' wins after *n* deuces, $n \in [(0, \infty)$ Probability of a deuce $=\frac{2}{3}.\frac{2}{3}+\frac{1}{3}.\frac{1}{3}=\frac{5}{9}$ (*A* wins his serve then *B* wins his serve or *A* loses his serve then *B* also loses his serve)

Now probability of '*A*' winning the game = $\sum_{n=0}^{\infty}(5/9)^n.\left(\frac{2}{3}\right)\frac{1}{3}=\frac{1}{1-(5/9)}.\frac{2}{9}=\frac{1}{2}$.

17.(B) Total number of ways to distribute the balls so that no box is empty are [(1, 1, 4), (2, 1, 3), (2, 2, 2)]

$$=\frac{3!}{2!}\left[\left({}^6C_1.\ {}^5C_1.\ {}^4C_4\right)\right]+3!\left({}^6C_2.\ {}^4C_1.\ {}^3C_3\right)+\left({}^6C_2.\ {}^4C_2.\ {}^2C_2\right) = 90 + 6.60 + 90 = 540$$

Required probability $=\frac{90}{540}=\frac{1}{6}$.

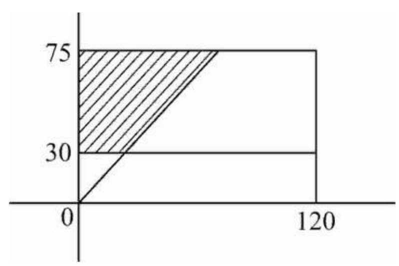
18.(D) First we select r rows and r columns which these squares will be consisting of by ${}^mC_r\ {}^nC_r$ ways.

Now for any one row any one out of r columns can be selected to get the first square, then for second row one column out of $(r-1)$ columns can be selected and so on in $r!$ ways.

Total number of ways are ${}^mC_r\ {}^nC_r\ r!$.   Hence probability is $\frac{{}^mC_r\ {}^nC_r\ r!}{{}^{mn}C_r}$.

19.(D) Mr. Walia calls at 2 hrs x minutes
Mr. Sharma calls at 2 hr y minutes
then $x\in(0,120),\ y\in(30,75)$
Favourable case $\equiv x<y$

Probability = $\frac{\frac{1}{2}(30+75)\times 45}{45\times 120}=\frac{7}{16}$



20.(C) $P(I\ \ II\cup I\ \ III)=P(I\ \ II)+P(I\ \ III)-P(I\ \ II\ III)$

$$= pq+\frac{p}{2}-\frac{pq}{2}=\frac{p}{2}(q+1)=\frac{1}{2}\quad\Rightarrow\quad p(q+1)=1$$

21.(C) Total number of ways $={}^6C_3=20$

Out of which only two equilateral triangle is possible, So required probability $=\frac{2}{20}=\frac{1}{10}$.

22.(A) $P(E_r) =$ Probability of getting r number of balls

$P(E) =$ The particular ball is first balls to pop up.

$$P(E)=\sum_{r=1}^{n} P(E_r)P\left(\frac{E}{E_r}\right)$$

$$P(E_r)=Kr \qquad \sum_{r=1}^{n} P(E_r)=1 \quad \Rightarrow \sum_{r=1}^{n} kr=1 \qquad \Rightarrow \qquad k=\frac{2}{n(n+1)}$$

$$P\left(\frac{E}{E_r}\right)=\frac{{}^{n-1}C_{r-1}(r-1)!}{{}^{n}C_r\ r!}=\frac{1}{n}$$

$$\Rightarrow \qquad P(E)=\sum_{r=1}^{n}\frac{1}{n}\times\frac{2r}{n(n+1)}=\frac{1}{n} \qquad \Rightarrow \qquad P\left(\frac{E_n}{E}\right)=\frac{P(E_n)\cdot P\left(\frac{E}{E_n}\right)}{P(E)}=\frac{\frac{2n}{n(n+1)}\times\frac{1}{n}}{\frac{1}{n}}=\frac{2}{n+1}$$

23.(D) Let d be the common difference

If d = 2, we can take 3 numbers as (1, 3, 5), (2, 4, 6), … ($4m - 3$, $4m - 1$, $4m + 1$).

In this case we have $4m - 3$ triplets.

If d = 4, we can take 3 numbers as

(1, 5, 9), (2, 6, 10), … ($4m - 7$, $4m - 3$, $4m + 1$) and in this case we have $4m - 7$ triplets.

If $d = 2m$, we have only one triplet (1, $2m + 1$, $4m + 1$)

Total number of ways of selecting such triplet is $4m - 3 + 4m - 7 + \ldots + 1 = \frac{m}{2}(2 + (m-1)4) = m(2m-1)$

Total number of ways of selecting 3 numbers out of 4m + 1 numbers is ${}^{4m+1}C_3$.

Required probability $= \frac{m(2m-1)}{{}^{4m+1}C_3}=\frac{6m(2m-1)}{(4m+1)4m(4m-1)}=\frac{3(2m-1)}{2(16m^2-1)}$.

24.(A) $P(A\cap B')=P(A)-P(A\cap B)=0.20$

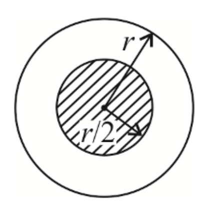
Also $\quad P(A'\cap B)=P(B)-P(A\cap B)=0.15, \qquad \Rightarrow \qquad P(A)+P(B)-2P(A\cap B)=0.35$

Now, $\quad P(A'\cap B')=1-P(A\cup B)$; $\qquad 0.1 = 1-P(A)-P(B)+P(A\cap B)$

$\Rightarrow \qquad P(A)+P(B)-P(A\cap B)=0.9 \quad \Rightarrow \qquad P(A\cap B)=0.9-0.35=0.55$

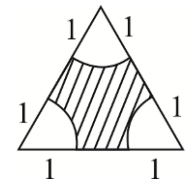
and $\quad P(A)=0.75P(B)=0.70 \qquad$ Now $\quad P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{0.55}{0.70}$

25.(C) For the favourable case, the points should lie inside the concentric circle of radius $\frac{r}{2}$.



So desired probability $=\frac{\text{area of smaller circle}}{\text{area of larger circle}}=\frac{\pi\left(\frac{r}{2}\right)^2}{\pi r^2}=\frac{1}{4}$

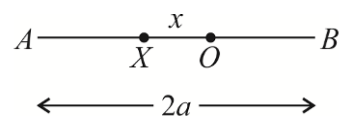
26.(B) The area of the triangle $=\dfrac{\sqrt{3}}{4}(3)^2=\dfrac{9}{4}\sqrt{3}$



The points should lie in the shaded region.

Area of any of the circular segment $=\left(\dfrac{\pi}{6}\right)(1)^2 \quad \therefore$ Desired probability $=1-\dfrac{3\times\dfrac{\pi}{6}}{\dfrac{9}{4}\sqrt{3}}=1-\dfrac{2\pi}{9\sqrt{3}}$

27.(A) AX, XB and AO can form a triangle if X lies in OA then



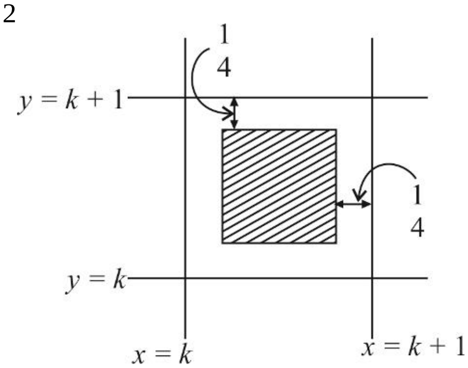
$AX+AO>XB \Rightarrow a-x+a>a+x$

$\therefore \qquad x<\dfrac{a}{2}$. Similarly if x lies in OB section, then again $x<\dfrac{a}{2}$

So, for favourable condition X should lie symmetrically about O, such that $OX<\dfrac{a}{2}$.

Hence the desired probability $=\dfrac{\text{Distance in which } X \text{ can be lie}}{\text{Entire length } AB}=\dfrac{a}{2a}=\dfrac{1}{2}$

28.(C) Adjacent line $x=k, x=k+1$



$y=k, y=k+1$ from a unit square

For the favourable event, the coin should fall into the shaded square of side length $\dfrac{1}{2}$

$\therefore \qquad$ Desired probability $=\dfrac{\text{area of shaded square}}{\text{area of larger square}}=\dfrac{1}{4}$

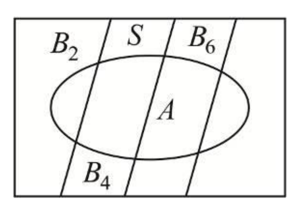
29.(D) If A got at least one head during the first 10 rounds, he will not be drained out at the 11th round.

30.(C) To finish at the 12th round he must have exactly 1 head in the first 10 rounds, and a tail at the 11th and the 12th round. The probability of this is ${}^{10}C_1\, pq^{11}$.

31.(A) Again A cannot be drained at the 13th round if A gets at least 2 heads in the first 10 rounds. To finish at the 14th round, A get exactly 2 heads in the first 10 rounds and a tail on all the round from 11th to 14th. This has a probability ${}^{10}C_2\, p^2q^{12}$ or if A get 1 head in first 10 & 1 head in 11th or 12th round i.e. ${}^{10}C_1 q^9 p\left(2pq^3\right)$

Therefore required probability $=q^{10}(1+10pq+65p^2q^2)$

32.(C) $P(A/B_E)=\dfrac{P(A\cap B_E)}{P(B_E)}$, where A = sum is 4.



$\therefore \qquad P(A/B_E)=\dfrac{P(A\cap B_2)+P(A\cap B_4)}{P(B_2)+P(B_4)+P(B_6)+....\infty}$

$$=\frac{P(B_2)P(A/B_2)+P(B_4)(A/B_4)}{\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+...\infty}=\frac{\frac{1}{4}\left(\frac{3}{36}\right)+\frac{1}{16}\left(\frac{1}{6^4}\right)}{\left(\frac{1}{4}\right)/\left(1-\frac{1}{4}\right)}=\left(\frac{3}{144}+\frac{1}{16\times1296}\right)\left(\frac{3}{1}\right)\cong\frac{1}{16}.$$

33.(A) $n(S) = 6.6.6 = 216$

Now greatest number is 4, no atleat one of the dice shown up 4;

$\therefore \quad n(A) = 4^3 - 3^3 = 37$. Hence $P(A) = \dfrac{37}{216}$.

34.(B) $P(B_2/S) = \dfrac{P(B_2 \cap A)}{P(A)} \Rightarrow P(B_2/S) = \dfrac{\frac{1}{4}\left(\frac{2}{36}\right)}{\frac{1}{2}\left(\frac{1}{6}\right)+\frac{1}{4}\left(\frac{2}{36}\right)+\frac{1}{8}\left(\frac{1}{216}\right)} = \dfrac{24}{169}$.

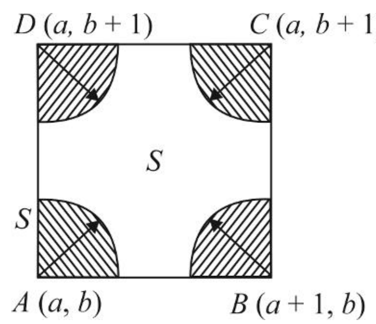
35.(ABCD) We have $P(X \geq a) = \sum_{r=a}^{\infty} pq^r = \dfrac{pq^a}{1-q} = q^a$

Next $P(X \geq a+b \mid X \geq a) = \dfrac{P[(X \geq a+b) \cap (X \geq a)]}{P(X \geq a)} = \dfrac{P(X \geq a+b)}{P(X \geq a)} = \dfrac{q^{a+b}}{q^a} = q^b = P(X \geq b)$

Similarly $P(X \geq a+b \mid x \geq b) = P(X \geq a)$

Again $P(X = a+b \mid X \geq a) = \dfrac{P[(X = a+b) \cap (X \geq a)]}{P(X \geq a)} = \dfrac{P(X = a+b)}{P(X \geq a)} = \dfrac{pq^{a+b}}{q^a} = pq^b = P(X = b)$

36.(ABC) Let S denote the set of points inside a square with corners $(a,b),(a,b+1),(a+1,b),(a+1,b+1), a$ and b are integers. Clearly each of the four points belong to the set X.



Let P denote the set of points in S with distance less than $\dfrac{1}{4}$ from any corner point. P consists of four quarter circles each of radius $\dfrac{1}{4}$.

A coin, whose centre falls in S, will cover a point of X if and only if its centre falls in P.

Hence, the required probability, $p = \dfrac{\text{area of } P}{\text{area of } S} = \dfrac{\pi(1/4)^2}{1 \times 1} = \dfrac{\pi}{16}$

37.(BD) $p_1 = \dfrac{2}{\pi}, \; p_2 = 1 - \dfrac{2}{\pi}$

38.(BC) $0 \leq \dfrac{1+4p}{4} \leq 1 \Rightarrow -\dfrac{1}{4} \leq p \leq \dfrac{3}{4}$ … (1) and $0 \leq \dfrac{1-p}{3} \leq 1 \Rightarrow -2 \leq p \leq 1$ … (2)

$0 \leq \dfrac{1-2p}{2} \leq 1 \Rightarrow -\dfrac{1}{2} \leq p \leq \dfrac{1}{2}$ … (3)

Also, $0 \leq P(A \cup B \cup C) \leq 1$

$\Rightarrow 0 \leq \dfrac{1+4p}{4} + \dfrac{1-p}{3} + \dfrac{1-2p}{2} \leq 1 \Rightarrow 0 \leq \dfrac{13-4p}{12} \leq 1 \Rightarrow \dfrac{1}{4} \leq p \leq \dfrac{13}{4}$ … (4)

From (1), (2), (3) and (4) : $\dfrac{1}{4} \leq p \leq \dfrac{1}{2}$.

39.(AD) If the match is won at the r^{th} game then the result of $(r-1)^{th}$ and r^{th} game will be a win for a particular player. The favourable outcomes of different games would look like this (starting from first and ending at r^{th} game).

	I	II	III	IV		$(r-2)^{th}$	$(r-1)^{th}$	r^{th}
	LW,	LW,	LW,	LW	.......,	LW,	LW,	WL
Or,	WL,	LW,	LW,	LW	…….,	LW,	LW,	WL
Or,	WL,	WL,	LW,	LW	…….,	LW,	LW,	WL
Or,	WL,	WL,	WL,	LW	…….,	LW,	LW,	WL.
	---	----	---	---	---	----	----	
	---	----	---	---	---	----	----	
Or,	WL,	WL,	WL,	WL	……..,	WL,	LW,	WL

Here 'L' stands for loss and 'W' for win 'LW' stands for loss of P_k and winof P_{k+1}, WL stands for win of P_k and loss of P_{k+1}.

That means there are $(r-1)$ favourable cases in which the match ends at r^{th} game. Also probability corresponding to each case is $\frac{1}{2}\frac{1}{2}\frac{1}{2}....'r'\text{ times} = \frac{1}{2^r}$ $\Rightarrow$ Required probability $= \frac{r-1}{2^r}$.

40.(AD) Let λ, μ and λ′, μ′, be the numbers of heads and tails by A and B respectively, so that $\lambda + \lambda' = n + 1$ and $\mu + \mu' = n$

The required probability P is the probability of inequality $\lambda > \mu$

The probability 1 – P of the opposite even $\lambda \le \cup \mu$ is at the same time the probability of the inequality $\lambda' > \mu'$ that is 1 – P is the probability that A will throw more tails than B.

As $\lambda \le \mu \Rightarrow n + 1 - \lambda' < n - \mu' \Rightarrow 1 - \lambda' \le -\mu' \Rightarrow \lambda' - 1 \ge \mu' \Rightarrow \lambda' \ge \mu + 1 > \mu'$

So by reason of symmetry $1 - p = p$ or $p = \frac{1}{2}$.

41.(ABCD) We have $P(A \cup B) \ge \max\{P(A), P(B)\} = \frac{2}{3}$,

Next, $P(A \cap B) = P(A) + P(B) - P(A \cup B) \ge P(A) + P(B) - 1 = 1/6$

And $P(A \cap B) \le \min\{P(A), P(B)\} = \frac{1}{2} \Rightarrow \frac{1}{6} \le P(A \cap B) \le \frac{1}{2}$

Also $P(A \cap B') = P(A) - P(A \cap B) \le \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

Lastly $P(A' \cap B) = P(B) - P(A \cap B)$

$$2/3 - 1/2 \le P(A' \cap B) \le \frac{2}{3} - \frac{1}{6} \Rightarrow 1/6 \le P(A' \cap B) \le ½$$

42.(AC) $\frac{6}{n(n+1)(2n+1)}$

Let E_i be the event that the bag contains exactly i white balls $(0 \le i \le n)$

$\Rightarrow$ $P(E_i) \propto i \Rightarrow P(E_1) = ki \Rightarrow P(E_0) = 0.$

Since $\{E_i\}$ is the set of mutually exclusive events.

$$\sum_{i=1}^{n} P(E_i) = 1 \Rightarrow \sum_{i=1}^{n} ki = 1 \Rightarrow k\frac{n(n+1)}{2} = 1 \Rightarrow k = \frac{2}{n(n+1)}. \Rightarrow P(E_i) = \frac{2i}{n(n+1)}$$

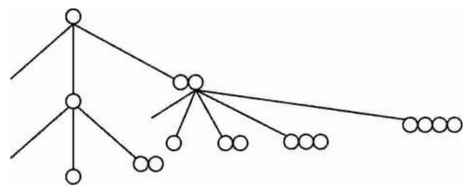
Let B be the event that the ball, drawn out, is white

$$\Rightarrow P\left(E_1/B\right)=\frac{P\left(E_1\right).P\left(B/E_1\right)}{\sum_{j=1}^{n}P\left(E_j\right).P\left(B/E_j\right)}=\frac{\frac{2}{n(n+1)}.\frac{1}{n}}{\sum_{j=1}^{n}\frac{2j}{n(n+1)}.\frac{j}{n}}=\frac{1}{\sum_{j=1}^{n}.j^2}=\frac{6}{n(n+1)(2n+1)}.$$

43.(AC) $p=\frac{\left({}^nC_0\right)^2+\left({}^nC_1\right)^2+\ldots+\left({}^nC_n\right)^2}{4^n}=\frac{{}^{2n}C_n}{2^{2n}}$

44.(AC) $P\left(X_2>0\right)=1-\frac{1}{4}-\frac{1}{2}\times\frac{1}{4}-\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{64-16-8-1}{64}=\frac{39}{64}$

$$P\left(X_1=2\mid X_2=1\right)=\frac{P\left(X_2=1\mid X_1=2\right).P\left(X_1=2\right)}{P\left(X_2=1\right)}$$



$$=\frac{\left(\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}\right).\frac{1}{4}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{4}\left(\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}\right)}=\frac{1}{5}$$

45.(ABCD) $A\subseteq A\cup B \quad \Rightarrow \quad P(A\cup B)\geq P(A)\Rightarrow P(A\cup B)\geq\frac{3}{4}$

$P(A\cap B)=P(A)+P(B)-P(A\cup B)\geq P(A)+P(B)-1\geq\frac{3}{4}+\frac{5}{8}-1\geq\frac{3}{8}$

$(A\cap B)\subseteq B \quad \Rightarrow \quad P(A\cap B)\leq P(B)\leq\frac{5}{8}$

$P(A\cap B')=P(A)-P(A\cap B) \quad \therefore \quad \frac{3}{4}-\frac{5}{8}\leq P(A\cap B')\leq\frac{3}{4}-\frac{3}{8} \quad \Rightarrow \quad \frac{1}{8}\leq P(A\cap B')\leq\frac{3}{8}$

Now $P(A'\cap B)=P(B)-P(A\cap B) \quad \Rightarrow \quad P(A\cap B)=P(B)-(A'\cap B)$

$\Rightarrow \quad \frac{3}{8}\leq P(B)-P(A'\cap B)\leq\frac{5}{8} \quad \Rightarrow \quad 0\leq P(A'\cap B)\leq\frac{1}{4}$

46.(ABCD) $P(E_0)=\frac{2^3}{3^3}$

$P(E_1)=\frac{{}^3C_1\cdot1\cdot2\cdot2}{3^3}$; $\quad P(E_2)=\frac{{}^3C_2\cdot1\cdot1\cdot2}{3^3}$; $P(E_3)=\frac{{}^3C_3\cdot1\cdot1\cdot1}{3^3}$

47.(BC) Let the number of green marbles be x.

Probability of drawing green values in one try = $\frac{1}{y}=\frac{x}{x+32} \quad \Rightarrow \quad x+32=yx \quad \Rightarrow \quad x=\frac{32}{y-1}$

Since x is integer, possible values of y are {33, 17, 9, 5, 3, 2}. Corresponding value of x are {1, 2, 4, 8, 16, 32.}

48.(ABCD) $P\left(E_1\right)=\frac{{}^6C_3}{216}=\frac{5}{54}$; $\quad P\left(E_2\right)=\frac{4+2}{216}=\frac{1}{36}$

[(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6), (1, 3, 5), (2, 4, 6)]

$E_1 \cap E_2 = E_2 \;\therefore \qquad P(E_1 \cap E_2) = P(E_2) = \dfrac{1}{36}$

$P(E_2 / E_1) = \dfrac{P(E_1 \cap E_2)}{P(E_1)} = \dfrac{1/36}{5/54} = \dfrac{54}{180} = \dfrac{3}{10}$

49.(AC) $P(A') = \dfrac{39}{52} = \dfrac{3}{4}$ (Event A' is that missing card is not a spade)

$P(S/A) = \dfrac{{}^{12}C_2}{{}^{51}C_2}$ (only 12 spades are left if missing card is spade)

$$P(A/S) = \frac{P(A \cap S)}{P(S)} = \frac{P(A)P(S/A)}{P(S)} = \frac{\frac{1}{4} \cdot \frac{{}^{12}C_2}{{}^{51}C_2}}{\frac{1}{4} \cdot \frac{{}^{12}C_2}{{}^{51}C_2} + \frac{3}{4} \cdot \frac{{}^{13}C_2}{{}^{51}C_2}} = \frac{11}{50}$$

50.(BD) If a is the radius of the circle, the area of the inscribed square $= 2a^2$

$\therefore \quad p_1 = \dfrac{2a^2}{\pi a^2} = 2/\pi$ and $p_2 = 1 - p = \dfrac{\pi - 2}{\pi} \qquad \therefore \qquad \pi - 2 < 2 \Rightarrow \dfrac{\pi - 2}{\pi} < \dfrac{2}{\pi} \Rightarrow p_2 < p_1$

$p_1^2 - p_2^2 = (p_1 + p_2)(p_1 - p_2) = \dfrac{4 - \pi}{\pi} < \dfrac{1}{3}$

51.(ABCD) Let A, B and C be the events that the student is successful in tests I, II and III, respectively.

Then P(the student is successful)

$= P[(A \cap B \cap C') \cup (A \cap B' \cap C') \cup (A \cap B \cap C)] = P(A \cap B \cap C') + p(A \cap B' \cap C') + P(A \cap B \cap C')$

$= P(A)P(B)P(C') + P(A)P(B')P(C') + P(A)P(B)P(C) \qquad [\because A, B$ and C are independent]

$= pq(1 - 1/2) + p(1-q)(1/2) + (pq)(1/2) = \dfrac{1}{2}[pq + p(1-q) + pq] = \dfrac{1}{2}p(1+q)$

$\therefore \quad \dfrac{1}{2} = \dfrac{1}{2}p(1+q) \quad \Rightarrow \quad p(1+q) = 1$

This equation is satisfied for all pairs of values in (A), (B) and (C). Also, it is satisfied for infinitely many values of p and q. For instance, when $p = n/(n+1)$ the nq = 1/n, where n is any positive integer.

52.(BCD) $P(E_1) = P(E_2) = \dfrac{\frac{10!}{2!}}{\frac{11!}{2!2!}} = \dfrac{2}{11}$

$P(E_1 \cap E_2)$ = probability that two I's are together, and two B's are together = $\dfrac{9!}{\frac{11!}{2!2!}} = \dfrac{2}{55}$

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \dfrac{2}{11} + \dfrac{2}{11} - \dfrac{2}{55} = \dfrac{18}{55}$

$P(E_1 / E_2) = \dfrac{P(E_1 \cap E_2)}{P(E_2)} = \dfrac{\frac{2}{55}}{\frac{2}{11}} = 1/5$

53.(ACD) We have $a^2+b^2+c^2=2$ where a, b, c are integers. $\Rightarrow$ $a=0, b=\pm1, c=\pm1$

Or $a=\pm1, b=0, c=\pm1$ Or $a=\pm1, b=\pm1, c=0$

There are 12 possibilities.

(a) Now $|A|\neq 0 \Rightarrow a+b+c\neq 0$

So, there are 6 possibilities. Probability = 1/2

(b) $|A|=0 \Rightarrow a+b+c=0$

So, there are 6 possibilities

But in no case the system has infinitely many solution

(c) If $a=0, b=1, c=-1, a=0, b=-1, c=1$

So there are two possibilities

(d) Range of $ab+bc+ca$ is $\{-1,\ 1\}$

54.(AC) (a) The probability of S_1 to be among the eight winners is equal to the probability of S_1 winning in the group, which is given by ½.

(b) If S_1 and S_2 are in the same pair-then exactly one wins.

If S_1 and S_2 are in two separate pairs, then for exactly one of S_1 and S_2 to be among the eight winners, S_1 wins and S_2 loses of S_1 lose and S_2 wins.

Now the probability of S_1, S_2 being in the same pair and one wins is (Probability of S_1, S_2 being in the same pair) $\times$ (Probability of any one winning in the pair). And the probability of S_1, S_2 being in the same pair is $\dfrac{n(E)}{n(S)}$

The number of ways 16 players are divided into 8 pairs is $n(S)=\dfrac{16!}{(2!)^8\times 8!}$

The number of ways in which 16 persons can be divided in 9 pairs so that S_1 and S_2 are in same pair is $n(E)=\dfrac{14!}{(2!)^7\times 7!}$

Therefore, the probability of S_1 and S_2 being in the same pair is $\dfrac{\dfrac{14!}{(2!)^7\times 7!}}{\dfrac{16}{(2!)^8\times 8!}}=\dfrac{2!\times 8}{16\times 15}=\dfrac{1}{15}$

The probability of any one winning in the pair of S_1, S_2 is P(certain event) = 1.

Hence, the probability that the pair of S_1, S_2 being in two pairs separately and any one of S_1, S_2 wins is given by the sum the probability of S_1, S_2 being in two pairs separately and S_1 wins, S_2 loses and the probability of S_1, S_2 being in two pairs separately and S_1 loses, S_2 wins. It is given by $\left[1-\dfrac{1}{15}\right]\times\dfrac{1}{2}\times\dfrac{1}{2}+\left[1-\dfrac{1}{15}\right]\times\dfrac{1}{2}\times\dfrac{1}{2}=\dfrac{1}{2}\times\dfrac{14}{15}=\dfrac{7}{15}$

Therefore, the required probability is $(1/15)+(7/15)=(8/15)$.

55.(AB) Given that A and B independent events.

$\therefore \quad p(A\cap B)=P(A)P(B)$ …(i)

Also given that

$P(A\cap B)=\dfrac{1}{6}$ …(ii)

And $P(\bar{A}\cap\bar{B})=\dfrac{1}{3}$

Also, $P(\bar{A}\cap\bar{B})=1-P(A\cup B)$ or $P(\bar{A}\cap\bar{B})=1-P(A)-P(B)+P(A\cap B)$

or $\dfrac{1}{3}=1-P(A)-P(B)+\dfrac{1}{6}$ or $P(A)+P(B)=\dfrac{5}{6}$

from eqs. (i) and (ii), we get $P(A)P(B)=\dfrac{1}{6}$

Let $P(A)=x$ and $P(B)=y$. Then eqs (iv) and (v) become $x+y=\dfrac{5}{6}$; $xy=\dfrac{1}{6}$

Solving we get x = 1/ 2 and y = 1/3 or x = 1/3 and y = ½. Thus, $P(A)$ = ½ or 1/3.

56.(AD) $P(A\cap B)=a,\ P(A)=a+d,\ P(B)=a+2d$ and $P(A\cup B)=a+3d$ also $a+d=d$

$\Rightarrow \quad a=0 \quad\Rightarrow\quad P(A\cap B)=0,\ P(A)=d,\ P(B)=2d$ and $P(A\cup B)=3d$

57.(ACD) $P(A)=\dfrac{{}^5C_1 2^6}{2^{10}}=\dfrac{5}{16},\ P(B)=\dfrac{{}^5C_1 2^6}{2^{10}}=\dfrac{5}{16}$

$P(A\cap B)=\dfrac{{}^5C_2\times 2\times 2^3}{2^{10}}=\dfrac{5}{32} \quad\Rightarrow\quad P(A\cup B)=\dfrac{15}{32}$

58.(ABCD) A, B are exhaustive events $\Rightarrow\ A\cup B=S$

$\Rightarrow \quad P(A\cup B)=1$

$2P(B)=P(A)=P(A\cup B)=K+1$

$\Rightarrow \quad P(B)=\dfrac{K+1}{2}-P(A\cup B)-P(A)-PB\ =1-K-\dfrac{(K+1)}{2}=\dfrac{2-2K-K-1}{2}$

$\therefore \quad P(A\cap B)=\dfrac{3K-1}{2} \qquad P(A'\cup B')=P\left((A\cap B)'\right)=1-P(A\cap B)=1-\dfrac{3K-1}{2}=\dfrac{3(1-K)}{2}$

59.(ABC) Let the no. of blue marbles be a and the no. of green marbles be b.

$\therefore \quad \dfrac{ab}{{}^{a+b}C_2}=\dfrac{1}{2} \quad\Rightarrow\quad (a+b)(a+b-1)=4ab \quad\Rightarrow\quad b^2-(2a+1)b+a^2-a=0$

But $b\in R\Rightarrow D=(2a+1)^2-4\left(a^2-a\right)=8a+1 \quad\therefore\quad 8a+1$ must be a perfect square

Hence possible values of a are 3, 6, 10

60.(AD) From the given condition, it follows that ${}^nC_4,\ {}^nC_5$ and ${}^nC_6$ are in A.P.

i.e, 2. ${}^nC_5={}^nC_4+{}^nC_6$

$$\frac{2n(n-1)(n-2)(n-3)(n-4)}{5!}=\frac{n(n-1)(n-2)(n-3)}{4!}+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!}$$

$$\Rightarrow \quad \frac{2(n-4)}{5}=1+\frac{(n-4)(n-5)}{5.6} \quad\Rightarrow\quad n^2-21n+98=0 \quad\Rightarrow\quad (n-7)(n-14)=0 \quad\Rightarrow\quad n=7,\ n=14$$

61.(BC) (a) Is false since $A_1, A_2, ..., A_n$ may be overlapping.

(b) If $A_1, A_2, ..., A_n$ are disjoint and exhaustive both then $\sum P(A_i) = 1$ if they are only exclusive then $\sum P(A_i) \le 1$

If exhaustive then $\sum P(A_i) \ge 1$ (choice (c) follows)

62.(AD) Clearly $P(A) = P(C)$

Event *A* is no boy or exactly one boy in family $\Rightarrow$ $P(A) = \left(\frac{1}{2}\right)^3 + {}^3C_1\left(\frac{1}{2}\right)^3 = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

Event *B* is 2 boys, 1 girl or 1 boy, 2 girls $P(B) = {}^3C_1\left(\frac{1}{2}\right)^3 + {}^3C_2\left(\frac{1}{2}\right)^3 = \frac{3}{4}$

Event *C* is no girl or exactly one girl $P(C) = \left(\frac{1}{2}\right)^3 + {}^3C_1\left(\frac{1}{2}\right)^3 = \frac{1}{2}$

Event $A \cap B$ is one boy and two girls $P(A \cap B) = {}^3C_1\left(\frac{1}{2}\right)^3 = \frac{3}{8}$

Event $B \cap C$ is one girl and two boys $P(B \cap C) = \frac{3}{8}$

$A \cap C \to \phi \quad \Rightarrow \quad A \cap B \cap C$ is also ϕ

$P(A \cap B) = \frac{3}{8} = P(A) \times P(B)$, so, A and B are independent neither $P(A \cap B \cap C) = P(A) \times P(B) \times P(B)$ nor $P(A \cap C) = P(A) \times P(C)$ So, *ABC* is not independent

63.(AC) We have $q = P(OH \text{ or } 2H \text{ or } 4H)$

$\therefore \quad q = p^4 + {}^4C_2 p^2 (1-p)^2 + (1-p)^4 = 8p^4 - 16p^3 + 12p^2 - 4p + 1 = \frac{(2p-1)^4 + 1}{2}$

Now put the values and check.

64.(ABC) Let event E_1 is first digit is '2' $\Rightarrow$ 211 or 222

$P(E_1) = \frac{2}{4} = \frac{1}{2}$

Let event E_2 is second digit is '2' $\Rightarrow$ 121, 222

$P(E_2) = \frac{1}{2}$

Let event E_3 is third digit is '2' $\Rightarrow$ 222 and 112

$P(E_3) = \frac{1}{2}$

$(E_1 \cap E_2) = 222 \Rightarrow P(E_1 \cap E_2) = \frac{1}{4} == P(E_1)P(E_2)$

Similarly E_2 and E_3, E_1 and E_3

Also, $E_1 \cap E_2 \cap E_3 \to 222 \quad \Rightarrow \quad P(E_1 \cap E_2 \cap E_3) = \frac{1}{4} \ne P(E_1)P(E_2)P(E_3)$

65. **[A-q, s] [B-p, t] [C-r, t]**

(A) $\lambda = 1 -$ Problem will not be solved

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C}) = 1 - P(\bar{A})P(\bar{B})P(\bar{C})$$

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = 1 - \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = 1 - \frac{1}{4} = \frac{3}{4}$$

and $\mu = P(\bar{A} \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap C)$

$$= P(A)P(\bar{B})P(\bar{C}) + P(\bar{A})P(B)P(\bar{C}) + P(\bar{A})P(\bar{B})P(C)$$

$$= \frac{1}{2} \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{2}\right) \cdot \frac{1}{3} \cdot \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdot \frac{1}{4} = \frac{6}{24} + \frac{3}{24} + \frac{2}{24} = \frac{11}{24}$$

$\therefore \lambda + \mu = \frac{3}{4} + \frac{11}{24} = \frac{29}{24}$ and $\lambda - \mu = \frac{7}{24} (Q, S)$

(B) Here, $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}, P(C) = \frac{1}{4}$

$\therefore \lambda = P(A \cap B \cap \bar{C}) + P(\bar{A} \cap B \cap C) + P(A \cap \bar{B} \cap C)$

$$= P(A)P(B)P(\bar{C}) + P(\bar{A})P(B)P(C) + P(A)P(\bar{B})P(C) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{6}{24}$$

and $\mu = \lambda + P(A) \cdot P(B) \cdot P(C) \quad = \lambda + \frac{1}{24} = \frac{7}{24}$

$\therefore$ and $\mu - \lambda = \frac{1}{24}$ and $\lambda + \mu = \frac{13}{24} (P, T)$

(C) $\lambda = \frac{2}{6} \times \frac{5}{8} = \frac{5}{24}$ and $\mu = \frac{4}{6} \times \frac{3}{8} = \frac{6}{24}$

$\therefore \quad \mu - \lambda = \frac{1}{24} (T)$ and $\mu + \lambda = \frac{11}{24} (R)$

66. **[A-t]** **[B-q]** **[C-s]** **[D-r]**

(A) A_1 : missing card is red, A_2 : missing card is black.

B: 1st 13 cards drawn are all red.

$$P\left(\frac{A_2}{B}\right) = \frac{P(A_2) \cdot P\left(\frac{B}{A_2}\right)}{P(A_1) \cdot P\left(\frac{B}{A_1}\right) + P(A_2) \cdot \left(\frac{B}{A_2}\right)}$$ 51 [$25^{red} + 26^{black}$]

$P\left(\frac{B}{A_1}\right) = \frac{{}^{25}C_{13}}{{}^{51}C_{13}}$ 51 [$26^{red} + 25^{black}$]

$P\left(\frac{B}{A_2}\right) = \frac{{}^{26}C_{13}}{{}^{51}C_{13}}$ $\quad = P\left(\frac{A_2}{B}\right) = \dfrac{\frac{1}{2} \times \frac{{}^{26}C_{13}}{{}^{51}C_{13}}}{\frac{1}{2}\left[\frac{{}^{25}C_{13}}{{}^{51}C_{13}} + \frac{{}^{26}C_{13}}{{}^{51}C_{13}}\right]} = \frac{{}^{26}C_{13}}{{}^{25}C_{13} + {}^{26}C_{13}}$.

(B) $\frac{25}{51}$

Let A be the event of drawing a red card when one card is drawn out of 51 cards (excluding missing card). Let A_1 be the event that the missing card is red and A_2 be the event that the missing card is black.

Now by Bayes's theorem, required probability,

$$P(A_1/A) = \frac{P(A_1).\ P(A/A_1)}{P(A_1).\ P(A/A_1) +\ P(A_2).P(A/A_2)} \qquad \ldots (1)$$

In a pack of 52 cards 26 are red and 26 are black.

Now $P(A_1) =$ probability that the missing card is red = $\frac{{}^{26}C_1}{{}^{52}C_1} = \frac{26}{52} = \frac{1}{2}$

$P(A_2) =$ probability that the missing card is black = $\frac{26}{52} = \frac{1}{2}$

$P(A/A_1) =$ probability of drawing a red card when the missing card is red.

= $\frac{25}{51}$ [$\because$ Total number of cards left is 51 out of which 25 are red and 26 are blacks as the missing card is red]

Again $P(A/A_2) =$ Probability of drawing a red card when the missing card is black = $\frac{26}{51}$. Now from (1), required probability, $P(A_1/A) = \frac{\frac{1}{2}.\frac{25}{51}}{\frac{1}{2}.\frac{25}{51} + \frac{1}{2}.\frac{26}{51}} = \frac{25}{51}$.

(C) $E_1 =$ Event that A has drawn a white ball, $E_2 =$ Event that A has drawn a black ball.

$E_3 =$ event that A has drawn a green ball.

$P(E_1) = \frac{2}{12}$, $P(E_2) = \frac{4}{12}$, $P(E_3) = \frac{6}{12}$.

E = event that B has drawn two green balls.

Hence $P\left(\frac{E}{E_1}\right) = \frac{{}^6C_2}{{}^{11}C_2} = P\left(\frac{E}{E_2}\right)$, and $P\left(\frac{E}{E_3}\right) = \frac{{}^5C_2}{{}^{11}C_2}$

$$\Rightarrow \qquad P\left(\frac{E_2}{E}\right) = \frac{P(E_2)P\left(\frac{E}{E_2}\right)}{\sum_{i=1}^{3} P(E_i)P\left(\frac{E}{E_i}\right)} = \frac{\frac{4}{12} \times \frac{{}^6C_2}{{}^{11}C_2}}{\frac{2}{12} \times \frac{{}^6C_2}{{}^{11}C_2} + \frac{4}{12} \times \frac{{}^6C_2}{{}^{11}C_2} + \frac{6}{12} \times \frac{{}^5C_2}{{}^{11}C_2}} = \frac{2}{5}.$$

(D) Suppose, there exist 3 rational points or more on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Therefore if $(x_1, y_1), (x_2, y_2)$ and (x_2, y_3) be those 3 points, then

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \ldots (1)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_1 + c = 0 \qquad \ldots (2)$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \qquad \ldots (3)$$

Solving (1), (2) and (3), we will get g, f, c as rational
Thus, centre of the circle $(-g, -f)$ is a rational point
$\Rightarrow$ both the coordinates of the centre are rational numbers.
Obviously, the possible values of p are 1, 2
Similarly, the possible values of q are 1, 2
$\Rightarrow$ for this case (p, q) may be in (2×2) i.e. 4 ways.
Now, (p, q) can be, without restriction, in (6×6) i.e. 36 ways.
Hence, the probability that at the most two rational points exist on the circle $= \dfrac{36-4}{36} = \dfrac{32}{36} = \dfrac{8}{9}$.

67. **[A-r]** **[B-s]** **[C-p]** **[D-q]**

(A) $\dfrac{{}^{11}C_5}{{}^{12}C_6} = \dfrac{1}{2}$.

(B) Let E_1 be the event that S_3 and S_4 are in same group
Let E_2 be the event that S_3 and S_4 are in different group

$P(E_1) = \dfrac{1}{11}$, $P(E_2) = \dfrac{10}{11}$

Let E be the event that exactly one of S_3 and S_4 is among the losers, then

$P(E) = P(E_1)P(E/E_1) + P(E_2)P(E/E_2)$

$= \dfrac{1}{11} \times 1 + \dfrac{10}{11} \times \left(\dfrac{1}{2} \cdot \dfrac{1}{2} + \dfrac{1}{2} \cdot \dfrac{1}{2}\right) = \dfrac{6}{11}$.

(C) S_2 and S_4 should be in different groups for both being winner

Required probability $= \dfrac{10}{11}\left(\dfrac{1}{2} \cdot \dfrac{1}{2}\right) = \dfrac{5}{22}$ or $\dfrac{{}^{10}C_4}{{}^{12}C_6} = \dfrac{5}{22}$.

68. **[A-r]** **[B-p]** **[C-q]** **[D-s]**

(A) The req. event will occur if last digit in all the chosen numbers is 1, 3, 7 or 9.

Therefor the req. prob. $= \left(\dfrac{4}{10}\right)^n$

(B) P(last digit is 1, 2, 3, 4, 6, 7, 8, 9) – P(last digit is 1, 3, 7, 9) $= \dfrac{8^n - 4^n}{10^n}$

(C) $P(1,3,5,7,9) - P(1,3,7,9) = \dfrac{5^n - 4^n}{10^n}$

(D) $P(0,5) - P(5) = \dfrac{(10^n - 8^n) - (5^n - 4^n)}{10^n}$

69. **[A-q]** **[B-p]** **[C-r]** **[D-q]**

(A) If E_i denotes the event that the bag contains i black balls and $12-i$ white balls $(i=0,1,2,....,12)$ and if A denotes the event that the four balls drawn are black,

$$P(E_i)=\frac{1}{13},\ \text{for } i=0,1,2,.......,12$$

$$P(A/E_i)=0,\ \text{for } i=0,1,2,3 \qquad \dots (1)$$

$$P(A/E_i)=\frac{{}^iC_4}{{}^{12}C_4},\ \text{for } 4\le i\le 12$$

$$\therefore \qquad P(A)=\sum_{i=0}^{12}P(E_i)P(A/E_i)=\frac{1}{13}\sum_{i=4}^{12}P(A/E_i)$$

$$=\frac{1}{13}\times\frac{1}{{}^{12}C_4}\left[{}^4C_4+{}^5C_4+{}^6C_4+.....+{}^{12}C_4\right]=\frac{1}{13\times{}^{12}C_4}{}^{13}C_5=\frac{13!\,4!\,8!}{13(12)!\,5!\,8!}=\frac{1}{5}$$

(B) $$P(A/E_{10})=\frac{{}^{10}C_4}{{}^{12}C_4}=\frac{10!\,4!\,8!}{12!\,4!\,6!}=\frac{7\times 8}{11\times 12}=\frac{14}{33}$$

(C) By Baye's theorem, $$P(E_{10}/A)=\frac{P(E_{10})P(A/E_{10})}{P(A)}=\frac{\frac{1}{13}\times\frac{14}{33}}{\frac{1}{5}}=\frac{70}{429}$$

(D) Let B denote the probability of drawing 2 white and 2 black balls.

$$P(B/E_i)=\frac{{}^iC_2\times{}^{(12-i)}C_2}{{}^{12}C_4},\ \text{for } i=2,3,4,......,10 \quad \text{and} \quad P(B/E_i)=0,\ \text{for } i=0,1,11,12$$

$$\therefore \qquad P(B)=\sum_{i=0}^{12}P(E_i)P(B/E_i)=\frac{1}{13}\times\frac{1}{{}^{12}C_4}\left[{}^2C_2\ {}^{10}C_2+{}^3C_2\ {}^9C_2+{}^4C_2\ {}^8C_2+.......+{}^{10}C_2\ {}^2C_2\right]$$

$$=\frac{1}{13\times 495}\left[2\left({}^2C_2\ {}^{10}C_2+{}^3C_2\ {}^9C_2+{}^4C_2\ {}^8C_2+{}^5C_2\ {}^7C_2\right)+{}^6C_2\ {}^6C_2\right]$$

$$=\frac{1}{13\times 495}\left[2(45+108+168+210)+225\right]=\frac{2(531)+225}{13\times 495}=\frac{1287}{13\times 495}=\frac{13\times 99}{13\times 99\times 5}=\frac{1}{5}$$

70.(7) Last digit of a^2 and b^2

Frequency	a^2	b^2
1	0	0
2	1	1
2	4	4
1	5	5
2	6	6
2	9	9

Favorable case = total number of cases when sum of last digit is 0 or 5.

= (0, 0), (0, 5), (1, 4), (1, 9), (4, 6), (5, 5), (6, 9)

Hence required probability = (1 × 1 × 1 + 1 × 1 × 2 + 2 × 2 × 2 + 2 × 2 × 2 + 2 × 2 × 2 + 1 × 1 × 1 + 2 × 2 × 2) / (10 × 10) = $\frac{9}{25}$.

71.(8) $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{2/x}$ $\quad$ (1^∞ form)

$$\ln k = \lim_{x\to 0}\frac{2}{x}\left(\frac{a^x+b^x}{2}-1\right) \quad \Rightarrow \quad k = ab.$$ So favorable case = (6, 1), (1, 6), (3,), (3, 2)

Hence required probability $=\frac{4}{36}=\frac{1}{9}$.

72.(3) Let E_1, E_2, E_3, E_4 be the events that first two drawn book are (math, math) (math, phy) (phy, math) and (phy, phy) respectively and A be the event that third drawn book is of math.

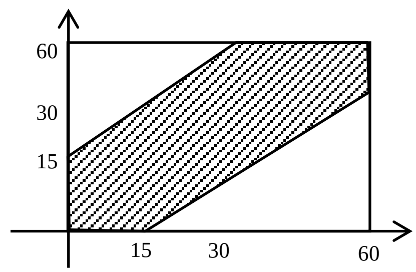
Here $P(E_1)=\frac{3}{5}\times\frac{5}{7}=\frac{3}{7}$, $P(E_2)=\frac{3}{5}\times\frac{2}{7}=\frac{6}{35}$, $P(E_3)=\frac{2}{5}\times\frac{3}{4}=\frac{3}{10}$, $P(E_4)=\frac{2}{5}\times\frac{1}{4}=\frac{1}{10}$

Also $P(A/E_1)=\frac{7}{9}; P(A/E_2)=5/6; P(A/E_3)=\frac{5}{6};\ P(A/E_4)=1$

Now by Baye's theorem $P(E_4/A)=\frac{P(E_4).P(A/E_4)}{P(A)}$

$$=\frac{\frac{1}{10}\times 1}{\frac{3}{7}\times\frac{7}{9}+\frac{6}{35}\times\frac{5}{6}+\frac{3}{10}\times\frac{5}{6}+\frac{1}{10}\times 1}=\frac{1}{\frac{10}{3}+\frac{10}{7}+\frac{5}{2}+1}=\frac{42}{140+60+105+42}=\frac{42}{347}.$$

73.(9) Let line of arrival of A be taken along x–axis and that of B along y–axis meeting between A and B taken place if and only if $|x-y|\le 15$.



Let x and y be the coordinates in the plane and scale be taken in minutes.

Therefore all possible out comes will be a square with side 60 the favourable out come will be in shaded region to the whole area $=\frac{60^2-45^2}{60^2}=\frac{7}{16}$.

74.(8) Let E_1 be the event that balls drawn from bag I are of different colours.

$E_2 \rightarrow$ both are white ; $E_3 \rightarrow$ both are red ; $A\rightarrow$ third drawn ball is red

$$P(E_2)=\frac{{}^5C_2}{{}^9C_2}=\frac{10}{36}=\frac{5}{18},\ P(E_3)=\frac{{}^4C_2}{{}^9C_2}=\frac{6}{36}=\frac{1}{6}$$

$$P(E_1)=1-(P(E_2)+P(E_3))=1-\left[\frac{5}{18}+\frac{1}{6}\right]=1-\frac{8}{18}=\frac{5}{9}$$

Now by Baye's Theorem $P(E_2/A)=\frac{P(E_2)\ P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+\quad P(E_3)P(A/E_3)}$

$$=\frac{\frac{5}{18}\times\frac{4}{7}}{\frac{5}{9}\times\frac{5}{9}+\frac{5}{18}\times\frac{4}{7}+\frac{1}{6}\times\frac{2}{7}}=\frac{45}{146}.$$

75.(9) Let E_1 = the event that one rupee coin is transferred from first purse to the second purse when 9 coins are transferred ; E_2 the event that one rupee coins is not transferred from first purse to the second purse when 9 coins are transferred ; E = the event that one rupee coin is found in the first purse after the second transfer.

Let $A_1 = E_1 \cap E, A_2 = E_2 \cap E$; Let $A = A_1 \cap A_2$

Since E_1 & E_2 are mutually exclusive events, therefore A_1 & A_2 are also mutually exclusive. Therefore required probability, $P(A) = P(A_1) + P(A_2) = P(E_1 \cap E) + P(E_2 \cap E) = P(E_1)P\left(\frac{E}{E_1}\right) + P(E_2)\cdot P\left(\frac{E}{E_2}\right) = P\left(\frac{E}{E_1}\right)$

= probability that one rupee coin will be transferred from second purse to the first purse when one rupee coin and 8 fifty paise coins are transferred from first purse to the second purse $= \frac{{}^1C_1 \cdot {}^{18}C_8}{{}^{19}C_9} = \frac{9}{19} =$ $\quad P\left(\frac{E}{E_1}\right) =$

probability that one rupee coin will be in the first purse when it has not been transferred to the second purse = 1

So ***P* (A)** = $\frac{{}^1C_1 \cdot {}^9C_8}{{}^{10}C_9} \cdot \frac{9}{19} + \frac{{}^9C_9}{{}^{10}C_9} 1 = \frac{9}{10} \cdot \frac{9}{19} + \frac{1}{10} = \frac{81+19}{190} = \frac{10}{19}$.

76.(8) The queens will be facing each other in case both are placed on the same column, the same row or the same diagonal of the chess board. There are eight rows, eight columns, two principal diagonals and four sets each containing diagonal lines of 7, 6, 5, 4, 3, 2, 1 squares. Hence the probability of the queens taking on each other is

$$P(\bar{A}) = \frac{18\,{}^8C_2 + 4\left({}^7C_2 + {}^6C_2 + {}^5C_2 + {}^4C_2 + {}^3C_2 + {}^2C_2\right)}{{}^{64}C_2}$$

$$= \frac{18\times 8\times 7 + 4[7\times 6 + 6\times 5 + 5\times 4 + 4\times 3 + 3\times 2 + 2\times 1]}{64\times 63}$$

$$= \frac{18\times 7 + [7\times 3 + 3\times 5 + 5\times 2 + 2\times 3 + 3 + 1]}{8\times 63} = \frac{18\times 7 + [7\times 3 + 3\times 7 + 14]}{8\times 63} \quad = \frac{18+6+2}{8\times 9} = \frac{26}{72} = \frac{13}{36}.$$

Hence the probability that the two queen to not take on each other is $P(A) = 1 - \frac{13}{36} = \frac{23}{36}$.

77.(3) Let p and q denote probability of things going to man and woman respectively.

Therefore $p = \frac{1}{1+\mu}$ and q = $\frac{\mu}{1+\mu}$.

Probability of men receiving r things is given by $P_r = {}^{\alpha}C_r . q^{\alpha - r} . p^r$.

So required probability is given by $P_1 + P_3 + P_5 + ...$ i.e. ${}^{\alpha}C_1\, q^{\alpha-1} p + {}^{\alpha}C_3\, q^{\alpha-3} p^3 + {}^{\alpha}C_5\, q^{\alpha-5}\, p^5 + ...$

$$= \frac{1}{2}\left[(q+p)^{\alpha} - (q-p)^{\alpha}\right] = \frac{1}{2}\left[1 - \left(\frac{\mu-1}{\mu+1}\right)^{\alpha}\right] = \frac{1}{2} - \frac{1}{2}\left(\frac{\mu-1}{\mu+1}\right)^{\alpha}.$$

By comparison, we have $\left(\frac{\mu-1}{\mu+1}\right) = \frac{1}{2} \Rightarrow 2\mu - 2 = \mu + 1$. Thus $\mu = 3$.

78.(1.44) Total number of outcomes = 6^6 . Number of ways of choosing 4 other different numbers is ${}^6C_2$ and choosing 2 out of remaining 4 can be lone in ${}^4C_2$ ways. Also, number of ways of arranging 6 numbers of which 2 are alike and 2 are alike is $\frac{6!}{212!}$

$\therefore$ Required probability = $\dfrac{{}^6C_2 \times {}^4C_2 \times \frac{6}{2!2!}}{6^6} = \frac{26}{72}$

79.(2.27) Probability of getting a six is 1/6. If Sanchita starts the same then the probability that she wins is $\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)+\ldots\infty=\frac{\frac{1}{6}}{1-\frac{25}{56}}=\frac{6}{11}$ and if Raj starts the game then probability that Sanchita, wins the game is $1-6/11=5/11$

80.(0.40) Probability of getting 5 is 4/36, = 1/9, Probability of getting 7 is 6/36 = 1/6 probability that neither 5 nor 7 will appear is $1-10/36=13/18$

Required probability is $\left(\frac{1}{9}\right)+\left(\frac{13}{18}\right)\left(\frac{1}{9}\right)+\left(\frac{13}{18}\right)\left(\frac{13}{18}\right)\left(\frac{1}{9}\right)+\ldots=(1/9)\left\{\frac{1}{1-\frac{13}{18}}\right\}=\frac{2}{5}=0.40$

81.(0.75) He will fail in exam in two cases:

Case (i) He studied and failed, probability of this case is (1/3)(1/2) = 1/6

Case (ii) He didn't studied and failed, probability of this case is (2/3)(3/4) = 1/2

So total probability is $1/6+1/2=4/6=2/3$

Then required probability = (1/2/(2/3) = 3/4

82.(0.45) Claim: on a rod of length $a+b+c,$ lengths a, b are measured at random. The probability that no point of the measured lines will coincide is $\frac{c^2}{(a+c)\,(b+c)}$.

Let AB be the line, and suppose $AP = x$ and $PQ = a$; also let a be measured from P towards B, so that x must be less than $b + c$. Again let $AP' = y, P'Q' = b,$ and suppose $P'Q'$ measured from P' towards B, then y must be less than $a + c$. Now in favorable case we must have $AP' > AQ$ or else $AP > AQ'$

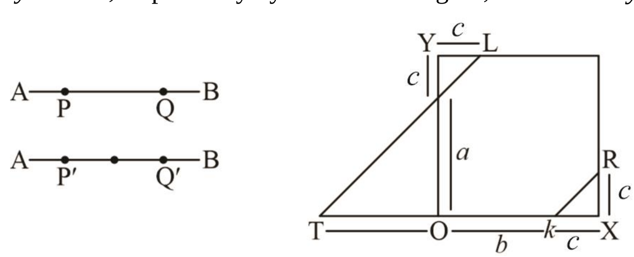
Hence $y > a + x$ or $x > b + y$ … (i)

We must have $x > 0$ and $< b + c$

$y > 0$ and $a + c$ … (ii)

Take a pair of rectangular axes and make OX equal to $b + c$, and OY equal to $a + c$.

Draw the line $y = a + x,$ respectively by TML in the figure; and $x = b + y$ represented by kR.



Then YM, KX are each equal to c, OM, OT are equal to a. The conditions (i) are only satisfied by points in the triangles MYL and KXR, while condition (2) are satisfied by any points within the rectangle OX, OY satisfied by any points within the rectangle OX, OY required probability $=\frac{c^2}{(a+c)\,(b+c)}$.

83.(0.9523) Here sample space is ${}^{10}C_6=(10!)/(6!)(4!)=(7\times8\times9\times10)/(24)=210$

Required answer is 200/210 = 20/21 = 0.9523

84.(1.25) $E: B_1 B_2 B_3 \times\times\times$, where $\times$ means B or G $\quad\therefore\quad P(E)=\frac{1}{2}.\frac{1}{2}.\frac{1}{2}=\frac{1}{8}=\frac{8}{64}$

85.(0.0130) In a throw of two dice probability that one will get 12 is 1/36 and will not get 12 is 35/36. As per the given condition in 1st 19 throw outcomes is not 12 and 20th throw outcome is 12 so required probability is $\left(\frac{35}{36}\right)^{19}\left(\frac{1}{36}\right)$ = 0.0130

86.(0.299) Required probability is $\left({}^{4}C_1\right)\left({}^{48}C_4\right)/\left({}^{52}C_5\right)=0.299$ (approx).

87.(0.1054) $\dfrac{\frac{(48!)}{(12!)^4}}{\frac{(52!)}{(13!)^4 4!}}=0.1054$

88.(0.422) Total number of ways of selecting 2 cards (6×13) and that for 3rd, 4th and 5th card is 48, 44 and 40 ways, so total number of favorable cases is $(78\times 48\times 44\times 40)$. Sample space is ${}^{52}C_5=2,598,560$

Required probability = $\dfrac{78\times 48\times 44\times 40}{2598560}=0.422$ (approx).

89.(0.018181) $=\dfrac{1}{55}$

90.(0.142857) Required probability is 720/(7!) = 1/7
0.142857

91.(1.35) This question is based on principle of inclusion and exclusion
Let X, Y and Z be the events that the student passes in Maths, Physics, Chemistry.
$P(X)=m$, $P(Y)=p$ and $P(Z)=c$ and P(passing in at least one) = $P(X\cup B\cup C)=0.75$ [given]

Now, $1-P(X'\cap Y'\cap Z')=0.75$, $P(X)=1-P(X')$ and $P(X\cup B\cup C)'=P(X\cap Y\cap Z)$

$\Rightarrow\quad 1-P(X')P(Y')P(Z')=0.75$

X, Y and Z are independent event therefore X', Y' and Z' are also independent.

$1-(1-m)(1-p)(1-c)=0.75\quad\Rightarrow\quad(1-m)(1-p)(1-c)=0.25$ …(i)

Also P(passing exactly in one subject) = 0.4 $\Rightarrow\quad P(X\cap Y\cap Z'\cup X\cap Y'\cap Z\cup X'\cap Y\cap Z)=0.4$

$\Rightarrow\quad P(X\cap Y\cap Z')\cup P(X\cap Y'\cap Z)\cup P(X'\cap Y\cap Z)=0.4$

$pm-pmc+pc-pmc+mc-pmc=0.4$ …(ii)

Again P(passing at least in two subjects) = 0.5

$\Rightarrow\quad P(X\cap Y\cap Z')\cup P(X\cap Y'\cap Z)\cup P(X'\cap Y\cap Z)\cup P(X\cap Y\cap Z)=0.5$

$\Rightarrow\quad(pm+pc+mc)-pcm=0.5$ …(iii)

From (ii) we get $(pm+pc+mc)-3pcm=0.4$ …(iv)

From (i) we get

$1-(m+p+c)+(pm+pc+cm)-pcm=0.25$ …(v)

Now from (iii), (iv) and (v) we get,

$p+m+c=1.35=27/20$ and pmc = 1/100.10

92.(0.1142) As $P_1, P_2, ...P_8$ (the eight players) the number of ways they can be paired in four pairs

$= 1/4!\left[\left({}^8C_2\right)\left({}^8C_2\right)\left({}^4C_2\right)\left({}^2C_2\right)\right]$ ways = 105 ways.

Now, at least two players certainly reach the second round in between P_1, P_2 and P_3, P_4 can reach in the final, if exactly two players play against each other in between P_1, P_2 and P_3 and remaining player will play against one of players from P_5, P_6, P_7, P_8 and P_4 plays against one of the remaining three from P_5 to P_8. Now this can be possible in ${}^3C_2 \times {}^4C_1 \times {}^2C_1 = 36$ ways. Required probability = 4/35

93.(1.33) $P\left(\left(\dfrac{A^C}{B^C}\right)^C\right) = 1 - P\left(\dfrac{A^C}{B^C}\right) = 1 - \dfrac{P\left(A^C \cap B^C\right)}{P\left(B^C\right)} = 1 - \dfrac{1-P(A\cup B)}{1-P(B)} = 1 - \dfrac{1-P(A)-P(B)+P(A\cap B)}{1-P(B)}$

$= 1 - \dfrac{0.65}{0.75} = 1 - \dfrac{13}{15} = \dfrac{2}{15}$

94.(0.20) Let C = the event of the selected number being composite,
E = the event of the there being no remainder.

$P(C) = \dfrac{n(C)}{n(S)} = \dfrac{15}{25} = \dfrac{3}{5},\ P(\bar{C}) = \dfrac{2}{5}$

$P(E/C) = \dfrac{4}{15},\ P(E/\bar{C}) = \dfrac{1}{10}$

$P(E) = P(C).\ P(E/C) + P(\bar{C}).P(E/\bar{C}) = \dfrac{3}{5}.\dfrac{4}{15} + \dfrac{2}{5}.\dfrac{1}{10} = \dfrac{4}{25} + \dfrac{1}{25} = \dfrac{1}{5} = 0.2$

95.(0.3488) Of the five bags, 3 belong to the first group and 2 to the second group. Hence $p_1 = \dfrac{3}{5},\ p_2 = \dfrac{2}{5}$

If a bag is selected from the first group, the chance of drawing a black ball is $\dfrac{2}{7}$. If from the second group, the chance is $\dfrac{4}{5}$. Thus $p, = \dfrac{2}{7},\ p_2 = \dfrac{4}{5},\ p_1p_1 = \dfrac{6}{35},\ p_2p_2 = \dfrac{8}{25}$.

Hence the chances that the black ball came from one of the first group is $\dfrac{6}{35} - \left(\dfrac{6}{35} + \dfrac{8}{25}\right) = \dfrac{15}{43}$.

96.(0.9722) There are two hypotheses, (i) their coincident testimony is true, (ii) it is false.

Hence, $p_1 = \dfrac{1}{6},\ p_2 = \dfrac{5}{6},\ p_1 = \dfrac{3}{4}\times\dfrac{7}{10},\ p_2 = \dfrac{1}{25}\times\dfrac{1}{4}\times\dfrac{3}{10}$.

For in estimating p_2 we must take into account the chance that As B will both select the while ball when it has not been drawn; this choice is $\dfrac{1}{5}\times\dfrac{1}{5}$ or $\dfrac{1}{25}$.

Now the probabilities of the two hypotheses are as 35:1; thus probability $= \dfrac{35}{36}$.

97.(4.17) The number of ways of drawing 7 balls (second drawn) = ${}^{10}C_7$. For each set of 7 balls of the second draw 3 must be common to the set of 5 balls of the first draw, i.e., 2 other balls can be drawn in ${}^3C_2$ ways. Thus, of making the first draw so that there are 3 balls common. Hence, the probability of having 3 balls in common = $\dfrac{{}^7C_3 \times {}^3C_2}{{}^{10}C_7} = \dfrac{5}{12}$

MATRICES & DETERMINANTS

1.(C) $A=\begin{pmatrix}1 & -1\\ 1 & -2\end{pmatrix}$ satisfies $A^2+A-I=0$

$A^2=I-A \quad\Rightarrow\quad A^3=A-A^2=2A-I$

$A^4=2A^2-A=2I-2A-A=2I-3A \quad\Rightarrow\quad A^5=2A-3A^2=2A-3I+3A=5A-3I$

$A^6=5A^2-3A=5I-5A-3A=5I-8A \Rightarrow n=6$

2.(D) $a_{rs}=(r-s)2^{i(r-s)} \quad\Rightarrow\quad A=\begin{bmatrix}0 & -2^{-i} & -2.2^{-2i} & -3.2^{-3i}\ldots\ldots\ldots\\ 2^{i} & 0 & -2^{-i} & -2.2^{-2i}\ldots\ldots\\ 2.2^{2i} & 2^{i} & 0 & \ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots & & & \end{bmatrix}$

$\overline{A}=\left[(r-s)2^{-i(r-s)}\right] \Rightarrow A=-(\overline{A})^T$

3.(C) A_1 is cofactor matrix of $A \Rightarrow A_1=(Adj(A))' \Rightarrow |A_1|=|Adj(A)|$

A_2 is cofactor matrix of $A_1 \Rightarrow |A_2|=|Adj(A_1)|=|Adj(Adj(A))|$

$\Rightarrow |A_n|=\left|\underbrace{Adj(Adj(\ldots\ldots\ldots(Adj(A))\ldots\ldots\ldots))}_{n\text{ times}}\right|=K=|A|^{(n-1)^n} \Rightarrow |A|=K^{\frac{1}{(n-1)^n}}$

4.(B) $A^2=A$ & $B=I-A$

$\Rightarrow \quad AB=A(I-A)=A-A^2=A-A=0$ and $BA=(I-A)A=A-A^2=A-A=0$

$B^2=(I-A).(I-A)=I-A-A+A^2=I-A=B$

5.(C) Trace (A) = $\sum_{i=1}^{n}a_{ii}=\sum_{i=1}^{n}(-1)^i({}^nC_i)^2=-({}^nC_1)^2+({}^nC_2)^2-({}^nC_3)^2+({}^nC_4)^2+\ldots+(-1)^n({}^nC_n)^2=-1$ (as n is odd)

6.(A) $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$; $B=\begin{pmatrix}a_{11} & \frac{1}{3}a_{12} & \frac{1}{9}a_{13}\\ 3a_{21} & a_{22} & \frac{1}{3}a_{23}\\ 9a_{31} & 3a_{32} & a_{33}\end{pmatrix} \Rightarrow |B|=|A|$ Similarly $|C|=|B|$

7.(D) $\frac{f(x)}{x}=\begin{vmatrix}x & 1+x^2 & x^2\\ \ln(1+x^2) & e^x & \frac{\sin x}{x}\\ \cos x & \tan x & \frac{\sin^2 x}{x}\end{vmatrix}$; $\lim_{x\to 0}\frac{f(x)}{x}=\begin{vmatrix}0 & 1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 0\end{vmatrix}=1$

8.(D) $ax+4y+z=0$ **;** $2y+3z=1$ **;** $3x-bz=-2$

$|A|=\begin{vmatrix}a & 4 & 1\\ 0 & 2 & 3\\ 3 & 0 & -b\end{vmatrix}=a(-2b)+3(10)=-2ab+30$

$\Rightarrow$ Unique solution if $ab \neq 15$

$$adj.(A).B = \begin{pmatrix} -2b & 4b & 10 \\ 9 & -ab-3 & -3a \\ -6 & 12 & 2a \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4b-20 \\ -ab-3+6a \\ 12-4a \end{pmatrix}$$

Infinitely many solutions if $ab = 15,\ b = 5,\ a = 3$

No solution if $ab = 15,\ a \neq 3$ or $b \neq 5$

9.(A) $\begin{vmatrix} 1 & 2\omega & 3\omega^2 \\ 2 & 3\omega & \omega^2 \\ 3 & \omega & 2\omega^2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 5 - 2(4-3) + 3(2-9) = 5 - 2 - 21 = -18$

10.(C) 3×3 symmetric matrix

$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$, let all diagonal elements be one

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Diagonal two zero's, one 1

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

By permuting diagonal elements, will get 6 more such matrices

Hence Total 12 symmetric matrices, of these nonsingular matrices are

$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ Hence probability = $\frac{1}{2}$

11.(C) $f(x) = \begin{vmatrix} x\cos x & 2x\sin x & x\tan x \\ 1 & x & 1 \\ 1 & 2x & 1 \end{vmatrix}$, $\frac{f(x)}{x^2} = \begin{vmatrix} \cos x & \frac{2\sin x}{x} & \tan x \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$ $\lim\limits_{x \to 0} \frac{f(x)}{x^2} = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -1$

12.(A) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3 = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 8$

As $a+b+c = -2$; $ab+bc+ca = 0$; $abc = -1$; $a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 4 - 2(0) = 4$

13.(A) $A^2 = 2A - I \Rightarrow A^3 = 2A^2 - A = 4A - 2I - A = 3A - 2I$

$A^4 = 3A^2 - 2A = 6A - 3I - 2A = 4A - 3I$; $A^n = nA - (n-1)I$

14.(C) $\begin{vmatrix} \lambda & \lambda+1 & \lambda-1 \\ \lambda+1 & \lambda & \lambda+2 \\ \lambda-1 & \lambda+2 & \lambda \end{vmatrix} = 0 = \begin{vmatrix} \lambda & \lambda+1 & \lambda-1 \\ 1 & -1 & 3 \\ -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \lambda & \lambda+1 & \lambda-1 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{vmatrix} = 4(-\lambda-\lambda-1) = 0 \Rightarrow \lambda = -\frac{1}{2}$

15.(A) $\begin{vmatrix} \cos X & \sin X & 0 \\ \cos Y & \sin Y & 0 \\ \cos Z & \sin Z & 0 \end{vmatrix} . \begin{vmatrix} \cos L & \cos M & \cos N \\ \sin L & \sin M & \sin N \\ 0 & 0 & 0 \end{vmatrix} = 0$

16.(B) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, P = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, Q = p^T AP$

$PQ^{2014}P^T = P(P^T AP)(P^T AP)\text{.........}(P^T AP)P^T = A^{2014}$

Note $PP^T = I$

$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}; \quad A^3 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad A^{2014} = \begin{pmatrix} 1 & 4028 \\ 0 & 1 \end{pmatrix}$

17.(C) $adj(adjA) = |A|^{n-2} A = |A| A$

18.(D) $\left|(xA)^{-1}\right| = \frac{1}{|xA|} = \frac{1}{x^3 K}$

19.(C) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & 5 & 10 \end{pmatrix} \qquad |A| = 0$

$Adj(A).(B) = \begin{pmatrix} 10 & -15 & 5 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ K \\ K^2 \end{pmatrix} \Rightarrow K^2 - 3K + 2 = 0 \Rightarrow K = 1 \text{ or } 2$

20.(C) **21.(B)** **22.(C)**

$A = \begin{bmatrix} 4 & 1 \\ -9 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ -9 & -3 \end{bmatrix}$

$\begin{bmatrix} 3 & 1 \\ -9 & -3 \end{bmatrix}^2 = \begin{bmatrix} 3 & -1 \\ -9 & -3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -9 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A^{100} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 100\begin{bmatrix} 3 & 1 \\ -9 & -3 \end{bmatrix} = \begin{bmatrix} 301 & 100 \\ -900 & -299 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = B$

$\Rightarrow \quad a + d = 2$

$\frac{c}{b} = -\frac{900}{100} = -9$; $\qquad A^{100}$ satisfies $B^2 - 2B + I = 0$ i.e. $A^{200} - 2A^{100} + I = 0$

23.(A) Characteristic equation of A is given by $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 4 \\ -1 & -1 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 - \lambda + 3 = 0 \quad \Rightarrow \lambda^2(\lambda-3) - (\lambda-3) = 0 \quad \Rightarrow \lambda = \pm 1, 3$

24.(C) Matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ have eigen values 1, 1

25.(B) Matrix A satisfies its characteristic equation given by $|A-\lambda I|=0$

i.e. $A^3-5A^2+11A-7I=0$

$A^7-4A^6+6A^5=(A^7-5A^6+11A^5-7A^4)+(A^6-5A^5+11A^4-7A^3)-4A^4+7A^3=-4A^4+7A^3$

$=-4A^4+20A^3-44A^2+28A-13A^3+44A^2-28A$

$=-13A^3+65A^2-143A+91I-21A^2+115A-91I=-21A^2+115A-91I \quad \Rightarrow \quad \alpha=-21, \quad \beta=115, \quad \gamma=-91$

26.(BCD) $A^5=B^5$ & $A^4B=B^4A, \quad A\neq B$

$A^5-A^4B=B^5-B^4A \quad \Rightarrow \quad A^4(A-B)=B^4(B-A)$

$(A^4+B^4)(A-B)=0$ & $A^5+A^4B=B^5+B^4A \Rightarrow A^4(A+B)=B^4(B+A) \Rightarrow (A^4-B^4).(A+B)=0$

Now if $|A^4+B^4|\neq 0$ then $(A^4+B^4)^{-1}$ exist hence

$A-B=0\Rightarrow A=B$ (contradicts $A\neq B$) Hence $|A^4+B^4|=0$

27.(ABC) $X^n-Y^n-X^{n+1}+Y^{n+1}=0$

$X^{n+1}-XY^n-X^{n+2}+XY^{n+1}=0$ **;** $X^nY-Y^{n+1}-X^{n+1}Y+Y^{n+2}=0$

$X^{n+1}-XY^n-X^{n+2}+XY^{n+1}+X^nY-Y^{n+1}-X^{n+1}Y+Y^{n+2}=0$

$\Rightarrow \quad XY^n-XY^{n+1}-X^nY+X^{n+1}Y=0 \quad \Rightarrow \quad (X^n-Y^n)(I-X)(I-Y)=0$

As X^n-Y^n is invertible $\Rightarrow (I-X)(I-Y)=0$. If $|I-X|\neq 0$ then $Y=I$ contradiction $\therefore |I-X|=|I-Y|=0$

28.(AB) $YZ=\begin{bmatrix}4&5\\3&4\end{bmatrix}\begin{bmatrix}4&-5\\-3&4\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$; $(YZ)^2=I^2=\begin{bmatrix}1&0\\0&1\end{bmatrix}.\begin{bmatrix}1&0\\0&1\end{bmatrix}=I \Rightarrow (YZ)^k=I$

$tr(X(YZ)^k)=tr(X)=3 \quad \Rightarrow \sum_{k=0}^{\infty}\frac{tr(X(YZ)^k)}{2^k}=3\sum_{k=0}^{\infty}\frac{1}{2^k}=\frac{3}{1-\frac{1}{2}}=6$ and $\sum_{k=1}^{\infty}\frac{tr(X(YZ)^k)}{2^k}=\frac{3\times 1/2}{1-1/2}=3$

29.(CD) $Adj(A)=\begin{pmatrix}1&4&4\\2&1&7\\1&1&3\end{pmatrix}$; $|Adj(A)|=\begin{vmatrix}1&4&4\\2&1&7\\1&1&3\end{vmatrix}=-4-4(6-7)+4(2-1)=4=|A|^2 \Rightarrow |A|=\pm 2$

30.(AD) In $f(x)$ $\quad C_2\to C_2-C_1-4C_3$

$C_2=0 \quad \Rightarrow f(4)=0$

In $g(x)$, $\quad C_2\to C_2-C_3$

$$g(x)=\begin{vmatrix}2x-2&0&x-1\\3x-4&x-2&x-1\\3x-5&0&2x-4\end{vmatrix}=(x-2)(4x^2-8x-4x+8-3x^2+3x+5x-5)$$

$$=(x-2)(x^2-4x+3)=(x-2)(x-1)(x-3)$$

$f(x)=g(x)\Rightarrow x=1,2,3$

31.(AC) $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} x^2-3 & 2 & x^2-12 \\ x^2-3 & x^2-12 & 3 \\ x^2-3 & 2 & 7 \end{vmatrix} = (x^2-3)\begin{vmatrix} 1 & 2 & x^2-12 \\ 1 & x^2-12 & 3 \\ 1 & 2 & 7 \end{vmatrix}$$

$$R_3 \to R_3 - R_1 = (x^2-3)\begin{vmatrix} 1 & 2 & x^2-12 \\ 1 & x^2-12 & 3 \\ 0 & 0 & 19-x^2 \end{vmatrix} = (x^2-3)(19-x^2)(x^2-14)$$

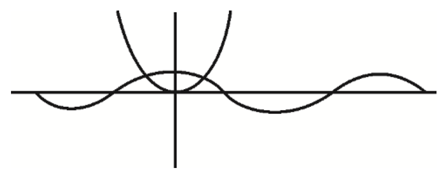
$f(x)=0$ has 6 real roots given by $\pm\sqrt{3}, \pm\sqrt{14}, \pm\sqrt{19}$

32.(ABC) $X \odot (Y+Z) = X \odot Y + X \odot Z$

33.(CD) $Adj(AB) = Adj(B).Adj(A) = |B|B^{-1}.|A|A^{-1} = |A||B|B^{-1}.A^{-1} = |A||B|(AB)^{-1}$

34.(AB) $f(0)=0 \Rightarrow (a)$; $\quad f'(0)=0 \Rightarrow (b)$

35.(BCD) $\begin{vmatrix} x^2+\sin x\cos x & x(1+\sin x) \\ x+\cos x & x+1 \end{vmatrix} = 0 \Rightarrow x^3 + x\sin x\cos x + x^2 + \sin x\cos x - x^2 - x\cos x - x^2\sin x - x\sin x\cos x = 0$



$\Rightarrow x(x^2-\cos x) - \sin x(x^2-\cos x) = 0 \Rightarrow x = \sin x$ or $x^2 = \cos x$

$\Rightarrow x = 0$ or $\alpha, -\alpha$ st. $\alpha^2 = \cos\alpha$

$\cos\frac{\pi}{6} > \left(\frac{\pi}{6}\right)^2$; $\cos\frac{\pi}{4} > \left(\frac{\pi}{4}\right)^2$ & $\cos\left(\frac{\pi}{3}\right) < \left(\frac{\pi}{3}\right)^2$

36.(AB) $M_1 = n^2 - n$, $\quad M_2 = \frac{n^2-n}{2} \Rightarrow \frac{M_1}{M_2} = 2$; $\quad M_1 = 2\sum_{k=1}^{n-1} k$

37.(AB) $a_{ij} = i^n - j^n = -(j^n - i^n) = -a_{ji} \Rightarrow$ A is skew symmetric matrix

38.(ACD) $f(x) = \begin{vmatrix} a^{-x} & a^{x} & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix}$; $\quad f(-x) = -f(x) \Rightarrow f(x)$ is an odd function

$\ln\left(\frac{a-x}{a+x}\right)$ is odd function

39.(ABCD) $A^m B^n = B^n A^m$

$(A-\lambda I).(B+\mu I) = (B+\mu I).(A-\lambda I)$; $\quad (A+\lambda I).(\mu I - B) = (\mu I - B).(A+\lambda I)$

$(A+B)^n = A^n + {}^nC_1 A^{n-1}B + \ldots\ldots\ldots\ldots + B^n$

40.(ABC) $adj(adj\,A) = |A|A = -A$

$|adj(AB)| = |AB|^2 = |A|^2|B|^2 = 576$; $\quad |adj(adj(adj(adj(A))))| = (-1)^{2^4} = 1$; $\quad |adj(B^{-1})| = |B^{-1}|^2 = \frac{1}{|B|^2} = \frac{1}{576}$

41.(ABD) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3 \neq 0$ for only trivial solution

$\Rightarrow a \neq b$ or $b \neq c$ or $c \neq a$ as $a, b, c \in R^+$ $\Rightarrow \dfrac{a^3+b^3+c^3}{3} > abc \Rightarrow a^3+b^3+c^3 > 3abc$

For non trivial solution $a = b = c$

For $a = b = c$ all 3 equations becomes $x + y + z = 0$

Distance of this plane from (1, 2, 3) is $\dfrac{1+2+3}{\sqrt{1+1+1}} = \dfrac{6}{\sqrt{3}}$

Hence minimum value of $(x-1)^2 + (y-2)^2 + (z-3)^2 = \dfrac{36}{3} = 12$

42.(ABCD) For each place we have 3 possibilities hence $O(A) = 3^9$

$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$ for $a+b+c=0,\ a,b,c \in \{0,1,-1\}$

Let $a = b = c = 0$ $\Rightarrow 3^3$ such matrices

Let $a = 1, b = -1,\ c = 0 \Rightarrow 3^3$ such matrices

$\Rightarrow 6 \times 3^3$ such matrices (by permuting a, b, c) $\Rightarrow 7 \times 3^3$ matrices with trace zero

$\begin{pmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{pmatrix}$ skew symmetric matrices = 3^3

Number of matrices in A such that each of 0,1,–1 occurs at least once

$$= 3^9 - {}^3C_1 \times 2^9 + {}^3C_2 \times 1^9 = 19683 - 3 \times 512 + 3 = 18150$$

Skew symmetric matrix of odd order has determinant zero.

43.(ABC) For symmetric $x = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \Rightarrow 5^2$ matrices

For skew symmetric $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ i.e. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ only possibility, already included in symmetric matrices.

$|X| = a^2 - b^2 = (a+b)(a-b),\ a, b \in \{0,1,2,3,4\}$

For it to be divisible by 5 either $a - b = 0$ *i.e.* $a = b$

$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$ i.e. 5 matrices or $a + b = 5$ *i.e.* $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$ or $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ or $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ or $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$

Trace of $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$ is $2a$, not divisible by 5 $\Rightarrow a \neq 0$

Determinant is $a^2 - bc$ for it to be divisible by 5 $b \neq 0,\ c \neq 0 \ \ \Rightarrow a,b,c \in \{1,2,3,4\}$

For any set of values of $a,b \in \{1,2,3,4\}$ we get a unique value of c $\Rightarrow 16$ such matrices.

$$\begin{pmatrix}1&1\\1&1\end{pmatrix},\begin{pmatrix}2&2\\2&2\end{pmatrix},\begin{pmatrix}3&3\\3&3\end{pmatrix},\begin{pmatrix}4&4\\4&4\end{pmatrix},\begin{pmatrix}1&4\\4&1\end{pmatrix},\begin{pmatrix}2&3\\3&2\end{pmatrix},\begin{pmatrix}3&2\\2&3\end{pmatrix},\begin{pmatrix}4&1\\1&4\end{pmatrix}$$
$$\begin{pmatrix}1&2\\3&1\end{pmatrix},\begin{pmatrix}1&3\\2&1\end{pmatrix},\begin{pmatrix}2&4\\1&2\end{pmatrix},\begin{pmatrix}4&2\\3&4\end{pmatrix},\begin{pmatrix}2&1\\4&2\end{pmatrix},\begin{pmatrix}4&3\\2&4\end{pmatrix},\begin{pmatrix}3&1\\4&3\end{pmatrix},\begin{pmatrix}3&4\\1&3\end{pmatrix}$$

Number of matrices in A whose determinant is not divisible by 5 = total – those which have determinant divisible by 5 $= 5^3 - (16+9) = 100$

9 comprises of $a=0,\ b=0,\ c\neq 0$ i.e. 4 ; $a=0, b\neq 0, c=0$ *i.e.* 4 ; $a=0, b=0, c=0$ *i.e.* 1

44.(AB) $x^a y^b = e^m$

$$x^c y^d = e^n \ ; \ \frac{x^{ac} y^{bc}}{x^{ca} y^{da}} = \frac{e^{mc}}{e^{na}} \Rightarrow y^{bc-da} = e^{mc-na} \Rightarrow y^R = e^Q \Rightarrow y = e^{Q/R} \Rightarrow R = Q\log_y e$$

$$\frac{x^{ad} y^{bd}}{x^{cb} y^{db}} = \frac{e^{md}}{e^{nb}} \Rightarrow x^{ad-bc} = e^{md-bn} \Rightarrow x^R = e^P \Rightarrow x = e^{P/R} \Rightarrow R = P\log_x e \ ; \ P\log_x e = Q\log_y e$$

45.(ACD) $AB = I \Rightarrow B = A^{-1} = \frac{1}{2}\begin{pmatrix}7&9&-1\\-19&-23&3\\2&2&0\end{pmatrix}$; $|Adj(B)| = |B|^2 = \frac{1}{|A|^2} = \frac{1}{4}$

46.(AC) $f(x) = \begin{vmatrix}n & n+1 & n+2\\ n! & (n+1)! & (n+2)!\\ 1 & 1 & 1\end{vmatrix} = \begin{vmatrix}n & 1 & 1\\ n! & n.n! & (n+1)(n+1)!\\ 1 & 6 & 0\end{vmatrix} = (n+1)(n+1)! - nn! = n!\left((n+1)^2 - n\right) = n!\left(n^2+n+1\right)$

47.(ABC) $|A(\theta)| = 1$ Hence $A(\theta)$ is invertible

$$A(\pi+\theta) = \begin{bmatrix}\sin(\pi+\theta) & i\cos(\pi+\theta)\\ i\cos(\pi+\theta) & \sin(\pi+\theta)\end{bmatrix} \ ; \ = \begin{bmatrix}-\sin\theta & -i\cos\theta\\ -i\cos\theta & -\sin\theta\end{bmatrix} = -A(\theta)$$

$$\text{adj } A(\theta) = \begin{bmatrix}\sin\theta & -i\cos\theta\\ -i\cos\theta & \sin\theta\end{bmatrix} \ ; \ A(\theta)^{-1} = \begin{bmatrix}\sin\theta & -i\cos\theta\\ -i\cos\theta & \sin\theta\end{bmatrix} = A(\pi-\theta)$$

48.(BD) Applying $C_1 \to c_1 - (\cot\phi)C_2$. we get

$$\Delta = \begin{vmatrix}0 & \sin\theta\sin\phi & \cos\theta\\ 0 & \cos\theta\sin\phi & -\sin\theta\\ -\frac{\sin\theta}{\sin\phi} & \sin\theta\cos\phi & 0\end{vmatrix} = -\frac{\sin\theta}{\sin\phi}\left[-\sin\phi\sin^2\theta - \cos^2\theta\sin\phi\right]$$

Expanding along C_1 = $\sin\theta$; Which is independent of ϕ, also

$$\frac{d\Delta}{d\theta} = \cos\theta \Rightarrow \left[\frac{d\Delta}{d\theta}\right]_{\theta=\frac{\pi}{2}} = \cos\frac{N}{2} = 0$$

49.(AB) Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Since $\Delta^c = \Delta^2$, where Δ^c is determinant of cofactors of Δ

$$\Rightarrow \quad \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$\Rightarrow \quad \begin{vmatrix} a^2+b^2+c^2 & ab+bc+ca & ac+ba+bc \\ ab+bc+ca & b^2+c^2+a^2 & bc+ac+ab \\ ca+ab+bc & bc+ca+ab & c^2+a^2+b^2 \end{vmatrix} = \begin{vmatrix} \alpha^2 & \beta^2 & \beta^2 \\ \beta^2 & \alpha^2 & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2 \end{vmatrix} \text{ where } \alpha^2 = a^2+b^2+c^2,\ \beta^2 = ab+bc+ca$$

50.(BCD) $ax + 2y = \lambda$

$3x - 2y = \mu$

(A) $a = -3$ gives ; $\lambda = -\mu$ or $\lambda + \mu = 0$ not for all λ, μ

(B) $a \neq -3 \Rightarrow \Delta \neq 0$ where $\Delta = \begin{vmatrix} a & 2 \\ 3 & -2 \end{vmatrix} = -2a - 6$ $\quad \therefore$ (B) is correct

(C) correct

(D) If $\lambda + \mu \neq 0 \Rightarrow -3x + 2y = \lambda$ and $3x - 2y = \mu$ Inconsistent (D) correct

51.(AC) $\Delta = \begin{vmatrix} a & a^2 & 0 \\ 1 & 2a+b & a+b \\ 0 & 1 & 2a+2b \end{vmatrix} = a \begin{vmatrix} 1 & a & 0 \\ 1 & 2a+b & a+b \\ 0 & 1 & 2a+3b \end{vmatrix} = a(a+b) \begin{vmatrix} 1 & a & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2a+3b \end{vmatrix} = a(a+b)(2a+3b-1)$

52.(BC) $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$

$$\begin{vmatrix} 1+\alpha^2+2\alpha & 1+4\alpha^2+4\alpha & 1+9\alpha^2+6\alpha \\ 4+\alpha^2+4\alpha & 4+4\alpha^2+8\alpha & 4+9\alpha^2+12\alpha \\ 9+\alpha^2+6\alpha & 9+4\alpha^2+12\alpha & 9+9\alpha^2+18\alpha \end{vmatrix} = -684\alpha$$

$R_2 \to R_2 - R_1$; $R_2 \to R_3 - R_2$

$$\begin{vmatrix} 1+\alpha^2+2\alpha & 1+4\alpha^2+4\alpha & 1+4\alpha^2+6\alpha \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 5+2\alpha & 5+4\alpha & 5+6\alpha \end{vmatrix} = -648\alpha \quad ; \quad \begin{vmatrix} \alpha^2-4 & 4\alpha^2-4 & 9\alpha^2-4 \\ -2 & -2 & -2 \\ 5+2\alpha & 5+4\alpha & 5+6\alpha \end{vmatrix} = -648\alpha$$

$$-2\begin{vmatrix} \alpha^2-4 & 4\alpha^2-4 & 9\alpha^2-4 \\ 1 & 1 & 1 \\ 5+2\alpha & 2\alpha & 4\alpha \end{vmatrix} = -648\alpha \ ; \quad -2\begin{vmatrix} \alpha^2-4 & 3\alpha^2 & 8\alpha^2 \\ 1 & 0 & 0 \\ 5+2\alpha & 2\alpha & 4\alpha \end{vmatrix} = -648\alpha$$

$$\Rightarrow \quad 2\left(12\alpha^3 - 16\alpha^3\right) = -648\alpha \Rightarrow 2\left(-4\alpha^3\right) = -648\alpha \Rightarrow 8\alpha^3 = 648\alpha \Rightarrow \alpha\left(\alpha^2 - 81\right) = 0 \ ; \ \alpha = 0, 9, -9$$

53.(AC) A is an orthogonal matrix

$\therefore \quad AA^T = I$

or $\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}, \frac{1}{3}\begin{bmatrix}1 & 2 & a\\ 2 & 1 & 2\\ 2 & -2 & b\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ or $\frac{1}{9}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}\begin{bmatrix}1 & 2 & a\\ 2 & 1 & 2\\ 2 & -2 & b\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}9 & 0 & a+4+2b\\ 0 & 9 & 2a+2-2b\\ a+4+2b & 2a+b-2b & a^2+4+b^2\end{bmatrix} = \begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix} \Rightarrow a+4+2b=0,\ 2a+2-2b=0 \,\&\, a^2+4+b^2=9$

$\Rightarrow \quad a+2b+4=0,\ a-b+1=0\ \ a^2+b^2=5 \ \Rightarrow\ a=-2,\ b=-1$

54.(A) $\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}a & b & g\\ b & e & h\\ c & f & i\end{bmatrix}$

$a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2+i^2=5$

Case-I. Five $(1's)$ and four $(0's)$

$^9C_5 = 126$

Case-II. One (2) and one $^9C_2.2! = 72$ $\quad\therefore\quad$ total 198

55.(BCD) $n=1$

$p=\left[w^2\right];\ \ p^2=\left[w^4\right]\neq 0$

$n=2$

$p=\begin{bmatrix}w^2 & w^3\\ w^3 & w^4\end{bmatrix} = \begin{bmatrix}w^2 & 1\\ 1 & w\end{bmatrix};\ \ p^2=\begin{bmatrix}w^4+1 & ..\\ .. & ..\end{bmatrix}\neq 0$

$n=3$

$P=\begin{bmatrix}w^2 & 1 & w\\ 1 & w & w^2\\ w & w^2 & 1\end{bmatrix}\begin{bmatrix}w^2 & 1 & w\\ 1 & w & w^2\\ w & w^2 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ Similarly $p^2 \neq 0$ when n is not multiple of 3.

56.(AC) $\begin{vmatrix}1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta\\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta\\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta\end{vmatrix}=0 \quad \Rightarrow \quad \begin{vmatrix}2 & \cos^2\theta & 4\sin 4\theta\\ 2 & 1+\cos^2\theta & 4\sin 4\theta\\ 1 & \cos^2\theta & 1+4\sin 4\theta\end{vmatrix}=0$ Applying $C_1 \to C_1+C_2$

$\Rightarrow \quad \begin{vmatrix}2 & \cos^2\theta & 4\sin 4\theta\\ 0 & 1 & 0\\ -1 & 0 & 0\end{vmatrix}=0$ Applying $R_2 \to R_2-R_1,\ R_3 \to R_3-R_1$

$\Rightarrow \quad 2+4\sin 4\theta=0 \ \Rightarrow\ \sin 4\theta = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right)$

The value of θ lying between 0 and $\frac{\pi}{2}$ are $\frac{7\pi}{24}$ and $\frac{11\pi}{24}$ for n = 1 & 2

57.(ABC) $\begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$

$= \begin{vmatrix} 2 & \cos^2 x & \sin 2x \\ 2 & 1+\cos^2 x & \sin 2x \\ 1 & \cos^2 x & 1+\sin 2x \end{vmatrix}$ apply $C_1 \to C_1 + C_2$ $= \begin{vmatrix} 2 & \cos^2 x & \sin^2 x \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$ $\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$

$=$ $2+\sin 2x$, max $\sin 2x = 1$; $\alpha = 3, \beta = 1$, $\min \sin 2x = -1$

Now, $\alpha - \beta = 2,\ \alpha + \beta = 4,\ \alpha + 3\beta = 6$; Thus, $(\alpha-\beta)+(\alpha+\beta) = \alpha + 3\beta$ so, $\alpha-\beta,\ \alpha+\beta,\ \alpha+3\beta$ cannot for a triangle.

58.(CD) (A) $\left(Y^3Z^4 - Z^4Y^3\right)^T$

$= \left(Y^3Z^4\right)^T - \left(Z^4Y^3\right)^T = \left(Z^4\right)^T.\left(Y^3\right)^T - \left(Y^3\right)^T\left(Z^4\right)^T = -Z^4Y^3 + Y^3Z^4 = Y^3Z^4 - Z^4Y^3$ $\therefore$ symmetric

(B) $\left(X^{44}+Y^{44}\right)^T = \left(X^{44}\right)^T + \left(Y^{44}\right)^T = X^{44}+Y^{44}$ $\therefore$ symmetric

(C) $\left(X^4Z^3 - Z^3 \times 4\right)^T$

$= \left(X^4Z^3\right)^T - \left(Z^3 \times 4\right)^T = \left(Z^3\right)^T\left(X^4\right)^T - \left(X^4\right)^T\left(Z^3\right)^T = -\left(X^4Z^3 - Z^3 \times 4\right)$ $\therefore$ skew symmetric

(D) $\left(X^{23}+Y^{23}\right)^T = -X^{23} - Y^{23}$ $\therefore$ skew symmetric

59.(AC) $A = B^2 \quad \Rightarrow \quad |A| = |B|^2 =$ positive

(A) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1(-1)$ = negative Matrix (a) can not be possible

(B) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1(1-0)$ = positive Matrix (b) can be possible

Ex. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$

(C) $\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1$ = negative Matrix (c) can not be possible

(D) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$ = positive Matrix (D) can be possible

60.(CD) $M=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

(A) $\begin{bmatrix} a \\ b \end{bmatrix}$ & [b, c] are transpose So $\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} b \\ c \end{bmatrix}$ is given $a = b = c$

$M=\begin{bmatrix} a & a \\ a & a \end{bmatrix} \Rightarrow |M|=0$

(B) [b, c] and $\begin{bmatrix} a \\ b \end{bmatrix}$ are transpose So $a = b = c$

(C) $M=\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \Rightarrow |M|=ac \neq 0$

(D) $M=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ given $=ac \neq \lambda^2$

61.(BD) $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}=0$

$\Rightarrow \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ 0 & 0 & -\alpha(a\alpha+b)-b\alpha-c \end{vmatrix}=0$ $[R_3 \to R_3-\alpha R_1-R_2]$ $\Rightarrow$ $-\left(a\alpha^2+2b\alpha+c\right)\left(ac-b^2\right)=0$

So the determinant va if $ac-b^2=0$ or $a\alpha^2+2b\alpha+c=0$, i,e. if a, b, c are in G.P or α is *a* root of $ax^2+2bx+c=0$

62.(ABC) $\begin{vmatrix} bc & ca & ab \\ ca & ab & bc \\ ab & bc & ca \end{vmatrix}=0$

or $(ab)^3+(bc)^3+(ca)^3-3(ab)(bc)(ca)=0$

or $\left(ab+bcw^2+caw\right)\left(abw+bcw^2+ca\right)\left(abw^2+bcw+ca\right)=0$

or $ab+bcw^2+caw=0$; $abw+bcw^2+ca=0$; $abw^2+bcw+ca=0$

$\Rightarrow$ $\frac{1}{cw^2}+\frac{1}{a}+\frac{1}{b\omega}=0,\ \frac{1}{cw}+\frac{1}{a}+\frac{1}{bw^2}=0$; $\frac{1}{c}+\frac{1}{aw}+\frac{1}{bw^2}=0$

63.(AB) The number of third order determinants = the number of arrangements of nine different number is nine places = 9! Corresponding to each determinant made there is a determinant obtained by interchanging, two consecutive rows (or columns) So, the sum of this pair will be 0. $\therefore$ the sum of all determinants = 0

64.(CD) (A) $\left(N^T MN\right)^T=N^T M^T N$ is symmetric if M is symmetric and skew symmetric if M is skew- symmetric

(B) $(MN-NM)^T = (MN)^T-(NM)^T = NM-MN$

$-(MN-NM)$ is skew symmetric

(C) $(MN)^T = N^T MT = NM \neq MN$

(D) standard result is $\text{adj}(MN)=\left[(\text{adj}\,N)\,(\text{adj}\,M)\right]$

$\neq (\text{adj}\,M)\,(\text{adj}\,N)$

65.(CD) For existence of non-trivial solution $\begin{vmatrix} 6 & 5 & \lambda \\ 3 & -1 & 4 \\ 1 & 2 & -3 \end{vmatrix} = 0 \Rightarrow \lambda = -5$

If $\lambda \neq -5,\ x = 0,\ y = 0,\ z = 0$ is the only solutions.

66. **[A-q, r]** **[B-p, s]** **[C-s]** **[D-r]**

A is idempotent $\Rightarrow A^2 = A$

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix};\ \begin{bmatrix} x^2 & xy+y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad x^2 = x \ \&\ xy + y = y \quad \Rightarrow x = 0 \text{ or } 1 \ \&\ xy = 0 \Rightarrow x = 0,\ y \in R \text{ or } x = 1,\ y = 0 \quad (A) \to Q, R$

A is involuntary

$\Rightarrow \quad A^2 = I$

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ \begin{bmatrix} x^2 & xy+y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow x^2 = 1 \quad \& \quad xy + y = 0 \ \Rightarrow x = 1,\ y = 0 \text{ or } x = -1,\ y \in R \qquad (B) \to P, S$

A is orthogonal $\Rightarrow AA' = I$

$\Rightarrow \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x^2+y^2 & y \\ y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow x^2 + y^2 = 1,\ y = 0 \ \Rightarrow x = \pm 1,\ y = 0 \qquad (C) \to S$

A is singular $\Rightarrow |A| = 0 \Rightarrow x = 0,\ y \in R \qquad (D) \to R$

67. **[A-q]** **[B-r, s]** **[C-p]** **[D-p]**

$$\frac{f(x)}{x} = \begin{vmatrix} (\alpha x+1)\cos^2 x & 1 & 1-x \\ \beta \sin x & x & 2x \\ (\gamma x^2+1)\tan x & 1 & 1-x^2 \end{vmatrix} \Rightarrow \frac{f(x)}{x^2} = \begin{vmatrix} (\alpha x+1)\cos^2 x & 1 & 1-x \\ \dfrac{\beta \sin x}{x} & 1 & 2 \\ (\gamma x^2+1)\tan x & 1 & 1-x^2 \end{vmatrix}$$

$\lim\limits_{x\to 0} \dfrac{f(x)}{x} = 0,\ \lim\limits_{x\to 0} \dfrac{f(x)}{x^2} = \begin{vmatrix} 1 & 1 & 1 \\ \beta & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -1 \qquad (A) \to Q$

$f(0) = f'(0) = 0 \Rightarrow \lim\limits_{x\to 0} \dfrac{f'(x)}{x} = \lim\limits_{x\to 0} f''(x) = f''(0)$

$f''(0) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2 \qquad (B) \to R, S$

$$A = \lim_{x\to 0} \frac{1}{x^6} \int_{x^2}^{x^3} f(x)dx = \lim_{x\to 0} \frac{f(x^3).3x^2 - f(x^2).2x}{6x^5}$$

$$= \lim_{x\to 0} \frac{3x f(x^3) - 2f(x^2)}{6x^4} = \lim_{x\to 0} \frac{3f(x^3) + 9x^3 f'(x^3) - 4x f'(x^2)}{24x^3}$$

$$= \lim_{x\to 0}\frac{3f(x^3)-4xf'(x^2)}{24x^3}+\frac{9}{24}f'(x^3) \;= \lim_{x\to 0}\frac{9x^2 f'(x^3)-4f'(x^2)-8x^2 f''(x^2)}{72x^2}+\frac{9}{24}f'(x^3)$$

$$= \lim_{x\to 0}\frac{-4f'(x^2)-8x^2 f''(x^2)}{72x^2}+\frac{1}{2}f'(x^3)$$

$$= \lim_{x\to 0}\frac{-8xf''(x^2)-16x f''(x^2)-16x^3 f'''(x^2)}{144x}+\frac{1}{2}f'(x^3)=\frac{48}{144}=\frac{1}{3}\Rightarrow[A]=0 \qquad (C)\to P$$

$$g(x)=\begin{vmatrix}\cos^2 x & 1 & 1-x\\ 0 & 1 & 2\\ \tan x & 1 & 1-x^2\end{vmatrix}=\cos^2 x(1-x^2-2)+\tan x(2-1+x)=-\cos^2 x(1+x^2)+\tan x(1+x)$$

$$g\left(\frac{\pi}{4}\right)=-\frac{1}{2}\left(1+\left(\frac{\pi}{4}\right)^2\right)+1+\frac{\pi}{4}=\frac{1}{2}+\frac{\pi}{4}-\frac{1}{2}\left(\frac{\pi}{4}\right)^2=0.97\text{ (Approx.)} \qquad \left[g\left(\frac{\pi}{4}\right)\right]=0 \qquad (D)\to P$$

68. **[A-s] [B-q] [C-r] [D-p]**

Symmetric A

$$\begin{pmatrix}a & d & e\\ d & b & f\\ e & f & c\end{pmatrix} \textit{ possible matrices } = 2^6=64; \begin{pmatrix}1 & -1 & 1\\ 1 & 1 & 1\\ -1 & -1 & 1\end{pmatrix} \text{ maximum value of } |A| \text{ is } 4$$

By interchanging any 2 rows or columns minimum value of $|A|$ is -4 ; Maximum value of trace of A is 3

69. **[A-p, q]** **[B-r]** **[C-s]** **[D-p, q]**

$A.Adj(A)=|A|I$

$\Rightarrow (Adj(A))^{-1}=\dfrac{A}{|A|} \qquad (A)\to P,Q$

$(Adj(KA))=K^{n-1}Adj(A) \qquad (B)\to R$

$Adj(Adj(KA))=Adj(K^{n-1}Adj(A))=K^{(n-1)^2}Adj(Adj(A))=K^{(n-1)^2}|A|^{n-2}A \qquad (C)\to S$

$A^{-1}.Adj(A^{-1})=|A^{-1}|I=\dfrac{1}{|A|}I\Rightarrow adj(A^{-1})=\dfrac{A}{|A|} \qquad (D)\to P,Q$

70. **[A-q] [B-p, s] [C-p, r] [D-p, r]**

$|A|=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \alpha\end{vmatrix}=(\alpha-3)\neq 0$ *i.e.* $\alpha\neq 3$ unique solution $\qquad (A)\to Q$

$\alpha=3,\ \beta\neq 10$ no solution $\qquad (B)\to P,S$

$\alpha=3,\ \beta=10$ infinitely many solution $\qquad (C)\to P,R$

Atleast 2 solution $\Rightarrow$ infinitely many solutions $\qquad (D)\to P,R$

71.(0) $(A-A^T)^{2015}$ is skew symmetric matrix of odd order $\Rightarrow |(A-A^T)^{2015}|=0$

72.(0) $z^6-1=0,\ z^{21}-1=0$ common roots are $1,\omega,\omega^2$

$R_1\to R_1+R_2+R_3$

$$\begin{vmatrix}x+y+z & x+y+z & x+y+z\\ 2y & y-x-z & 2y\\ 2z & 2z & z-x-y\end{vmatrix}=0$$

73.(7) $P^T P = I = PP^T$

$PQ^{2014}P^T = P(P^T AP)(P^T AP)..........(P^T AP).P^T$

$$= A^{2014} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{2014} = \begin{bmatrix} 1 & 2014 \\ 0 & 1 \end{bmatrix} \Rightarrow b = 2014 \Rightarrow \text{sum of digits of } b \text{ is } 7$$

74.(2) $S = \begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$

$|S| = (1 - c\omega) - \omega(a - b\omega) + \omega^2(ac - b) = 1 - c\omega - a\omega + b\omega^2 + ac\omega^2 - b\omega^2 = 1 - a\omega - c\omega + ac\omega^2$

$a = \omega, c = \omega \qquad |S| = -3\omega^2 \neq 0, \quad b = \omega$ or ω^2

$a = \omega, c = \omega^2 \qquad |S| = 0$

$a = \omega^2, c = \omega \qquad |S| = 0$

$a = \omega^2, c = \omega^2 \qquad |S| = 0 \qquad \Rightarrow$ 2 non singular matrices

$a = \omega, b = \omega, c = \omega$ & $a = \omega, b = \omega^2, c = \omega$

75.(9) $f(x) = \begin{vmatrix} x^2 + 3x & x - 1 & x + 3 \\ x + 1 & -2x & x - 4 \\ x - 3 & x + 4 & 3x \end{vmatrix} = ax^4 + bx^3 + cx^2 + dx + e$

$a + b + c + d + e = f(1) = \begin{vmatrix} 4 & 0 & 4 \\ 2 & -2 & -3 \\ -2 & 5 & 3 \end{vmatrix} = 4(-6 + 15) + 4(10 - 4) = 36 + 24 = 60 \quad ; \; e = f(0) = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = 0$

a = coefficient of $x^4 = -7$

$-\left[\dfrac{a + b + c + d + e}{a + e}\right] = -\left[\dfrac{60}{-7}\right] = 9$

76.(0) Use ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

77.(6) $\frac{1}{2}|[x]| = \{x\} \quad x \in [-2, 2] \Rightarrow x = 0, -\frac{1}{2}, \frac{3}{2}$ using graph

$|\sin y| = \{y\} \quad y \in (-\pi, \pi) \Rightarrow y$ can take 7 values, using graph

$\cos z = \{z\} \quad z \in (-\pi, \pi) \Rightarrow z$ can take 2 values, using graph $\quad \Rightarrow (x, y, z)$ can take $3 \times 7 \times 2$ i.e. 42 triplets

78.(7) $A^2 - 5A - 2I = 0$; $A^2 = 5A + 2I$; $A^{2014} = 5A^{2013} + 2A^{2012} \Rightarrow \lambda + \mu = 7$

79.(2) $\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$

Symmetric matrices

Diagonal all 'a', or 1 'a', 2 'b'

$\begin{pmatrix} a & a & b \\ a & a & b \\ b & b & a \end{pmatrix}$ or $\begin{pmatrix} a & b & a \\ b & a & b \\ a & b & a \end{pmatrix}$ or $\begin{pmatrix} a & b & b \\ b & a & a \\ b & a & a \end{pmatrix} \equiv 3$

$$\begin{pmatrix} a & b & a \\ b & b & a \\ a & a & b \end{pmatrix} \text{ or } \begin{pmatrix} a & a & b \\ a & b & a \\ b & a & b \end{pmatrix} \text{ or } \begin{pmatrix} a & a & a \\ a & b & b \\ a & b & b \end{pmatrix} \equiv 3\times 3 \text{ (permuting } a, b, b\text{)}$$

$\Rightarrow$ 12 symmetric matrices $\Rightarrow k = 12$, number of zeroes at end of 12! is 2

80.(1) $A^2 = 3A - 2I \;;\; A^3 = 3A^2 - 2A = 3(3A-2I) - 2A = 7A - 6I$

$$A = 3I - 2A^{-1} \Rightarrow A^{-1} = \frac{3I - A}{2} = \frac{3I - \left(\frac{A^3+6I}{7}\right)}{2} = \frac{21I - A^3 - 6I}{14} = \frac{15I - A^3}{14} \;;\; \lambda + k\mu = 15 - 14 = 1$$

81.(2.25) In $\Delta ABC \begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$

$\Rightarrow \quad 1\left(c^2 - ab\right) - a(c-a) + b(b-c) = 0 \;\Rightarrow\; a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow \quad 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0 \;\Rightarrow\; (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

$\Rightarrow \quad a = b = c \;\Rightarrow\; \Delta ABC$ is equilateral $\Rightarrow \angle A = \angle B = \angle C = 60^\circ$

$\therefore \quad \sin^2 A + \sin^2 B + \sin^2 C = \sin^2 60 + \sin^2 60 + \sin^2 60 = 3\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4}$

82.(3) Using $C_1 \to C_1 + C_3$, we get $\begin{vmatrix} 1 & \tan\theta + \sec^2\theta & 3 \\ 0 & \cos\theta & \sin\theta \\ 0 & -4 & 3 \end{vmatrix} = 3\cos\theta + 4\sin\theta$

$$\frac{d\Delta}{d\theta} = -3\sin\theta + 4\cos\theta \;;\; \frac{d\Delta}{d\theta} = 0 \Rightarrow 3\sin\theta - 4\cos\theta = 0$$

$$\tan\theta = \frac{4}{3} \Rightarrow \sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5} \text{ min } \Delta = 3$$

83.(103) $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$

$$P^2 = \begin{vmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 16(1+2) & 8 & 1 \end{vmatrix} ;\; P^3 = \begin{vmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 16(1+2+3) & 12 & 1 \end{vmatrix} ;\; P^{50} = \begin{vmatrix} 1 & 0 & 0 \\ 4\times 50 & 1 & 0 \\ 16(1+2+\ldots+50) & 4\times 50 & 1 \end{vmatrix} ;\; P^{50} = \begin{vmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 20400 & 200 & 1 \end{vmatrix}$$

$\therefore \quad P^{50} - Q = I$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 20400 & 200 & 1 \end{bmatrix} - \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad 200 - q_{21} = 0 \Rightarrow q_{21} = 200$

$20400 - q_{31} = 0 \Rightarrow q_{31} = 20400$ and $200 - q_{32} = 0 \Rightarrow q_{32} = 200 \quad \therefore$

$$\frac{q_{31} + q_{32}}{q_{21}} = \frac{20400 + 200}{200} = 103$$

84.(3) We have $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = \begin{vmatrix} x^3 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} + \begin{vmatrix} 1 & x^2y & x^2z \\ 0 & y^3+1 & y^2z \\ 0 & yz^2 & z^3+1 \end{vmatrix}$

$$= x\begin{vmatrix} x^2 & x^2y & x^2z \\ y^2 & y^3+1 & y^2z \\ z^2 & yz^2 & z^3+1 \end{vmatrix} + \left(y^3+1\right)\left(z^3+1\right) - y^3z^3$$

$$C_2 \to C_2 - yC_1;\;\; C_3 \to C_3 - zC_1 \quad = x\begin{vmatrix} x^2 & 0 & 0 \\ y^2 & 1 & 0 \\ z^2 & 0 & 1 \end{vmatrix} + y^3z^3 + z^3 + y^3 + 1 - y^3z^3 \; = \; x^3 + y^3 + z^3 + 1$$

Given $x^3+y^3+z^3+1=30$; $x^3+y^3+z^3=29$ $\quad\therefore\quad$ solutions are (3, 1, 1), (1, 3, 1) (1, 1, 3)

85.(99) $A = \begin{bmatrix} 1 & \omega^2 \\ \omega & 1 \end{bmatrix} \quad \therefore \quad A^2 = \begin{bmatrix} 1 & \omega^2 \\ \omega & 1 \end{bmatrix}\begin{bmatrix} 1 & \omega^2 \\ \omega & 1 \end{bmatrix} = \begin{bmatrix} 1+\omega^3 & 2\omega^2 \\ 2\omega & \omega^3+1 \end{bmatrix} = \begin{bmatrix} 2 & 2\omega^2 \\ 2\omega & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & \omega^2 \\ \omega & 1 \end{bmatrix} = 2^1.\,A$

Similarly $A^3 = A^2 \times A \; = 2A \times A \; = 2A^2 \; = 4\,A \; = 2^2 A \quad \therefore \quad A^{100} = 2^{99}.A \qquad \therefore$ k = 99

86.(10) Determenant D = 0

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0 \;\Rightarrow -3k+8-k(-9+4)+3(12-2k)=0 \;\Rightarrow\; -3k+8+5k+36-6k=0 \;\Rightarrow\; -4k=-44 \;\Rightarrow\; k = 11$$

$\therefore$ we have, $x+11y+3z=0$; $3x+11y-2z=0$; $2x+4y-3z=0$, Put $z=t$; $x+11y=-3t$; $3x+11y=2t$

Solving, $x = \dfrac{5t}{2},\; y = -\dfrac{t}{2} \qquad \therefore \quad \dfrac{xz}{y^2} = \dfrac{\frac{5t}{2}\times t}{\frac{t^2}{4}} = \dfrac{5}{2}\times 4 = 10$

87.(2) $\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = \begin{vmatrix} 3 & \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 \\ \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 & \alpha^3+\beta^3+\gamma^3 \\ \alpha^2+\beta^2+\gamma^2 & \alpha^2+\beta^3+\gamma^3 & \alpha^4+\beta^4+\gamma^4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} . \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}^2 = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$$

88.(6) $B.C = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 9-8 & -12+12 \\ 6-6 & -8+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BC = I$; $\qquad tr(A) + tr\left(\dfrac{A.BC}{2}\right) + tr\left(\dfrac{A(BC)^2}{4}\right) + ...\infty$

$$= tr(A) + tr\left(\frac{A}{2}\right) + tr\left(\frac{A}{4}\right) + ... \quad = tr\left(A + \frac{A}{2} + \frac{A}{4} + \frac{A}{8} + ...\right) \; = tr\left(\frac{A}{1-\frac{1}{2}}\right) = tr(2A) \; = 2tr(A) = 2\times 3 = 6$$

89.(6) $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2 \Rightarrow \begin{vmatrix} 5x-4 & 2x & 2x \\ 5x-4 & x-4 & 2x \\ 5x-4 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$

$C_1 \to C_1 + C_2 + C_3$

$\Rightarrow \quad (5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x-4 & 2x \\ 1 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2 \Rightarrow \quad (5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -x-4 & 0 \\ 0 & 0 & -x-4 \end{vmatrix} = (A+Bx)(x-A)^2$

$\therefore \quad R_2 \to R_2 - R_1;\ R_3 \to R_3 - R_1$

$(5x-4)(x+4)^2 = (A+Bx)(x-A)^2;\quad A=-4,\ B=5\ ;\ A+2B=10-4=6$

90.(9) The number of third order determinants = the number of arrangements of nine different numbers in places = 9!

$\therefore \quad (a+b)! = 9!\quad ;\quad a+b=9$

91.(4) Let $A = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$

$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 2\sqrt{k} & 1+2k & -2k \\ -2\sqrt{k} & 2k+1 & -1 \end{vmatrix} C_2 \to C_2 - C_3 \ = \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 4\sqrt{k} & 0 & 1-2k \\ -2\sqrt{k} & 2k+1 & -1 \end{vmatrix} R_2 \to R_2 - R_3 \ = (2k+1)3$

$\because$ B is a skew-symmetric matrix of odd order therefore det B = 0

Now, det $(\text{adj} A)$ + det$(\text{adj} B) = 10^6$

$\left\{(2k+1)^3\right\}^2 + 0 = 10^6 \quad \Rightarrow \quad 2k+1=10$ as $k > 0$; k = 4.5 ; [k] = 4

92.(1) D = 0

$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{vmatrix} = 0\ ;\ \begin{vmatrix} 1+\alpha+\alpha^2 & 2\alpha+1 & \alpha^2+\alpha+1 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{vmatrix} = 0$

$\begin{vmatrix} 1+\alpha+\alpha^2 & 2\alpha+1 & 0 \\ \alpha & 1 & 0 \\ \alpha^2 & \alpha & 1-\alpha^2 \end{vmatrix} = 0 \Rightarrow (1-\alpha^2)(1+\alpha+\alpha^2-2\alpha^2-\alpha) = 0 \Rightarrow (1-\alpha^2)^2 = 0\ ;\ \alpha=-1$ or 1

For $\alpha = 1$, system of linear equation has no solutions $\quad\therefore\quad \alpha=-1\ ;\ 1+\alpha+\alpha^2 = 1$

93.(4) $\Delta = \begin{vmatrix} 1 & x+y & x+y+z \\ 2 & 3x+2y & 4x+3y+2z \\ 3 & 6x+3y & 10x+6y+3z \end{vmatrix} \quad = x^2 \begin{vmatrix} 1 & 1 & x+y \\ 2 & 3 & 4x+3y \\ 3 & 6 & 10x+6y \end{vmatrix}$

$C_3 \to C_3 - zC_1;\ C_2 \to C_2 - yC_1$

$= x^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} \ = x^3(6-8+3) = 64\ = x^3 = 64 \Rightarrow x = 4$

94.(2) Using $C_3 \to C_3 - (C_1 + C_2)$ in D_1 and D_2, we have, $\left|\dfrac{D_1}{D_2}\right| = \left|\dfrac{+2b(ad-bc)}{b(ad-bc)}\right| = +2$

95.(9) $M=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $a_{12}=-1$

$a_{11}-a_{12}=1 \Rightarrow a_{11}=0$; $a_{11}+a_{12}+a_{13}=0 \Rightarrow a_{13}=1$

$a_{22}=2,\ a_{21}-a_{22}=1 \Rightarrow a_{21}=3$; $a_{21}+a_{22}+a_{23}=0 \Rightarrow a_{21}=3$; $a_{21}+a_{22}+a_{23}=0 \Rightarrow a_{23}=-5$

$a_{32}=3,\ a_{31}-a_{32}=1 \Rightarrow a_{31}=2$; $a_{31}+a_{32}+a_{33}=12 \Rightarrow a_{33}=7$

Hence sum of diagonal of M is $a_{11}+a_{22}+a_{23}=0+2+7=9$

96.(1) $\omega=\cos\frac{2N}{3}+i\sin\frac{2\pi}{3}$

By $C_1+C_1+C_2+C_3$

$$\begin{vmatrix} z & \omega & \omega^2 \\ z & z+\omega^2 & 1 \\ z & 1 & z+\omega \end{vmatrix}=0 \Rightarrow \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix}=0 \Rightarrow \begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & z+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & z+\omega-\omega^2 \end{vmatrix}=0$$

$R_2 \to R_2-R_1$; $R_3 \to R_3-R_1$

$\Rightarrow \left(z-\omega^2-\omega\right)\left(z+\omega-\omega^2\right)-\left(1-\omega\right)\left(1-\omega^2\right)=0 \Rightarrow z^2=0$ $\therefore$ only one solution

97.(4) $|\text{adj } A|=|A|^{n-1}=(-2)^2=4$ where $|A|=-2$

98.(1) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}=1$$

99.(0) Taking e^{x_1} common from R_1, e^{x_4} common from R_2 and e^{x_1} from R_3, we get,

$\Delta=e^{x_1+x_2+x_3}\,\Delta_1$

Where $\Delta_1=\begin{vmatrix} 1 & e^{3d} & e^{6d} \\ 1 & e^{3d} & e^{6d} \\ 1 & e^{3d} & e^{6d} \end{vmatrix}=0$ Where d is common difference of AP $\therefore$ $\Delta=0$

100.(1) We can write the following Δ_1 as product of two determinants as follows:

$$\Delta_1=\begin{vmatrix} x^2 & -2x & 1 \\ y^2 & -2y & 1 \\ z^2 & -2z & 1 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

We interchange C_1 and C_3 in the first determinant and absorb the resulting negative sign in C_2 to get

$$\Delta_1=\begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=\begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}=\Delta_2$$

STATISTICS

Errata for Questions

6. If $x_1, x_2, ..., x_{18}$ are observations such that $\sum_{j=1}^{18}(x_j - 8) = 9$ *and* $\sum_{j=1}^{18}(x_j - 8)^2 = 45$, then the standard deviation of these observations is:

(A) 1.2 **(B)** 2.5 **(C)** 2 **(D)** 1.5

13. If r is the variance and σ is the standard deviation, then:

(A) $r = 1/\sigma^2$ **(B)** $r = 1/\sigma$ **(C)** $r = \sigma^2$ **(D)** $r^2 = \sigma$

17. A batsman scores runs in 10 innings as 38, 70, 48, 34, 42, 55, 63, 46, 54 and 44. The mean deviation about mean is:

(A) 8.88 **(B)** 6.4 **(C)** 10.6 **(D)** 7.6

55. Show that the mean deviation from the mean of the A.P $a, a+d, a+2d, ..., a+2nd$ is dependent of the common difference of A.P.

59. The mean annual salaries paid to 1000 employees of a company was Rs. 2400. The mean annual salaries paid to male and female employees were Rs. 2000 and Rs. 4000 respectively. Determine the percentage of males and females employed by the company.

Errata for Answer key

42.	n = 20, mean = 49.5	**44.**	23.6	**48.**	71	**52.**	28.50
54.	**(a)** $(n!)^{1/n} \cdot G$	**55.**	$\frac{(n+1)n}{2n+1}d$	**59.**	80% and 20%	**60.**	Proof based problem

Solutions to Statistics

1.(A) $\text{Corr. coeff.} = \frac{CoV.}{\sigma_1.\sigma_2} = \frac{8}{3 \times 4} = \frac{2}{3}$

2.(C) First it is divided by α. Then add 10

3.(A) Total (n + 1) terms

n is even

So, $\left(\frac{n+1+1}{2}\right)^{th}$ term is median ; Median = ${}^{2n}C_{\left[\left(\frac{n}{2}+1\right)-1\right]} = {}^{2n}C_{\frac{n}{2}}$

4.(D) Median is same.

5.(D) $\frac{400}{n} - 5^2 = 0 \quad \Rightarrow \quad n = 16$

6.(D) $\sum_{i=1}^{18} x_i - 8 \times 18 = 9 \quad \Rightarrow \quad \sum_{i=1}^{18} x_i = 153$

$$\sum_{i=1}^{18} x_i^2 - 2 \times 8 \sum_{i=1}^{18} x_i + 64 \times 18 = 45$$

$\Rightarrow \quad \sum_{i=1}^{18} x_i^2 = 45 + 16 \times 153 - 64 \times 18$

$Var. = \frac{45 + 16 \times 153 - 64 \times 18}{18} - \left(\frac{153}{18}\right)^2 = 2.25$

S.D. = 1.5

7.(A) $\frac{n(n+1)\,(2n+1)}{6 \times n} = \frac{46n}{11} \quad \Rightarrow \quad \frac{(n+1)\,(2n+1)}{6} = \frac{46n}{11} \quad \Rightarrow \quad n = 11$

8.(C) $\frac{100}{n} - \left(\frac{20}{n}\right)^2 = 2^2$

9.(B) Shifting has no effect only multiplication factor is important.

10.(B) $\frac{400}{n} - \left(\frac{80}{n}\right)^2 = \sigma^2 \quad \Rightarrow \quad \frac{400}{n} - \left(\frac{80}{n}\right)^2 \geq 0 \quad \Rightarrow \quad \frac{400}{n} \geq \frac{80}{n} \cdot \frac{80}{n} \quad \Rightarrow \quad n \geq 16$

11.(D) Formula based

12.(A) Formula based

13.(C) Standard Formula

14.(D) Mean deviations is minimum when deviations are taken from median as median is the middle most value of the series and disperse the whole series into minimal value.

15.(A) $\frac{\sum fx^2}{\sum f} = \frac{{}^nC_0.1 + {}^nC_1.a^2 + {}^nC_2.a^4 + ...{}^nC_n.a^{2n}}{{}^nC_0 + {}^nC_1 + ...{}^nC_n} = \frac{(1+a^2)^n}{2^n}$

$\left(\frac{\sum fx}{\sum f}\right)^2 = \frac{({}^nC_0.1 + {}^nC_1.a + {}^nC_n.a^n)^2}{(2^n)^2} = \left(\frac{1+a}{2}\right)^{2n}$

16.(C) Mean = $a + \frac{2n(2n+1)d}{2.(2n+1)} = a + nd$

Mean deviation about mean $= \frac{d[n + (n-1) + ...1] \times 2}{2n+1} = \frac{n(n+1)d}{2n+1}$

17.(A) Mean = $\frac{494}{10} = 49.4$

Mean deviation $= \frac{\sum |x_i - \bar{x}|}{10}$ = 8.88

18.(C) Mean = 5

Mean dev. $= \frac{\sum |x_i - \bar{x}|}{5} = 1.2$

19.(A) Formula based

20.(C) Formula based

21.(D) $\sigma_{new} = \frac{8}{|-2|} = 4$

22.(B) $\left|\frac{a}{c}\right| = -\frac{a}{c}$

23.(D) $\frac{400}{n} - \left(\frac{80}{n}\right)^2 \geq 0 \Rightarrow \frac{400}{n} \geq \frac{80}{n} \times \frac{80}{n} \Rightarrow n \geq 16$

24.(B) $\frac{70 \times 75 + 30 \times n}{100} = 72 \Rightarrow 7 \times 75 + 3n = 72 \times 10 \Rightarrow 7 \times 25 + n = 240 \Rightarrow n = 240 - 175 = 65$

25.(D) No change

26.(A) Shifting has no change on variance.

27.(B) Learn as result

28.(C) $0.45 = \frac{\sigma}{12} \Rightarrow \sigma = 12 \times 0.45 = 5.4$

29.(A) Correct $\sum x = 170 - 20 + 30 = 180$

Correct $\sum x^2 = 2830 - 20^2 + 30^2 = 3330$

$Var. = \frac{3330}{15} - \left(\frac{180}{15}\right)^2 = 222 - 144 = 78$

30.(C) $\frac{a+b+8+5+10}{5} = 6 \Rightarrow a + b = 7$

$\frac{a^2+b^2+64+25+100}{5} - 6^2 = 6.80 \Rightarrow a^2 + b^2 = 25$

$b = 7 - a \Rightarrow a^2 + 49 + a^2 - 14a = 25 \Rightarrow 2a^2 - 14a + 24 = 0 \Rightarrow a^2 - 7a + 12 = 0 \Rightarrow (a-3)(a-4) = 0$

31.(B) Correct mean = $\frac{40 \times 200 - 50 + 40}{200} = 39.95$

Old var $= 15^2 = 225 \Rightarrow \frac{\sum x^2}{200} - 40^2 = 225 \Rightarrow \sum x^2 = (225 + 1600) \times 200$

Correct $\sum x^2 = (225 + 1600) \times 200 - 50^2 + 40^2$

32.(C) Min. at $\frac{-b}{2a}$

33.(B) Median $= \frac{52+53}{2} = 52.5$

34.(B) Mean dev. from median $= \frac{[51.5 + 50.5 + ...0.5] \times 2}{104} = \frac{1+3+5+.....103}{104} = \frac{52 \times 52}{104} = 26$

35. Mean $= \frac{{}^nC_0 + {}^nC_1 + ...{}^nC_n}{n+1} = \frac{2^n}{n+1}$

36.(B) $\sum x = a\sum U + b\sum V$

$\frac{\sum x}{n} = \frac{a\sum U}{n} + \frac{b\sum V}{n} \Rightarrow \bar{x} = a\bar{U} + b\vec{V}$

37. Let $n = 2m$

$M = 1, 2, 3, \ldots\ldots. n$

Mean $= \frac{\sum 2m}{n} = \frac{2\times n.(n+1)}{2.n} = n+1$

$$Var. = \frac{\sum 4m^2}{n} - (n+1)^2 = \frac{4.n.(n+1)\,(2n+1)}{6.n} - (n+1)^2 = (n+1)\left[\frac{4}{3}n + \frac{2}{3} - n - 1\right] = \frac{(n+1)\,(n-1)}{3} = \frac{n^2-1}{3}$$

38. **(a)** New var. $= K^2.\sigma^2$

(b) Same S.D. as all items are being increased. Median will increase by 2.

39. **(a)** $0.50 = \frac{21.2}{\text{Mean}} \Rightarrow$ Mean = 42.4

(b) $\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2}$

$$\sigma^2 = \frac{1}{(n_1+n_2)}\left[n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)\right]$$

$d_1 = \bar{x} - \bar{x}_1$ $\quad$ $d_2 = \bar{x} - \bar{x}_2$

$\bar{x}_1 = 25$ $\quad$ $n_1 = 200$ $\quad$ $\bar{x}_2 = 10$ $\quad$ $n_2 = 300$

$\bar{x} = 16$ $\quad$ $d_1 = -9$ $\quad$ $d_2 = 6$

$$\sigma^2 = \frac{1}{500}[200(9+81) + 300(16+36)]$$

$$\sigma^2 = 67.2$$

40. New mean $= \frac{n\bar{x} + n(a-b)}{n} = \bar{x} + (a-b)$

41. $T_1 + T_2 + \ldots + T_{10} = 10\times 1445$

New mean $= \frac{10\times1445+1500}{11} = \frac{11\times1445+55}{11} = 1445+5 = 1450$

42. $\sum(x_i - 50) = -10$

$\Rightarrow \sum x_i - 50n = -10 \Rightarrow \sum x_i = 50n - 10$ ……. (1)

Similarly, $\sum x_i = 46n + 70$ …….. (2)

$\Rightarrow 50n - 10 = 46n + 70 \Rightarrow 4n = 80 \Rightarrow n = 20$

Mean $= \frac{50n-10}{n} = 50 - \frac{10}{20} = 49.5$

43. LHS = $x_1 + x_2 + \ldots x_{10} - 10\bar{x} = 10\bar{x} - 10\bar{x} = 0$

44. Correct total $= 200\times50 - 92 - 8 + 192 + 88$ = 1000 + 180= 1180

Correct mean $= \frac{1180}{50} = 23.6$

46. Correct total $= 100 \times 40 - 83 + 53 = 3970$

Mean $= \dfrac{3970}{100} = 39.7$

47. $\sum x_i - 12n = -10$

$\Rightarrow \sum x_i = 12n - 10$ (1)

and $\sum x_i = 3n + 62$ (2)

$\Rightarrow$ $12n - 10 = 3n + 62 \Rightarrow 9n = 72 \Rightarrow$ $n = 8$

$\bar{x} = \dfrac{12n - 10}{n} = \dfrac{86}{8} = 10.75$

48. Data in ascending order 41, 43, 51, 58, 61, 71, 92, 99, 127

Median is $T_5 = 61$

New data 41, 43, 51, 61, 71, 85, 92, 99, 127

New median is 71

49. A.T.P. $x + x + 2 = 2 \times 63 \Rightarrow 2x + 2 = 126 \Rightarrow 2x = 124 \Rightarrow x = 62$

50. Combined mean $= \dfrac{29 \times 71 + 31 \times 48}{29 + 31} = 59.12$

51. Let initial price be 100 units. (in 1997)

Price in $2000 = 100 \times 1.05 \times 1.08 \times 1.53$

Average increase $= \dfrac{73.502}{3} = 24.5\%$

52. Correct total $= 4 \times 22 - 14 + 40$; Correct mean $= \dfrac{114}{4} = 28.5$

53. $w_1T_1 + w_2T_2 + ... w_9T_9 + w_{10} \times 60 = (w_1 + w_2 + ... + w_9 + w_{10}) \times 36$

$w_1T_1 + w_2T_2 + ... w_9T_9 + w'_{10} \times 40 = (w_1 + w_2 + ... + w_9 + w'_{10}) \times 36$

Subtracting $w_{10} \times 60 - w'_{10} \times 40 = w_{10} \times 36 - w'_{10} \times 36 \Rightarrow w_{10} \times 24 = w'_{10} \times 4 \Rightarrow \dfrac{w'_{10}}{w_{10}} = \dfrac{6}{1}$

54. (a) $G_1G_2........ G_n = G^n$

Now $G_1 \cdot 2G_2 \cdot 3G_3 .... \cdot nG_n = n!G^n$

Mean $= \left(n!G^n\right)^{1/n} = (n!)^{1/n} \cdot G$

(b) Let $T_1 + T_2 + ... + T_k = k.A$

Now $nT_1 + nT_2 + ... + nT_k = nk.A$

Mean $= \dfrac{nk.A}{k} = nA$

55. Mean $= a + \dfrac{d(2n)(2n+1)}{(2n+1)2} = a + nd$

Mean dev. From mean

$$= \frac{\{(a+nd-a)+(a+nd-a-d)+...[a+nd-a-(n-1)d]\}\times 2}{2n+1} = \frac{d\cdot[n+(n-1)+...1]\times 2}{2n+1} = \frac{d\cdot n\cdot(n+1)}{2n+1}$$

56. Standard property

57. ($\bar{X}_{12} = 51.67;\ \sigma_{12} = 7.6$)

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2}$$

$n_1 = 50$ $\quad$ $n_2 = 100$

$\bar{x}_1 = 54.8$ $\quad$ $\bar{x}_2 = 50.3$

$d_1 = \bar{x} - \bar{x}_1$ $\quad$ $d_2 = \bar{x} - \bar{x}_2$

$\sigma_1 = 8$ $\quad$ $\sigma_2 = 7$

$$\sigma^2 = \frac{1}{(n_1+n_2)}\left[n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)\right]$$

58. $G_1G_2 .... G_6 = 11^6$

Let $G_1 = 1$

Now G_1 is changed to 64

So, new $G_1G_2 ....G_6 = 11^6 \times 64$

Mean $= (11^6 \times 2^6)^{1/6} = 11 \times 2 = 22$

59. $\dfrac{x\times 2000 + (1000-x)\times 4000}{1000} = 2400 \quad \Rightarrow \quad x = 800$

80% males ; 20% females

60. $A.M. = \dfrac{x+2x+3x+4x}{4} = 2.5x\,\text{m/s}$

$$H.m. = \left(\frac{\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x}+\frac{1}{4x}}{4}\right)^{-1} = \left(\frac{12+6+4+3}{12x\times 4}\right)^{-1} = \frac{48x}{25}$$

$t_1 = \dfrac{d}{x};\ t_2 = \dfrac{d}{2x};\ t_3 = \dfrac{d}{3x};\ t_4 = \dfrac{d}{4x}$

Total time $= \dfrac{d}{x}\left(1+\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}\right)$

Avg. speed $= \dfrac{4d}{\frac{d}{x}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)} = \dfrac{48x}{25}$

61. ($\bar{X} = 39.9$; $\sigma = 4.9$)

Correct total $= 4000 - 50 + 40 = 3990$

Old var. = 25

$$\Rightarrow \frac{\sum x_i^2}{100} - 40^2 = 25 \quad \Rightarrow \sum x_i^2 = 1625 \times 100$$

Correct $\sum x_i^2 = 1625 \times 100 - 50^2 + 40^2$

$$\text{Var.} = \frac{\text{Correct} \sum x_i^2}{100} - \left(\frac{3990}{100}\right)^2$$

62. Sum of squares of deviations

$$= (x_1 - \alpha)^2 + (x_2 - \alpha)^2 + ... + (x_n - \alpha)^2 = n\alpha^2 - 2\alpha\left(\sum x_1\right) + \sum x_1{}^2$$

Let this be $f(\alpha)$

It is min. at $\alpha = -\frac{b}{2a}$

$$= -\frac{(-2)\left(\sum x_1\right)}{2n} = \frac{\sum x_1}{n} = \text{mean.}$$

63.(Mean = 30.005, Standard Deviation = 0.01)

Class Marks (x_i)	f	$f_i x_i$	$f_i x_i^2$
5	12		
10	28		
15	65		
20	121		
25	175		
30	198		
35	176		
40	120		
45	66		
50	27		
55	9		
60	3		

Calculate $f_i x_i$ for each row and $f_i x_i^2$ for each row

Mean $= \frac{\sum f_i x_i}{\sum f_i}$; Variance $= \frac{\sum f_i x_i^2}{\sum f_i} - (mean)^2$; S.D. $= \sqrt{var}$.

64. **(a)** $x_1 + x_2 + ...x_{10} = 17 \times 10$

Let $x_{10} = 26$

So, $x_1 + x_2 + ...x_9 = 170 - 26$

New Mean $= \dfrac{144}{9} = 16$

$\dfrac{x_1^2 + x_2^2 + ... + x_{10}^2}{10} - 17^2 = 33$; $x_{10} = 26$ so find $x_1^2 + x_2^2 + ...x_9^2$

$Var. = \dfrac{x_1^2 + x_2^2......x_9^2}{9} - (New\,Mean)^2$

(b) Similar to previous

(c) 43 was to be taken as 42.5

53 was taken as 52.5

So 10 will get subtracted from total

Mean $= 40 - \dfrac{10}{200} = 39.95$

$\dfrac{\sum f_i x_i^2}{200} - 40^2 = 15^2 \Rightarrow \sum f_i x_i^2 = 1825 \times 200$

Corrected $\sum f_i x_i^2 = 1825 \times 200 - (52.5)^2 + (42.5)^2 = 364050$

$Var. = \dfrac{364050}{200} - (39.95)^2$

65.(4, 9) $\dfrac{1+2+6+a+b}{5} = 4.4 \Rightarrow a+b=13$; $\dfrac{1^2+2^2+6^2+a^2+b^2}{5} - (4.4)^2 = 8.24$

66. **(a)** compare S.D. **(b)** Compare S.D.

67. (Combined mean = 51.57, Combined S.D. = 7.5 approx.)

$\bar{x} = \dfrac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$

$d_1 = \bar{x} - \bar{x}_1$ $d_2 = \bar{x} - \bar{x}_2$

$\sigma^2 = \dfrac{1}{(n_1 + n_2)}\left[n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2) \right]$

69. Mean $= \dfrac{1+2+....+n}{n} = \dfrac{n+1}{2}$

Var. $= \dfrac{n(n+1)(2n+1)}{6 \times n} - \left(\dfrac{n+1}{2}\right)^2$

70. **(** n is odd, M.D. = $\frac{n^2-1}{4n}$; n is even, M.D. = $\frac{n}{4}$;

Variance = $\frac{n^2-1}{12}$**)**

Let frequency $=x$

Mean $=\frac{x\times 0.5+x\times 1.5+....+x\times\left(\frac{2n-1}{2}\right)}{nx}=\frac{(1+3+...+2n-1)}{2\cdot n}=\frac{n^2}{2n}=\frac{n}{2}$

Mean deviation (if n is even)

$$=\frac{\left[\left(\frac{n}{2}-\frac{1}{2}\right)+\left(\frac{n}{2}-\frac{3}{2}\right)+...+\left[\frac{n}{2}-\left(\frac{n-1}{2}\right)\right]\right]\times 2}{n}$$

Mean deviation (if n is odd)

$$=\frac{\left[\left(\frac{n}{2}-\frac{1}{2}\right)+\left(\frac{n}{2}-\frac{3}{2}\right)+...+\frac{n}{2}-\left(\frac{n-2}{2}\right)\right]\times 2}{n}$$

For variance we can use data 1, 2, 3 ………………n as shifting has no effect

Variance = $\frac{n^2-1}{12}$

71. Mean $=\frac{\sum\frac{ax-b}{c}}{n}=\frac{a}{c}\frac{\sum x}{n}-\frac{n\cdot b}{c\cdot n}=\frac{a\bar{x}-b}{c}=\frac{am-b}{c}$

S.D. will get multiplied by a factor of $\left|\frac{a}{c}\right|$.

72. $\bar{x}=\frac{m_1x_1+m_2x_2+....}{m_1+m_2....}=\frac{\Sigma m_r x_r}{\Sigma m_r}$

$Var.=\frac{\Sigma m_r x_r^2}{\Sigma m_r}-\left(\frac{\Sigma m_r x_r}{\Sigma m_r}\right)^2$

Now, $\frac{\Sigma m_r (k-x_r)^2}{\Sigma m_r}-\delta^2$

$$=\frac{k^2\Sigma m_r+\Sigma m_r x_r^2-2k\Sigma m_r x_r}{\Sigma m_r}-(\bar{x}-k)^2=k^2+\frac{\Sigma m_r x_r^2}{\Sigma m_r}-2k\bar{x}-\bar{x}^2-k^2+2k\bar{x}=\frac{\Sigma m_r x_r^2}{\Sigma m_r}-\left(\frac{\Sigma m_r x_r}{\Sigma m_r}\right)^2$$

74. $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_1^2 - \bar{x}^2 = \frac{1}{n}(x_1^2 + x_2^2 + ... + x_n^2) - \frac{T^2}{n^2}$

Where $T = x_1 + x_2 + ... + x_n$

Let σ_1^2 be the corrected variance. Then $\sigma_1^2 = \frac{1}{n}\{x_1^2 + x_2^2 + ... + x_n^2\} - \left\{\frac{T - x_1 + x_1'}{n}\right\}^2$

Adjustment to the variance to correct the error is:

$$\sigma_1^2 - \sigma^2 = \frac{1}{n}\{x_1'^2 - x_1^2\} - \frac{1}{n^2}\left\{(T - x_1 + x_1')^2 - T^2\right\} = \frac{1}{n}\{x_1' + x_1\}\{x_1' - x_1\} - \frac{1}{n^2}\{(x_1' - x_1) \times (2T - x_1 + x_1')\}$$

75. $N = \Sigma f = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = (1+1)^n = 2^n$

$$\Sigma fx = 0 \cdot {}^nC_0 + 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 + ... + n \cdot {}^nC_n$$

$$= \left\{1 + (n-1) + \frac{(n-1)(n-2)}{2!} + ... + 1\right\} = n(1+1)^{n-1} = n \cdot 2^{(n-1)}$$

Hence mean $(\bar{x}) = \frac{1}{N}\Sigma fx = \frac{n \cdot 2^{(n-1)}}{2^n} = \frac{n}{2}$

The mean square deviation s^2 (say), about the point $x = 0$ is given by

$$s^2 = \frac{1}{N}\Sigma fx^2 = \frac{1}{2^n}[1^2 \cdot {}^nC_1 + 2^2 \cdot {}^nC_2 + 3^2 \cdot {}^nC_2 + ... + n^2 \cdot {}^nC_n]$$

$$= \frac{n}{2^n}\left[1 + 2(n-1) + \frac{3}{2}(n-1)(n-2) + ... + n\right]$$

$$= \frac{n}{2^n}\left(\left\{1 + (n-1) + \frac{(n-1)(n-2)}{2!} + ... + 1\right\} + \left\{(n-1) + (n-1)(n-2) + ... + (n-1)\right\}\right)$$

$$= \frac{n}{2^n}\left[({}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + ... + {}^{n-1}C_{n-1}) + \left\{(n-1)({}^{n-2}C_0 + {}^{n-2}C_1 + ... + {}^{n-2}C_{n-2})\right\}\right]$$

$$= \frac{n}{2^n}\left[(1+1)^{n-1} + (n-1)(1+1)^{n-2}\right] = \frac{n(n+1)}{4} \qquad \therefore \qquad \sigma^2 = \frac{n(n+1)}{4} - \frac{n^2}{4} = \frac{n}{4}$$

76. Since $x_i - \bar{x} \le r,\ i = 1, 2, ..., n,$ we have

$$s^2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 \le \frac{1}{N}\sum_{i=1}^{n} f_i(r^2) \Rightarrow s^2 \le i^2 \frac{1}{N}\sum_{i=1}^{n} f_i = r^2 \qquad \Rightarrow \qquad s \le r$$